

# On Some Differential Inequalities and the Uniqueness of Global Semiclassical Solutions to the Cauchy Problem for Weakly-coupled Systems\*

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We combine our previous method of multifunctions and differential inclusions with the technique of Carathéodory comparison equations and consider some partial differential inequalities of Haar type. In this way, certain new uniqueness criteria for global semiclassical solutions to weakly-coupled systems will be derived.

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## 1. INTRODUCTION

The main applications of differential inequality theory concern questions such as: estimates of solutions of differential equations, criteria of uniqueness and of continuous dependence on initial data and right sides of equations for solutions . . . The subjects have been well studied by many authors; see, e.g., Chaplygin [8], Deimling [13], Haar [15],

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Hartman [16], Kamke [18], Lakshmikantham and Leela [21], Nagumo [22,23], Szarski [24], Walter [31], Ważewski [32,33], etc. For recent results in functional setting, we refer the reader to [3–6] and the references therein.

In [21,24] a differential inequality of Haar type

$$|u_t| \leq M|\nabla_x u| + \rho(t, |u|),$$

with  $\rho \in C^0$  the right side of a certain *comparison differential equation*, was considered. The estimate of  $u$  was established by means of extremal solutions of the corresponding comparison equation  $w' = \rho(t, w)$ . From this, one can derive some uniqueness criteria of classical solutions to the Cauchy problem for evolution partial differential equations. We emphasize here that these criteria may be used only locally.

Let us mention that the global existence and uniqueness of generalized solutions for convex Hamilton–Jacobi equations were well studied by several methods: variational method [10], method of envelopes [1], vanishing viscosity method [14,19], etc. The global theory for nonconvex Hamilton–Jacobi equations has recently been considered by Crandall and coworkers [11,12] and Ishii [17], etc. They have introduced the notion “viscosity solutions” to define generalized solutions and characterized their properties. By these contributions, the global existence and uniqueness of generalized solutions have been established almost completely. However, it should be noted that viscosity solutions of partial differential equations are, as regular as possible, in general just continuous. They may therefore contain singularities. So what kinds of phenomena would appear when we extend the classical (local) solutions? In such a procedure, we must go back (for this see [26]) to Haar’s lemma [16, Chapter VI, Lemma 10.1]. Of course, furthermore, the *a priori* estimates from the lemma (or something like it) are of much interest from various points of view.

Recently, Van and Thai Son [29,30] have provided a new method, based on the theory of multifunctions and differential inclusions, to integrate the differential inequality of the form

$$|u_t| \leq \ell(t)[(1 + |x|)|\nabla_x u| + \mu(x)|u|],$$

with  $\mu$  a function locally bounded on  $\mathbb{R}^n$  and  $\ell \in L^1(0, +\infty)$ . The result plays a key role in investigating the uniqueness of the so-called *global*

*semiclassical solutions* to the Cauchy problem for a single first-order partial differential equation with time-measurable Hamiltonian. Particularly, an answer to an open uniqueness problem of Kruzhkov [20] is therein given by the study of such solutions, whose existence has been considered in [27,28].

In this paper we combine the method of multifunctions and differential inclusions in [29,30] with the technique of Carathéodory comparison equations and prove some new uniqueness criteria for weakly-coupled systems. The paper will be organized as follows. In Section 2 the notion of comparison equation in [24] is extended to the Carathéodory case. Section 3 concerns a system of differential inequalities of Haar type. Finally, in Section 4 we give some results of uniqueness of global semiclassical solutions to systems of first-order partial differential equations. They are new even when restricted to the classical case of a single equation.

From now on  $n, m$  stand for certain positive integers,  $0 < T < +\infty$ , and

$$\Omega_T \stackrel{\text{def}}{=} (0, T) \times \mathbb{R}^n = \{(t, x) : 0 < t < T, x \in \mathbb{R}^n\}.$$

The notation  $\nabla_x$  will denote the gradient  $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ . Let  $|\cdot|$  and  $\langle \cdot, \cdot \rangle$  be the Euclidean norm and scalar product in  $\mathbb{R}^n$ , respectively.

Denote by the set of all locally Lipschitz continuous functions  $u$  defined on  $\Omega_T$  by  $\text{Lip}(\Omega_T)$ . Further, set  $\text{Lip}([0, T) \times \mathbb{R}^n) \stackrel{\text{def}}{=} \text{Lip}(\Omega_T) \cap C([0, T) \times \mathbb{R}^n)$ . For every function  $u$  defined on  $\Omega_T$ , we put

$$\text{Dif}(u) \stackrel{\text{def}}{=} \{(t, x) \in \Omega_T : u \text{ is differentiable at } (t, x)\}.$$

We shall be concerned with the following class of Lipschitz continuous functions:

$$\begin{aligned} V(\Omega_T) \stackrel{\text{def}}{=} \{u \in \text{Lip}([0, T) \times \mathbb{R}^n) : \\ \exists G \subset [0, T] \text{ mes}(G) = 0, \text{Dif}(u) \supset \Omega_T \setminus (G \times \mathbb{R}^n)\}. \end{aligned}$$

(Here, “mes” signifies the Lebesgue measure on  $\mathbb{R}^1$ .) In other words, a function  $u \in \text{Lip}([0, T) \times \mathbb{R}^n)$  belongs to  $V(\Omega_T)$  if and only if for almost all  $t$ , it is differentiable at any point  $(t, x)$ .

