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# On the Kolmogorov–Stein Inequality*

HA HUY BANG [a,†] and HOANG MAI LE [b]

[a] Institute of Mathematics, P.O. Box 631, Bo Ho, Hanoi, Vietnam; [b] Thai Nguyen Provincial Pedagogic Secondary School, Vietnam

Dedicated to Ju. A. Dubinskii on the occasion of his sixtieth birthday

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In this paper, we prove the Kolmogorov–Stein inequality for norms generated by concave functions (with the same constants).

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## 1. INTRODUCTION

Kolmogorov [1] has given the following result: Let $f(x), f'(x), \dots, f^{(n)}(x)$ be continuous and bounded on $\mathbb{R}$. Then

$$\|f^{(k)}\|_\infty^n \leq C_{k,n}\|f\|_\infty^{n-k}\|f^{(n)}\|_\infty^k,$$

where $0 < k < n$, $C_{k,n} = K_{n-k}^n / K_n^{(n-k)}$,

$$K_i = \frac{4}{\pi}\sum_{j=0}^{\infty}(-1)^j/(2j+1)^{i+1}$$

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† Corresponding author.

for even $i$, while

$$K_i = \frac{4}{\pi}\sum_{j=0}^{\infty} 1/(2j+1)^{i+1}$$

for odd $i$. Moreover the constants are best possible.

This result has been extended by Stein [2] to $L_p$-norm and by Ha Huy Bang [3] to any Orlicz norm. The Kolmogorov–Stein inequality and its variants are a problem of interest for many mathematicians and have various applications (see, for example [4,5] and their references).

In this paper, modifying the methods of [2,3] we prove this inequality for another norm generated by concave functions. Note that the Orlicz norm is generated by convex functions and here we must overcome some essential difficulties because of the difference between the convex and concave functions.

## 2. RESULTS

Let $\mathcal{L}$ denote the family of all non-zero concave functions $\Phi(t):[0,\infty)\to[0,\infty]$, which are non-decreasing and satisfy $\Phi(0)=0$. For an arbitrary measurable function $f$, $\Phi\in\mathcal{L}$ then we define

$$\|f\|_{N_\Phi} = \int_0^{\infty} \Phi\big(\lambda_f(y)\big)\,\mathrm{d}y,$$

where $\lambda_f(y)=\operatorname{mes}\{x\colon\ |f(x)|>y\}$, $(y\geq 0)$. If the space $N_\Phi=N_\Phi(\mathbb{R})$ consists of measurable functions $f(x)$ such that $\|f\|_{N_\Phi}<\infty$ then $N_\Phi$ is a Banach space. Denote by $M_\Phi=M_\Phi(\mathbb{R})$, the space of measurable functions $g(x)$ such that

$$\|g\|_{M_\Phi} = \sup\left\{\frac{1}{\Phi(\operatorname{mes}\Delta)}\int_\Delta |g(x)|\,\mathrm{d}x\colon\ \Delta\subset\mathbb{R},\ 0<\operatorname{mes}\Delta<\infty\right\}<\infty.$$

Then $M_\Phi$ is a Banach space, too [6,7].

We have the following results [6]:

Lemma 1 *If $\Phi\in\mathcal{L}$, there is an isometric order-preserving isomorphism $J: M_\Phi\to N_\Phi^*$ (of $M_\Phi$ onto $N_\Phi^*$) such that*

$$J(g)(f) = \int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{d}x,\quad (f\in N_\Phi,\ g\in M_\Phi).$$

LEMMA 2 *If $f \in N_\Phi$, $g \in M_\Phi$ then $fg \in L_1$ and*

$$\int_{-\infty}^{\infty} |f(x)g(x)|\,\mathrm{d}x \le \|f\|_{N_\Phi}\|g\|_{M_\Phi}.$$

Now, we give the main theorem:

THEOREM 1 *Let $\Phi \in \mathcal{L}$, $f(x)$ and its generalized derivative $f^{(n)}(x)$ be in $N_\Phi$. Then $f^{(k)}(x) \in N_\Phi$ for all $0<k<n$ and*

$$\|f^{(k)}\|_{N_\Phi}^{n} \le C_{k,n}\|f\|_{N_\Phi}^{n-k}\|f^{(n)}\|_{N_\Phi}^{k}. \tag{1}$$

*Proof* We begin to prove (1) with the assumption that $f^{(k)}(x) \in N_\Phi$, $0 \le k \le n$.

