

L^∞ -Estimates for Nonlinear Elliptic Problems with p -growth in the Gradient*

V. FERONE^{a,†}, M.R. POSTERARO^a and J.M. RAKOTOSON^b

^a*Dipartimento di Matematica e Applicazioni "R. Caccioppoli",
Università di Napoli "Federico II", Via Cintia – Complesso Monte S. Angelo,
80126 Napoli, Italia;*

^b*Département de Mathématiques, Université de Poitiers,
40, Avenue du Recteur Pineau, 86022 Poitiers, France*

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We consider the Dirichlet problem for a class of nonlinear elliptic equations whose model is $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p + g - \operatorname{div} f$. We give *a priori* L^∞ -estimates using symmetrization methods. An obstacle problem for nonlinear variational inequalities is also studied.

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1. INTRODUCTION

Let us consider the following model problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p + g - \operatorname{div} f, & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases} \quad (1.1)$$

where $g(x) \in L^m(\Omega)$ with $m > n/p$ and $f(x) \in (L^q(\Omega))^n$, $q > n/(p-1)$, $p > 1$. Our main result is an L^∞ -estimate for a solution u of (1.1) which is also Hölder-continuous. In the case $f \equiv 0$ similar results are contained in [8], where the statements are given for any $p > 1$, but the proof seems to work only for $p \geq 2$.

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† Corresponding author.

The typical result we prove can be stated as follows: *if the measure of Ω and the norms of f and g satisfy a suitable smallness condition, then any solution of (1.1) is bounded in L^∞ by a constant which depends only on the data.* According to [4,5,7,28,29] once we have an L^∞ -estimate, we immediately obtain the existence of a solution for problem (1.1). We remark that in general the boundedness and then the existence of u cannot be expected if one does not put any restriction on $|\Omega|$, f and g . As a matter of fact one can exhibit problems like (1.1) which do not have any solution (see [1,13,20]).

After some preliminary results in Section 3 we study a problem in the general form

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = H(x, u, \nabla u) - \operatorname{div} f, & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases} \quad (1.2)$$

where $a(x, \eta, \xi)$, $H(x, \eta, \xi)$ are Carathéodory functions satisfying suitable growth conditions on $|\xi|$. The main tools we use are symmetrization methods based on rearrangement properties (see e.g. [1,14,29,31,32]). In Section 4 we show how the same method permits to study a class of variational inequalities with an obstacle in the form

$$\begin{cases} u \in K(\eta), \\ \int_{\Omega} a(x, u, \nabla u) \nabla(v - u) \, dx \geq \int_{\Omega} H(x, u, \nabla u)(v - u) \, dx \\ \quad + \int_{\Omega} f \nabla(v - u) \, dx, \quad \forall v \in K(\eta), \end{cases}$$

where $\eta \in L^\infty(\Omega)$ and $K(\eta) = \{v \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega), v \geq \eta \text{ a.e. in } \Omega\}$.

As regards the case of an equation, the problem of finding *a priori* estimates for solution of problems like (1.2) has been studied by many authors, under various assumptions on H (see e.g. [1,14,20] for $p=2$, and [8,18,21,25] for $p > 1$). We would like to remark that there exist various papers where estimates and existence results are proved for problems of the form (1.2) when H satisfies a sign condition (see e.g. [3–5,7,10,28]). Estimates, existence and regularity results (like Hölder-continuity) for variational problems are contained for example in [6,26,27,29].

2. NOTATIONS AND PRELIMINARY RESULTS

Let Ω be an open bounded set of \mathbb{R}^n , $n \geq 1$, and let $w: \Omega \rightarrow \mathbb{R}$ be a measurable function from Ω into \mathbb{R} . If one denotes by $|E|$ the Lebesgue measure of a set E , one can define the distribution function $\mu_w(t)$ of w as:

$$\mu_w(t) = |\{x \in \Omega: w(x) > t\}|, \quad t \in \mathbb{R}.$$

The decreasing rearrangement w^* of w is defined as the generalized inverse function of μ_w :

$$w^*(s) = \inf\{t \in \mathbb{R}: \mu_w(t) \leq s\}, \quad s \in (0, |\Omega|).$$

We recall that w and w^* are equimeasurable, i.e.,

$$\mu_w(t) = \mu_{w^*}(t), \quad t \in \mathbb{R}.$$

This implies that for any Borel function ψ it holds that

$$\int_{\Omega} \psi(w(x)) \, dx = \int_0^{|\Omega|} \psi(w^*(s)) \, ds,$$

and, in particular,

$$\|w^*\|_{L^p(0,|\Omega|)} = \|w\|_{L^p(\Omega)}, \quad 1 \leq p \leq \infty. \tag{2.1}$$

The theory of rearrangements is well known and exhaustive treatments of it can be found for example in [12,19,22,30].

Now we recall two notions which allow us to define a “generalized” concept of rearrangement of a function f with respect to a given function w .