Finally, consider the class

$$V^m(\Omega_T) \stackrel{\text{def}}{=} \underbrace{V(\Omega_T) \times \cdots \times V(\Omega_T)}_{m \text{ times}}.$$

Each element of  $V^m(\Omega_T)$  is therefore a vector function, namely  $u = (u_1, \dots, u_m)$  from  $\Omega_T \subset \mathbb{R}^{n+1}$  into  $\mathbb{R}^m$  such that  $u_j$  belongs to  $V(\Omega_T)$  for every  $j \in \{1, \dots, m\}$ .

## 2. CARATHÉODORY COMPARISON DIFFERENTIAL EQUATIONS

For our next discussions, we need to extend the notion of *comparison equations* given in [24] to the Carathéodory case. Consider an ordinary differential equation

$$w' = \rho(t, w), \quad (2.1)$$

where the function  $\rho$  is defined on  $D_+ \stackrel{\text{def}}{=} (0, +\infty) \times [0, +\infty) = \{(t, w) : t > 0, w \geq 0\}$ . The following Carathéodory conditions are always assumed:

- (1) For almost every  $t \in (0, +\infty)$  the function  $[0, +\infty) \ni w \mapsto \rho(t, w)$  is continuous.
- (2) For each  $w \in [0, +\infty)$  the function  $(0, +\infty) \ni t \mapsto \rho(t, w)$  is measurable.
- (3) For any  $r \in (0, +\infty)$  there exists a function  $m_r \in L^1_{\text{loc}}(0, +\infty)$  with

$$|\rho(t, w)| \leq m_r(t) \quad \forall w \in [0, r]$$

for almost every  $t \in (0, +\infty)$ .

In this situation we call (2.1) a *Carathéodory differential equation* on  $D_+$ . A *solution* of it on an interval  $I \subset (0, +\infty)$ , with  $\text{int } I \neq \emptyset$ , is meant a nonnegative function  $w(\cdot)$  absolutely continuous on each compact interval  $J \subset I$  (*absolutely continuous on  $I$*  for short) such that

$$w'(t) = \rho(t, w(t))$$

almost everywhere in  $I$ . We refer to [9] for what concerns the local existence of a solution of (2.1) through any given point  $(t^0, w^0) \in \text{int } D_+$ .

Moreover, every such solution can be extended (as a solution) over a [left, right] maximal interval of existence.

**DEFINITION A** Carathéodory differential equation (2.1), with  $\rho(t, w) \geq 0$  on  $D_+$  and  $\rho(t, 0) = 0$  for almost all  $t > 0$ , will be called a *comparison equation* if  $w = w(t) \equiv 0$  is in every interval  $(0, \gamma)$  the only solution satisfying the condition  $\lim_{t \rightarrow 0} w(t) = 0$ .

*Remark* Let  $\ell$  be a nonnegative function Lebesgue integrable on each bounded interval  $(0, \gamma) \subset \mathbb{R}$ , and  $\sigma \in C[0, +\infty)$  be such that  $\sigma(0) = 0$ ,  $\sigma(w) > 0$  as  $w > 0$ , and  $\int_0^\delta (1/\sigma(w)) dw = +\infty$  for every  $\delta > 0$ . Then (cf. [24, Example 14.2])

$$w' = \ell(t)\sigma(w) \tag{2.2}$$

is a comparison equation. In fact, assume the contrary that (2.2) admits a nonzero solution  $w(\cdot)$  on some interval  $(0, \gamma)$  with  $\lim_{t \rightarrow 0} w(t) = 0$ . Letting  $w(0) \stackrel{\text{def}}{=} 0$ , from this we easily find a nonempty subinterval  $(t^1, t^2]$  of  $(0, \gamma)$  such that  $w(t^1) = 0$  and  $w(t) > 0$  for all  $t \in (t^1, t^2]$ . It follows that

$$\int_0^{w(t^2)} \frac{dv}{\sigma(v)} = \int_{t^1}^{t^2} \frac{w'(t)}{\sigma(w(t))} dt = \int_{t^1}^{t^2} \ell(t) dt < +\infty,$$

a contradiction. Therefore (2.2) must be a comparison equation. Motivated by this fact, we propose the following:

**PROPOSITION 2.1** *Let  $\sigma \in C[0, +\infty)$ , and  $\ell \geq 0$  a function Lebesgue integrable on each bounded interval  $(0, \gamma) \subset \mathbb{R}$  with  $\int_0^{+\infty} \ell(t) dt = +\infty$ .*