By virtue of Lemma 1, it is clear that $N_\Phi^* = M_\Phi$, and if $f \in N_\Phi$, $g \in M_\Phi$ then

$$\langle f, g\rangle = J(g)(f) = \int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{d}x.$$

Therefore, since $\|x\|_X = \|x\|_{X^{**}}$ for any normed space $X$ [9, p. 113], we have

$$\begin{aligned}\|f^{(k)}\|_{N_\Phi} &= \sup_{\|g\|_{M_\Phi}=1} |\langle f^{(k)}, g\rangle| \\ &= \sup_{\|g\|_{M_\Phi}=1} \left|\int_{-\infty}^{\infty} f^{(k)}(x)g(x)\,\mathrm{d}x\right|.\end{aligned} \tag{2}$$

Let $\epsilon > 0$. We choose a function $h(x) \in M_\Phi$ such that $\|h\|_{M_\Phi} = 1$ and

$$\left|\int_{-\infty}^{\infty} f^{(k)}(x)h(x)\,\mathrm{d}x\right| \ge \|f^{(k)}\|_{N_\Phi} - \epsilon. \tag{3}$$

Put

$$F(x) = \int_{-\infty}^{\infty} f(x+y)h(y)\,\mathrm{d}y.$$

By Lemma 2

$$\begin{aligned}|F(x)| = \left|\int_{-\infty}^{\infty} f(x+y)h(y)\,\mathrm{d}y\right| &\le \int_{-\infty}^{\infty} |f(x+y)h(y)|\,\mathrm{d}y \\ &\le \|f(x+\cdot)\|_{N_\Phi}\|h\|_{M_\Phi} = \|f\|_{N_\Phi},\end{aligned}$$

where the last equality holds because of (2) and the definition $\|\cdot\|_{M_\Phi}$. Then $F(x) \in L_\infty(\mathbb{R})$, and arguing as in [3] we get

$$F^{(r)}(x) = \int_{-\infty}^{\infty} f^{(r)}(x+y)h(y)\,\mathrm{d}y, \quad 0 \le r \le n \tag{4}$$

in the distribution sense.

For all $x \in \mathbb{R}$, clearly

$$|F^{(r)}(x)| \le \|f^{(r)}(x+\cdot)\|_{N_\Phi}\|h\|_{M_\Phi} = \|f^{(r)}\|_{N_\Phi}.$$

Now we prove continuity of $F^{(r)}(x)$ on $\mathbb{R}$ $(0 \le r \le n)$. We show this for $r = 0$ by contradiction: Assume that for some $\epsilon > 0$, point $x^0$ and subsequence $|t_k| \to 0$

$$\left| \int_{-\infty}^{\infty} (f(x^0 + t_k + y) - f(x^0 + y))h(y)\,\mathrm{d}y \right| \ge \epsilon, \quad k \ge 1. \tag{5}$$

Since $f \in N_\Phi$ we get easily $f \in L_{1,loc}(\mathbb{R})$. Then for any $m = 1, 2, \ldots, f(t_k + y) \to f(y)$ in $L_1(-m, m)$. Therefore, there exists a subsequence, denoted again by $\{t_k\}$, such that $f(t_k + y) \to f(y)$ a.e. in $(-m, m)$. Therefore, there exists a subsequence (for simplicity of notation we assume that it is coincident with $\{t_k\}$) such that $f(x^0 + t_k + y) \to f(x^0 + y)$ a.e. in $(-\infty, \infty)$.

On the other hand, $\{f(x^0 + t_k + y)\}$ is bounded in $N_\Phi$ because of

$$\|f(x^0 + t_k + \cdot)\|_{N_\Phi} = \|f\|_{N_\Phi}, \quad k \ge 1.$$

So $\{f(x^0 + t_k + y)\}$ is a weak precompact sequence. Therefore, there exist a subsequence denoted by $\{f(x^0 + t_k + y)\}$ and a function $f_*(y) \in N_\Phi$ such that

$$\langle f(x^0 + t_k + y), v(y)\rangle \to \langle f_*(y), v(y)\rangle$$

when $k \to \infty$, $\forall v(y) \in N_\Phi^*$.

It means

$$\int_{-\infty}^{\infty} f(x^0 + t_k + y)v(y)\,\mathrm{d}y \to \int_{-\infty}^{\infty} f_*(y)v(y)\,\mathrm{d}(y), \quad \forall v(y) \in M_\Phi. \tag{6}$$

Let $u(x)$ be an arbitrary function in $C_0^\infty(\mathbb{R})$, then $u(x) \in M_\Phi$. Therefore, by (6) we get

$$\int_{-\infty}^{\infty} f(x^0 + t_k + y)u(y)\,\mathrm{d}y \to \int_{-\infty}^{\infty} f_*(y)u(y)\,\mathrm{d}y, \quad \forall u \in C_0^\infty(\mathbb{R}).$$