(i) *If (2.2) is a comparison equation, then so is the equation*

$$w' = \sigma(w). \tag{2.3}$$

(ii) *Conversely, under the condition  $\text{ess inf}_{t \in (0, +\infty)} \ell(t) > 0$ , if moreover (2.3) is a comparison equation, then so is (2.2).*

*Proof* (i) Let  $w^1(\cdot)$  be a solution of (2.3) on some interval  $(0, \gamma^1)$  with  $\lim_{t \rightarrow 0} w^1(t) = 0$ . Find a number  $\gamma^2 > 0$  such that

$$\gamma^1 = \int_0^{\gamma^2} \ell(\tau) d\tau. \tag{2.4}$$

Setting  $w^2(t) \stackrel{\text{def}}{=} w^1(\int_0^t \ell(\tau) d\tau)$ , we see that  $w^2(\cdot)$  is a solution of (2.2) on  $(0, \gamma^2)$  with  $\lim_{t \rightarrow 0} w^2(t) = 0$ . By assumption,  $w^2(t) \equiv 0$  on  $(0, \gamma^2)$ . Hence  $w^1(t) \equiv 0$  on  $(0, \gamma^1)$ . This shows that (2.3) is a comparison equation.

(ii) Let  $(0, +\infty) \ni t \mapsto \hat{\ell}(t)$  be the inverse of  $(0, +\infty) \ni t \mapsto \int_0^t \ell(\tau) d\tau$ , and  $w^2(\cdot)$  be a solution of (2.2) on some interval  $(0, \gamma^2)$  with  $\lim_{t \rightarrow 0} w^2(t) = 0$ . First, define a number  $\gamma^1 > 0$  by (2.4). Then setting  $w^1(t) \stackrel{\text{def}}{=} w^2(\hat{\ell}(t))$ , we also see that  $w^1(\cdot)$  is a solution of (2.3) on  $(0, \gamma^1)$  with  $\lim_{t \rightarrow 0} w^1(t) = 0$  (cf. [13, Proposition 3.4(c)]). The rest of the proof runs as before.

In the sequel, for each function  $g$  defined and continuous in a certain interval  $(0, t^0)$ , let  $P_g$  denote the open set  $\{t \in (0, t^0) : g(t) > 0\}$ . Here is an elementary property of comparison equations:

**PROPOSITION 2.2** *Let (2.1) be a comparison equation and  $g$  be a given function absolutely continuous on some interval  $(0, t^0)$  such that  $\lim_{t \rightarrow 0} g(t) \leq 0$  and that  $g'(t) \leq \rho(t, g(t))$  almost everywhere in  $P_g$ . Then  $g(t) \leq 0$  for all  $t \in (0, t^0)$ .*

*Proof* On the contrary, suppose that there exists  $t^1 \in (0, t^0)$  with  $\bar{w} \stackrel{\text{def}}{=} g(t^1) > 0$ . Setting  $g(0) \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} g(t)$  and  $t^2 \stackrel{\text{def}}{=} \sup\{t \in [0, t^1] : g(t) = 0\}$ , we see that  $0 \leq t^2 < t^1$ ,  $g(t^2) = 0$  and  $(t^2, t^1) \subset P_g$ . Hence, by assumption,

$$g'(t) \leq \rho(t, g(t)) \quad \text{almost everywhere in } (t^2, t^1). \tag{2.5}$$

Now take

$$\hat{\rho}(t, w) \stackrel{\text{def}}{=} \begin{cases} \rho(t, \max\{0, g(t)\}) & \text{if } t^2 < t < t^0, w \geq \max\{0, g(t)\}, \\ \rho(t, w) & \text{if } t^2 < t < t^0, 0 \leq w < \max\{0, g(t)\}. \end{cases} \tag{2.6}$$

The Carathéodory conditions (1)–(3) mentioned earlier are clearly satisfied for  $\hat{\rho}$  on  $(t^2, t^0) \times [0, +\infty)$ . Let  $w(\cdot)$  be a solution through  $(t^1, \bar{w})$  of (2.1) with  $\hat{\rho}$  in place of  $\rho$ , and let  $(t^3, t^1) \subset (t^2, t^1]$  be its left maximal interval of existence. We next claim that

$$(0 \leq) w(t) \leq g(t) \quad \forall t \in (t^3, t^1]. \tag{2.7}$$

Assume (2.7) is false. Then one would find a nonempty interval  $(t^4, t^5) \subset (t^3, t^1)$  such that

$$w(t) > g(t) \quad \forall t \in (t^4, t^5), \tag{2.8}$$

with

$$w(t^5) = g(t^5). \tag{2.9}$$

It follows from (2.5), (2.6) and (2.8) that  $g'(t) \leq \rho(t, g(t)) = \hat{\rho}(t, w(t)) = w'(t)$  almost everywhere in  $(t^4, t^5)$ . Thus (2.9) implies that  $g(t) \geq w(t)$  for all  $t \in (t^4, t^5)$ , which contradicts (2.8). So (2.7) must hold.

We proceed to show that  $t^3 = t^2$ . In fact, if  $(0 \leq) t^2 < t^3$ , then (2.6) together with Carathéodory's condition (3), where  $r \stackrel{\text{def}}{=} \max\{g(t) : t \in [t^3, t^1]\}$ , prove that  $\lim_{t \rightarrow t^3} w(t)$  exists and is finite; hence  $w(\cdot)$  could be extended (as a solution of (2.1) with  $\hat{\rho}$  in place of  $\rho$ ) over an interval  $(t^6, t^1) \supset [t^3, t^1]$ , which is impossible.

Finally, (2.6) and (2.7) show that  $w(\cdot)$  is indeed a solution through  $(t^1, \bar{w})$  of (2.1) on  $(t^2, t^1]$  with  $\lim_{t \rightarrow t^2} w(t) = g(t^2) = 0$ . Setting  $w(t) \stackrel{\text{def}}{=} 0$  for  $t \in [0, t^2]$ , we obtain a nonzero solution of (2.1) on  $(0, t^1)$  which tends to 0 as  $t$  tends to 0; thus we arrive at a contradiction. This completes the proof.

### 3. DIFFERENTIAL INEQUALITIES OF HAAR TYPE

We can now combine the method of multifunctions and differential inclusions in [29,30] with the technique of Carathéodory comparison equations and prove the following:

**THEOREM 3.1** *Let  $u$  be a vector function in  $V^m(\Omega_T)$  with  $u_j(0, x) \equiv 0$  ( $j = 1, \dots, m$ ), and (2.1) be a comparison equation. If there exists a nonnegative function  $\ell \in L^1(0, T)$  such that*

$$|\partial u_j(t, x) / \partial t| \leq \ell(t)(1 + |x|) \cdot |\nabla_x u_j(t, x)| + \rho\left(t, \max_{k=1, \dots, m} |u_k(t, x)|\right) \quad (j = 1, \dots, m) \tag{3.1}$$

*for almost every  $t \in (0, T)$  and for all  $x \in \mathbb{R}^n$ , then  $u_j(t, x) \equiv 0$  in  $\Omega_T$  ( $j = 1, \dots, m$ ).*

*Proof* For an arbitrary point  $(t^0, x^0) \in \Omega_T$ , it suffices to prove that

$$\max_{j=1, \dots, m} |u_j(t^0, x^0)| = 0. \tag{3.2}$$

Let  $\bar{B}_r = \bar{B}_r^{\text{def}} \{x \in \mathbb{R}^n: |x| \leq r\}$ ,  $r \geq 0$ . Denote by  $\Sigma_I(t^0, x^0)$  the set of all absolutely continuous functions  $x(\cdot)$  from  $I \stackrel{\text{def}}{=} [0, t^0]$  into  $\mathbb{R}^n$  which satisfy almost everywhere in  $I$  the differential inclusion  $dx(t)/dt \in \hat{B}_{\ell(t) \cdot (1+|x(t)|)}$  subject to the constraint  $x(t^0) = x^0$ . From [7, Theorem VI-13], it follows that  $\Sigma_I(t^0, x^0)$  is a nonempty compact set in  $C(I, \mathbb{R}^n)$ . The sets  $Z(t, t^0, x^0) \stackrel{\text{def}}{=} \{x(t): x(\cdot) \in \Sigma_I(t^0, x^0)\}$  and  $\Gamma(t^0, x^0) \stackrel{\text{def}}{=} \{(\tau, x): \tau \in I, x \in Z(\tau, t^0, x^0)\}$  are therefore compact sets in  $\mathbb{R}^n$  and  $\mathbb{R}^{n+1}$ , respectively, for all  $t \in I$ . Moreover, by the converse of Ascoli's theorem, the multifunction

$$Z(\cdot, t^0, x^0): I \rightsquigarrow \mathbb{R}^n$$

is continuous. We now define

$$g(t) \stackrel{\text{def}}{=} \max_{k=1, \dots, m} g^k(t) \quad (3.3)$$

for  $t \in I$ , where

$$g^k(t) \stackrel{\text{def}}{=} \max\{|u_k(t, x)|: x \in Z(t, t^0, x^0)\} \quad (k = 1, \dots, m). \quad (3.4)$$

Then according to the Maximum Theorem (see [2, Theorem 1.4.16]), the fact that  $u \in C(\Gamma(t^0, x^0), \mathbb{R}^m)$  implies that  $g, g^1, \dots, g^m \in C(I)$ . In addition, it follows from [29, Lemma 1] that for any number  $\theta \in (0, t^0)$  each function  $g^k$  is absolutely continuous on  $[\theta, t^0]$  and so is the function  $g$ .