Because each $u \in C_0^\infty(\mathbb{R})$ has a finite support, then it follows from $f(x^0 + t_k + y) \to f(x^0 + y)$ a.e. that

$$\int_{-\infty}^{\infty} f(x^0 + t_k + y)u(y)\,\mathrm{d}y \to \int_{-\infty}^{\infty} f(x^0 + y)u(y)\,\mathrm{d}y, \quad \forall u \in C_0^\infty(\mathbb{R}). \tag{7}$$

Combining (6), (7), we have

$$\int_{-\infty}^{\infty} f(x^0 + y)u(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f_*(y)u(y)\,\mathrm{d}(y), \quad \forall u \in C_0^\infty(\mathbb{R}).$$

Then it is known that [8, p. 15]:

$$f(x^0 + y) = f_*(y) \quad \text{a.e.}$$

Therefore,

$$\int_{-\infty}^{\infty} f(x^0 + t_k + y)h(y)\,\mathrm{d}y \to \int_{-\infty}^{\infty} f(x^0 + y)h(y)\,\mathrm{d}y$$

because of (6), which contradicts (5). The cases $1 \le r \le n$ are proved similarly. The continuity of $F^{(r)}(x)$ has thus been proved.

The functions $F^{(r)}(x)$ are continuous and bounded on $\mathbb{R}$. Therefore, it follows from the Kolmogorov inequality and (3), (4) that

$$\begin{aligned}\left(\|f^{(k)}\|_{N_\Phi} - \epsilon\right)^n &\le |F^{(k)}(0)|^n \le \|F^{(k)}\|_\infty^n \\ &\le C_{k,n}\|F\|_\infty^{n-k}\|F^{(n)}\|_\infty^k.\end{aligned} \tag{8}$$

On the other hand,

$$\|F\|_\infty \le \|f(x+y)\|_{N_\Phi}\|h(y)\|_{M_\Phi} = \|f\|_{N_\Phi}, \tag{9}$$

$$\|F^{(n)}\|_\infty \le \|f^{(n)}(x+y)\|_{N_\Phi}\|h(y)\|_{M_\Phi} = \|f^{(n)}\|_{N_\Phi}. \tag{10}$$

Combining (8)–(10), we get

$$\left(\|f^{(k)}\|_{N_\Phi} - \epsilon\right)^n \leq C_{k,n}\|f\|_{N_\Phi}^{n-k}\|f^{(n)}\|_{N_\Phi}^{k}.$$

By letting $\epsilon \to 0$ we have (1).

To complete the proof, it remains to show that $f^{(k)} \in N_\Phi$, $0<k<n$ if $f, f^{(n)} \in N_\Phi$.

Let $\psi_\lambda(x) \in C_0^\infty(\mathbb{R})$, $\psi_\lambda(x) \geq 0$, $\psi_\lambda(x)=0$ for $|x| \geq \lambda$ and $\int \psi_\lambda(x)\,\mathrm{d}x = 1$. We put $f_\lambda = f * \psi_\lambda$. Then $f_\lambda \in C^\infty(\mathbb{R})$ because of $f \in L_{1,loc}(\mathbb{R})$. Therefore, $f_\lambda^{(k)} = f * \psi_\lambda^{(k)}$, $k \geq 0$ and it is easy to check that $f_\lambda^{(n)} = f^{(n)} * \psi_\lambda$. Now we prove $f_\lambda^{(k)} = f * \psi_\lambda^{(k)} \in N_\Phi$, $k \geq 0$. Actually, for $k=0$ it follows that

$$\begin{aligned}
\|f * \psi_\lambda\|_{N_\Phi} &= \sup_{\|g\|_{M_\Phi}=1} \left| \int_{-\infty}^{\infty} (f * \psi_\lambda)(x) g(x)\,\mathrm{d}x \right| \\
&= \sup_{\|g\|_{M_\Phi}=1} \left| \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x-y)\psi_\lambda(y)\,\mathrm{d}y \right) g(x)\,\mathrm{d}x \right| \\
&= \sup_{\|g\|_{M_\Phi}=1} \left| \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x-y) g(x)\,\mathrm{d}x \right) \psi_\lambda(y)\,\mathrm{d}y \right| \\
&\leq \sup_{\|g\|_{M_\Phi}=1} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-y) g(x)\,\mathrm{d}x \right| |\psi_\lambda(y)|\,\mathrm{d}y \\
&\leq \sup_{\|g\|_{M_\Phi}=1} \int_{-\infty}^{\infty} \|f(\cdot - y)\|_{N_\Phi} |\psi_\lambda(y)|\,\mathrm{d}y \\
&= \sup_{\|g\|_{M_\Phi}=1} \left( \|f\|_{N_\Phi} \int_{-\infty}^{\infty} |\psi_\lambda(y)|\,\mathrm{d}y \right) \\
&= \|f\|_{N_\Phi} \|\psi_\lambda\|_1.
\end{aligned}$$