Going back to the proof of Theorem 3.1, we set now

$$h(t) \stackrel{\text{def}}{=} \int_0^t \ell(\tau) \, d\tau \quad \text{for } t \in [0, T].$$

It is well known that there exists a set  $G_1 \subset (0, T)$  of Lebesgue measure 0 with the property that

$$dh(t)/dt = \ell(t) \quad \forall t \in (0, T) \setminus G_1.$$

Obviously, (3.2) will be obtained if one can verify that  $g(t^0) = 0$ . Since  $g$  is a nonnegative function absolutely continuous on  $(0, t^0]$ , with  $\lim_{t \rightarrow 0} g(t) = g(0) = 0$  (by assumption), Proposition 2.2 shows that we need only claim that

$$g'(t) \leq \rho(t, g(t)) \quad \text{almost everywhere in } (0, t^0). \quad (3.5)$$



By the hypothesis of the theorem, one finds a set  $G_2 \subset (0, T)$  of Lebesgue measure 0 such that

$$\Omega_T \setminus (G_2 \times \mathbb{R}^n) \subset \bigcap_{k=1}^m \text{Dif}(u_k) \tag{3.6}$$

and that (3.1) holds for any  $t \in (0, T) \setminus G_2, x \in \mathbb{R}^n$ . Assume without loss of generality that  $g$  is differentiable at any point of  $(0, t^0) \setminus G$ , where  $G \stackrel{\text{def}}{=} G_1 \cup G_2$ . Now fix an arbitrary point  $t_* \in (0, t^0) \setminus G$  and take  $1 \leq j \leq m$  such that

$$g(t_*) = g^j(t_*) = |u_j(t_*, x_*)| = \varepsilon \cdot u_j(t_*, x_*), \quad \varepsilon \stackrel{\text{def}}{=} \text{sign } u_j(t_*, x_*) \tag{3.7}$$

for some  $x_* \in Z(t_*, t^0, x^0)$ . Then one may find a function  ${}^*x(\cdot) \in \Sigma_I(t^0, x^0)$  so that  ${}^*x(t_*) = x_*$ . Next, choose a unit vector  $e \in \mathbb{R}^n$  with

$$\langle e, \varepsilon \cdot \nabla_x u_j(t_*, x_*) \rangle = -|\nabla_x u_j(t_*, x_*)|. \tag{3.8}$$

Let  $y(\cdot)$  be a continuously differentiable  $\mathbb{R}^n$ -valued function of  $s \in \mathbb{R}^1$  such that  $y(h(t_*)) = x_*$  and  $dy/ds = (1 + |y|) \cdot e$ ; and let  $x(t) \stackrel{\text{def}}{=} y(h(t))$  for  $t \in [0, T]$ . Of course,  $x(\cdot)$  is absolutely continuous on  $[0, T]$ ,  $x(t_*) = x_*$ , and

$$\frac{dx}{dt}(t) = \ell(t) \cdot (1 + |x(t)|) \cdot e \quad \forall t \in (0, T) \setminus G. \tag{3.9}$$

Moreover, the function  $\bar{x}(\cdot)$  defined by

$$\bar{x}(t) \stackrel{\text{def}}{=} \begin{cases} x(t) & \text{if } 0 \leq t \leq t_*, \\ {}^*x(t) & \text{if } t_* \leq t \leq t^0 \end{cases}$$

clearly belongs to  $\Sigma_I(t^0, x^0)$ ; hence,

$$x(t) \in Z(t, t^0, x^0) \quad \forall t \in [0, t_*].$$

This, together with (3.3) and (3.4), implies

$$\varepsilon \cdot u_j(t, x(t)) \leq |u_j(t, x(t))| \leq g^j(t) \leq g(t) \quad \text{for all } t \in [0, t_*]. \tag{3.10}$$

Besides that, by (3.7),

$$\varepsilon \cdot u_j(t_*, x(t_*)) = |u_j(t_*, x_*)| = g^j(t_*) = g(t_*). \tag{3.11}$$

Therefore, since  $t_* \in (0, t^0) \setminus G$ , it may be deduced from (3.10) and (3.11) that

$$g'(t_*) \leq \frac{d}{dt} [\varepsilon \cdot u_j(t, x(t))] \Big|_{t=t_*}.$$

Consequently, by (3.1) and (3.7)–(3.9), we conclude that

$$\begin{aligned} g'(t_*) &\leq \varepsilon \cdot \left( \partial u_j(t_*, x(t_*)) / \partial t \right) + \left\langle \frac{dx}{dt}(t_*), \varepsilon \cdot \nabla_x u_j(t_*, x(t_*)) \right\rangle \\ &\leq |\partial u_j(t_*, x_*) / \partial t| - \ell(t_*) (1 + |x_*|) \cdot |\nabla_x u_j(t_*, x_*)| \\ &\leq \rho(t_*, |u_j(t_*, x_*)|) = \rho\left(t_*, \max_{k=1, \dots, m} |u_k(t_*, x_*)|\right) \\ &= \rho(t_*, g(t_*)). \end{aligned}$$

Finally, because  $G$  has measure 0 and  $t_* \in (0, t^0) \setminus G$  is arbitrarily chosen, (3.5) must hold. This completes the proof.

**THEOREM 3.2** *Let  $u$  be a vector function in  $V^m(\Omega_T)$  with  $u_j(0, x) \equiv 0$  ( $j = 1, \dots, m$ ), and (2.3) be a comparison equation. If there exist a nonnegative function  $\mu$  locally bounded on  $\mathbb{R}^n$  and a nonnegative function  $\ell \in L^1(0, T)$  such that*

$$\begin{aligned} |\partial u_j(t, x) / \partial t| &\leq \ell(t) \left[ (1 + |x|) \cdot |\nabla_x u_j(t, x)| \right. \\ &\quad \left. + \mu(x) \sigma\left(\max_{k=1, \dots, m} |u_k(t, x)|\right) \right] \quad (j = 1, \dots, m) \end{aligned} \tag{3.12}$$

for almost every  $t \in (0, T)$  and for all  $x \in \mathbb{R}^n$ , then  $u_j(t, x) \equiv 0$  in  $\Omega_T$  ( $j = 1, \dots, m$ ).

*Proof* For an arbitrary point  $(t^0, x^0) \in \Omega_T$ , it suffices to prove (3.2). Let us continue using the method (and notations) introduced in the proof of Theorem 3.1. We may extend the function  $\ell$  over the whole  $(0, +\infty)$  and assume  $\text{ess inf}_{t \in (0, +\infty)} \ell(t) > 0$ . Then by (3.12) (instead of (3.1)) we get

$$g'(t) \leq C \ell(t) \sigma(g(t)) \quad \text{almost everywhere in } (0, t^0)$$

(instead of (3.5)) for some positive constant  $C$ . By Proposition 2.1(ii), the Carathéodory differential equation

$$w' = C \ell(t) \sigma(w)$$

must also be a comparison one. Thus (3.2) is straightforward as before.

**4. UNIQUENESS OF GLOBAL SEMICLASSICAL SOLUTIONS TO THE CAUCHY PROBLEM FOR WEAKLY-COUPLED SYSTEMS**

The results in Section 3 can be used to investigate the uniqueness of global semiclassical solutions to the Cauchy problem for weakly-coupled systems of first-order partial differential equations, i.e., the problem of the form

$$\partial u_j / \partial t + f_j(t, x, u, \nabla_x u_j) = 0 \quad \text{in } \Omega_T \quad (j = 1, \dots, m), \quad (4.1)$$

$$u(0, x) = (\phi_1(x), \dots, \phi_m(x)) \quad \text{on } \{t = 0, x \in \mathbb{R}^n\}. \quad (4.2)$$

Here, the initial data  $\phi = (\phi_1, \dots, \phi_m)$  is a given vector function continuous on  $\mathbb{R}^n$ . Each Hamiltonian  $f_j = f_j(t, x, u, p^j)$  is always assumed to be measurable in  $t \in (0, T)$  and continuous in  $(x, u, p^j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ . First, we repeat the definition (in [25]) of the solutions to be considered.

**DEFINITION** A vector function  $u \in V^m(\Omega_T)$  is called a *global semiclassical solution* of (4.1)–(4.2) if it satisfies (4.1) for all  $x \in \mathbb{R}^n$  and almost all  $t \in (0, T)$  and if  $u(0, x) = \phi(x)$  for all  $x \in \mathbb{R}^n$ .

In this section we have:

**THEOREM 4.1** *Let (2.1) be a comparison equation. Suppose that  $f_j$  ( $j = 1, \dots, m$ ) satisfy the following conditions: there exists a nonnegative function  $\ell \in L^1(0, T)$  such that*

$$\begin{aligned} & |f_j(t, x, u, p^j) - f_j(t, x, v, q^j)| \\ & \leq \ell(t)(1 + |x|)|p^j - q^j| + \rho\left(t, \max_{k=1, \dots, m} |u_k - v_k|\right) \end{aligned} \quad (4.3)$$

*for almost every  $t \in (0, T)$  and for all  $(x, u, p^j), (x, v, q^j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$  ( $j = 1, \dots, m$ ). If  $u^1$  and  $u^2$  are global semiclassical solutions to the Cauchy problem (4.1)–(4.2), then  $u^1(t, x) \equiv u^2(t, x)$  in  $\Omega_T$ .*

*Proof* Consider the vector function  $u = u(t, x) \stackrel{\text{def}}{=} u^1(t, x) - u^2(t, x)$ . Then  $u(0, x) \equiv 0$ . Furthermore, by (4.3) and the definition of global

semiclassical solutions, we have

$$\begin{aligned} \left| \frac{\partial u_j}{\partial t}(t, x) \right| &= \left| f_j(t, x, u^1(t, x), \nabla_x u_j^1(t, x)) \right. \\ &\quad \left. - f_j(t, x, u^2(t, x), \nabla_x u_j^2(t, x)) \right| \\ &\leq \ell(t)(1 + |x|) \cdot \left| \nabla_x u_j^1(t, x) - \nabla_x u_j^2(t, x) \right| \\ &\quad + \rho\left(t, \max_{k=1, \dots, m} |u_k^1(t, x) - u_k^2(t, x)|\right) \\ &= \ell(t)(1 + |x|) \cdot |\nabla_x u_j(t, x)| \\ &\quad + \rho\left(t, \max_{k=1, \dots, m} |u_k(t, x)|\right) \end{aligned}$$

for almost every  $t \in (0, T)$  and for any  $x$  ( $j = 1, \dots, m$ ). Now it follows from Theorem 3.1 that  $u(t, x) \equiv 0$  in  $\Omega_T$ . This proves the theorem.