The cases $k>0$ are proved similarly. Therefore, by the fact proved above, we have

$$\|f_\lambda^{(k)}\|_{N_\Phi}^n \leq C_{k,n}\|f_\lambda\|_{N_\Phi}^{n-k}\|f_\lambda^{(n)}\|_{N_\Phi}^k, \quad 0<k<n.$$

Therefore, since

$$\|f_\lambda\|_{N_\Phi} \leq \|f\|_{N_\Phi}\|\psi_\lambda\|_1 = \|f\|_{N_\Phi}$$

and

$$\|f_\lambda^{(n)}\|_{N_\Phi} \le \|f^{(n)}\|_{N_\Phi}\|\psi_\lambda\|_1 = \|f^{(n)}\|_{N_\Phi},$$

we get that, for any $0 \le k \le n$, the sequence $\{f_\lambda^{(k)}\}$ is bounded in $N_\Phi$. Now we prove that, for any $0 \le k \le n$, there exists a subsequence, which is weakly convergent to some $g_k \in N_\Phi$. We will show, for example, the fact that $f_\lambda$ is weakly convergent to $f$ by contradiction: Assume that for some $\epsilon_0 > 0$, $g \in M_\Phi$ and a subsequence $\lambda_k \to 0$,

$$\left| \int_{-\infty}^{\infty} (f_{\lambda_k}(x) - f(x))g(x)\,\mathrm{d}x \right| \ge \epsilon_0, \quad k \ge 1. \tag{11}$$

Then, it is known that $f_\lambda \to f$, $\lambda \to 0$ in $L_{1,loc}(\mathbb{R})$. Therefore, there exists a subsequence $\{k_m\}$ (for simplicity we assume that $k_m = m$) such that $f_{\lambda_k}(x) \to f(x)$ a.e.

On the other hand, $\{f_{\lambda_k}\}$ is bounded in $N_\Phi$ because of $\|f_{\lambda_k}\|_{N_\Phi} \le \|f\|_{N_\Phi}$. So $\{f_{\lambda_k}\}$ is a weak precompact sequence. Therefore, there exists a subsequence, denoted again by $\{f_{\lambda_k}\}$, and a function $f_*(x) \in N_\Phi$ such that

$$\int_{-\infty}^{\infty} f_{\lambda_k}(x)v(x)\,\mathrm{d}x \to \int_{-\infty}^{\infty} f_*(x)v(x)\,\mathrm{d}(x), \quad \forall v(x) \in M_\Phi. \tag{12}$$

By an argument similar to the previous one, we get

$$f(x) = f_*(x) \quad \text{a.e.}$$

Therefore,

$$\int_{-\infty}^{\infty} f_{\lambda_k}(x)v(x)\,\mathrm{d}x \to \int_{-\infty}^{\infty} f(x)v(x)\,\mathrm{d}x$$

because of (12), which contradicts (11).

Finally, it follows from weak convergence $f_\lambda \rightharpoonup f$ that for any $\varphi \in C_0^\infty(\mathbb{R})$

$$\begin{aligned} \langle f_\lambda^{(k)}(x), \varphi(x)\rangle &= (-1)^k\langle f_\lambda(x)\varphi^{(k)}(x)\rangle \to (-1)^k\langle f(x), \varphi^{(k)}(x)\rangle \\ &= \langle f^{(k)}(x), \varphi(x)\rangle. \end{aligned}$$

Therefore, since the weak convergence of some subsequence of $\{f_\lambda^{(k)}\}$ to $g_k \in N_\Phi$, we get $f^{(k)} = g_k \in N_\Phi$ $(0 < k < n)$. So we have proved the fact that $f^{(k)} \in N_\Phi$ for all $0 < k < n$ if $f, f^{(n)} \in N_\Phi$. The proof is complete.

*Remark* For periodic functions we have:

THEOREM 2 *Let $\Phi(t) \in \mathcal{L}$, $f(x)$ and its generalized derivative $f^{(n)}(x)$ be in $N_\Phi(\mathbb{T})$. Then $f^{(k)}(x) \in N_\Phi(\mathbb{T})$ for all $0 < k < n$ and*

$$|||f^{(k)}|||^n_{N_\Phi(\mathbb{T})} \leq C_{k,n}|||f|||^{n-k}_{N_\Phi(\mathbb{T})}|||f^{(n)}|||^k_{N_\Phi(\mathbb{T})},$$

*where $\mathbb{T}$ is the torus and $|||.|||_{N_\Phi(\mathbb{T})}$ the corresponding norm.*

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