Analogously, Theorem 3.2 implies the following:

**THEOREM 4.2** *Let (2.3) be a comparison equation. Suppose that  $f_j$  ( $j = 1, \dots, m$ ) satisfy the following conditions: there exist a nonnegative function  $\mu$  locally bounded on  $\mathbb{R}^n$  and a nonnegative function  $\ell \in L^1(0, T)$  such that*

$$\begin{aligned} &|f_j(t, x, u, p^j) - f_j(t, x, v, q^j)| \\ &\leq \ell(t) \left[ (1 + |x|) \cdot |p^j - q^j| + \mu(x) \sigma \left( \max_{k=1, \dots, m} |u_k - v_k| \right) \right] \end{aligned} \tag{4.4}$$

for almost every  $t \in (0, T)$  and for all  $(x, u, p^j), (x, v, q^j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$  ( $j = 1, \dots, m$ ). If  $u^1$  and  $u^2$  are global semiclassical solutions to the Cauchy problem (4.1)–(4.2), then  $u^1(t, x) \equiv u^2(t, x)$  in  $\Omega_T$ .

A useful uniqueness criterion for global semiclassical solutions with essentially bounded derivatives is given by the next sharpening.

**THEOREM 4.3** *Let (2.3) be a comparison equation. Suppose that  $f_j$  ( $j = 1, \dots, m$ ) satisfy the following conditions: for any compact sets  $K_1 \subset \mathbb{R}^m, K_2 \subset \mathbb{R}^n$  there exist a nonnegative function  $\ell_{K_2} \in L^1(0, T)$  and a nonnegative function  $\mu_{K_1, K_2}$  locally bounded on  $\mathbb{R}^n$  such that (4.4) with  $\ell_{K_2}$  and  $\mu_{K_1, K_2}$  in place of  $\ell$  and  $\mu$ , respectively, hold for almost every  $t \in (0, T)$  and for all  $(x, u, p^j), (x, v, q^j) \in \mathbb{R}^n \times K_1 \times K_2$  ( $j = 1, \dots, m$ ). If  $u^1$  and  $u^2$*

are global semiclassical solutions to the problem (4.1)–(4.2) with

$$\max_{i=1,2} \max_{j=1,\dots,m} \operatorname{ess\,sup}_{(t,x) \in \Omega_T} |\nabla_x u_j^i(t, x)| < +\infty,$$

then  $u^1(t, x) \equiv u^2(t, x)$  in  $\Omega_T$ .

*Proof* According to the definition of  $V(\Omega_T)$ , [30, Lemma 4.1] shows that

$$\sup_{x \in \mathbb{R}^n} \left| \frac{\partial u_j^i}{\partial x_k}(t, x) \right| = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} \left| \frac{\partial u_j^i}{\partial x_k}(t, x) \right|$$

$$(i = 1, 2; j = 1, \dots, m; k = 1, \dots, n)$$

for almost all  $t \in (0, T)$ . Taking the essential supremum over  $t \in (0, T)$ , we find that

$$\operatorname{ess\,sup}_{t \in (0, T)} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial u_j^i}{\partial x_k}(t, x) \right| = \operatorname{ess\,sup}_{(t,x) \in \Omega_T} \left| \frac{\partial u_j^i}{\partial x_k}(t, x) \right|.$$

Consequently, by assumption,

$$r \stackrel{\text{def}}{=} \max_{i=1,2} \max_{j=1,\dots,m} \operatorname{ess\,sup}_{t \in (0, T)} \sup_{x \in \mathbb{R}^n} |\nabla_x u_j^i(t, x)| < +\infty. \tag{4.5}$$

Let  $u = u(t, x)$  be as in the proof of Theorem 4.1, and let

$$K_2 \stackrel{\text{def}}{=} \bar{B}_r^n \subset \mathbb{R}^n, \quad \ell \stackrel{\text{def}}{=} \ell_{K_2}; \tag{4.6}$$

$$X^k \stackrel{\text{def}}{=} \underbrace{(-k, k) \times \dots \times (-k, k)}_{n \text{ times}} \subset \mathbb{R}^n \quad (k = 1, 2, \dots). \tag{4.7}$$

For an arbitrarily fixed  $T' \in (0, T)$ , we consider the sequence  $\{P^k\}_{k=1}^{+\infty}$  of the following parallelepipeds:

$$P^k \stackrel{\text{def}}{=} (0, T') \times X^k = \{(t, x): 0 < t < T', x \in X^k\}. \tag{4.8}$$

Obviously,  $P^1 \subset P^2 \subset \dots \subset P^k \subset \dots$  and  $\cup_{k=1}^{+\infty} P^k = \Omega_{T'}$ . Next, take

$$s^k \stackrel{\text{def}}{=} \max_{i=1,2} \max_{j=1,\dots,m} \max_{(t,x) \in P^k} |u_j^i(t, x)|,$$

$$K_1^k \stackrel{\text{def}}{=} \underbrace{[-s^k, s^k] \times \dots \times [-s^k, s^k]}_{m \text{ times}} \subset \mathbb{R}^m. \tag{4.9}$$

We now define a function  $\mu$  on  $\mathbb{R}^n$  by setting

$$\mu(x) \stackrel{\text{def}}{=} \begin{cases} \mu_{K_1^1, K_2}(x) & \text{if } x \in X^1, \\ \mu_{K_1^{k+1}, K_2}(x) & \text{if } x \in X^{k+1} \setminus X^k \text{ (for } k = 1, 2, \dots). \end{cases} \quad (4.10)$$

It follows that  $\mu$  is locally bounded on  $\mathbb{R}^n$ . Moreover, according to (4.5)–(4.10) and the hypothesis of the theorem, we have

$$\begin{aligned} & \left| \frac{\partial u_j}{\partial t}(t, x) \right| \\ &= \left| f_j(t, x, u^1(t, x), \nabla_x u_j^1(t, x)) - f_j(t, x, u^2(t, x), \nabla_x u_j^2(t, x)) \right| \\ &\leq \ell(t) \left[ (1 + |x|) \cdot |\nabla_x u_j(t, x)| + \mu(x) \sigma \left( \max_{k=1, \dots, m} |u_k(t, x)| \right) \right] \end{aligned}$$

for almost every  $t \in (0, T)$  and for any  $x \in \mathbb{R}^n, j = 1, \dots, m$ . (We may check these inequalities first for  $(t, x)$  in  $P^1$ , and then for  $(t, x)$  in each  $P^{k+1} \setminus P^k$ .) Theorem 3.2 therefore shows that  $u(t, x) \equiv 0$  in  $\Omega_{T'}$ . Since  $T' \in (0, T)$  is arbitrarily chosen, the conclusion follows.

**COROLLARY 4.4** *Let  $f_j$  be measurable in  $t \in (0, T)$ , continuous in  $x \in \mathbb{R}^n$ , and differentiable in  $(u, p^j) \in \mathbb{R}^m \times \mathbb{R}^n$  such that for any compact set  $K \subset \mathbb{R}^n$  the function*

$$\ell_K = \ell_K(t) \stackrel{\text{def}}{=} 1 + \max_{j=1, \dots, m} \sup_{(x, u, p^j) \in \mathbb{R}^n \times \mathbb{R}^m \times K} |\nabla_{p^j} f_j(t, x, u, p^j) / (1 + |x|)|$$

*is Lebesgue integrable on  $(0, T)$ , and the function*

$$\nu_K = \nu_K(x, u) \stackrel{\text{def}}{=} \max_{k, j=1, \dots, m} \operatorname{ess\,sup}_{t \in (0, T)} \sup_{p^j \in K} \left| \frac{\partial f_j}{\partial u_k}(t, x, u, p^j) / \ell_K(t) \right|$$

*is locally bounded on  $\mathbb{R}^n \times \mathbb{R}^m$ . If  $u^1$  and  $u^2$  are global semiclassical solutions to the Cauchy problem (4.1)–(4.2) with*

$$\max_{i=1, 2} \max_{j=1, \dots, m} \operatorname{ess\,sup}_{(t, x) \in \Omega_T} |\nabla_x u_j^i(t, x)| < +\infty,$$

*then  $u^1(t, x) \equiv u^2(t, x)$  in  $\Omega_T$ .*

*Proof* We take here  $\sigma(w) \stackrel{\text{def}}{=} m|w|$ . Then (2.3) is a comparison equation (see the remark preceding Proposition 2.1). Let us introduce the notation

$$\mu_{K_1, K_2}(x) \stackrel{\text{def}}{=} \sup_{u \in K_1} \nu_{K_2}(x, u)$$

for any right parallelepipeds  $K_1 \subset \mathbb{R}^m$ ,  $K_2 \subset \mathbb{R}^n$ . Then it is easy to check that (4.4) with  $\ell_{K_2}$  and  $\mu_{K_1, K_2}$  in place of  $\ell$  and  $\mu$ , respectively, hold for almost every  $t \in (0, T)$  and for all  $(x, u, p^j)$ ,  $(x, v, q^j) \in \mathbb{R}^n \times K_1 \times K_2$  ( $j = 1, \dots, m$ ). The corollary thereby follows from Theorem 4.3.

*Remark* Theorem 4.3 and Corollary 4.4 generalize corresponding results in [30] to the case of weakly-coupled systems (of first-order partial differential equations with time-measurable Hamiltonian). They are new even when restricted to the classical case of a single equation.

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