DEFINITION 2.1 (see [2]) Let $f \in L^1(\Omega)$ and $w \in L^1(\Omega)$. We will say that a function $\tilde{f}_w \in L^1(0, |\Omega|)$ is a *pseudo-rearrangement* of f with respect to w if there exists a family $\{D(s)\}_{s \in (0,|\Omega|)}$ of subsets of Ω satisfying the properties:

- (i) $|D(s)| = s$,
- (ii) $s_1 < s_2 \Rightarrow D(s_1) \subset D(s_2)$,
- (iii) $D(s) = \{x \in \Omega: w(x) > t\}$ if $s = \mu_w(t)$,

such that

$$\bar{f}_w(s) = \frac{d}{ds} \int_{D(s)} f(x) dx, \quad \text{in } \mathcal{D}'(\Omega).$$

DEFINITION 2.2 (see [23,24]) Let $f \in L^1(\Omega)$ and $w \in L^1(\Omega)$. The following limit exists:

$$\lim_{\lambda \searrow 0} \frac{(w + \lambda f)^* - w^*}{\lambda} = f_w^*,$$

where the convergence is in $L^p(\Omega)$ -weak, if $f \in L^p(\Omega)$, $1 \leq p < \infty$, and in $L^\infty(\Omega)$ -weak*, if $f \in L^\infty(\Omega)$. The function f_w^* is called the *relative rearrangement* of f with respect to w . Moreover, one has

$$f_w^*(s) = \frac{dG}{ds}, \quad \text{in } \mathcal{D}'(\Omega),$$

where

$$G(s) = \int_{w > w^*(s)} f(x) dx + \int_0^{s - |\{w > w^*(s)\}|} (f|_{\{w = w^*(s)\}})(\sigma) d\sigma.$$

The two notions are equivalent in some precise sense (see [11,12]). For this reason we will denote both \bar{f}_w and f_w^* by F_w . We only recall a few results which hold for both the pseudo- and the relative rearrangements. If f and w are non-negative and $w \in W_0^{1,p}(\Omega)$ it is possible to prove the following properties:

$$-\frac{d}{dt} \int_{w > t} f(x) dx = F_w(\mu_w(t))(-\mu_w'(t)), \quad \text{for a.e. } t > 0; \quad (2.2)$$

$$\|F_w\|_{L^p(0,|\Omega|)} \leq \|f\|_{L^p(\Omega)}, \quad 1 \leq p \leq \infty. \quad (2.3)$$

The proofs of (2.2) and (2.3) can be found in [2] (for pseudo-rearrangements) and in [28,29] (for relative rearrangements).

We finally recall the following chain of inequalities which holds for any non-negative $w \in W_0^{1,p}(\Omega)$:

$$\begin{aligned} nC_n^{1/n} \mu_w(t)^{1-1/n} &\leq -\frac{d}{dt} \int_{w > t} |\nabla w| dx \\ &\leq (-\mu_w'(t))^{1/p'} \left(-\frac{d}{dt} \int_{w > t} |\nabla w|^p dx \right)^{1/p}, \end{aligned} \quad (2.4)$$

where C_n denotes the measure of the unit ball in \mathbb{R}^n . It is a consequence of the Fleming–Rishel formula [15], the isoperimetric inequality [9] and the Hölder’s inequality.

3. A CLASS OF NONLINEAR EQUATIONS

In this section we will show how it is possible to obtain uniform L^∞ -estimates for bounded solutions of (1.2) under smallness assumptions on the data. Let u be a solution of the problem.

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = H(x, u, \nabla u) - \operatorname{div} f & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases} \quad (3.1)$$

where Ω is a bounded open set of \mathbb{R}^n , and the following assumptions are made:

- $a(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Carathéodory function which satisfies, for a.e. $x \in \Omega$, any $s \in \mathbb{R}$ and any $\xi \in \mathbb{R}^n$,

$$\begin{cases} a(x, s, \xi)\xi \geq \alpha|\xi|^p, \\ |a(x, s, \xi)| \leq \beta[b(x) + |s|^{p-1} + |\xi|^{p-1}], \end{cases} \quad (3.2)$$

for some $\alpha > 0, \beta > 0, 1 < p \leq n, b \in L^{p'}(\Omega)$;

- $H(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a Carathéodory function which satisfies, for a.e. $x \in \Omega$, any $s \in \mathbb{R}$ and any $\xi \in \mathbb{R}^n$,

$$|H(x, s, \xi)| \leq \gamma(x)|\xi|^p + g(x), \quad (3.3)$$

for some $\gamma \in L^\infty(\Omega), 0 \leq \gamma(x) \leq \lambda$ a.e., $g \in L^m(\Omega), m > n/p, g(x) \geq 0$ a.e.;

- $f(x) : \Omega \rightarrow \mathbb{R}^n$ satisfies

$$f \in (L^q(\Omega))^n, \quad q > \frac{n}{p-1}. \quad (3.4)$$

LEMMA 3.1 *Let u be a solution of (3.1) under assumptions (3.2)–(3.4). Define*

$$w = \frac{e^{k|u|} - 1}{k}, \quad k = \frac{\lambda p'}{\alpha(p-1)}. \tag{3.5}$$

Then the decreasing rearrangement of w satisfies the following differential inequality:

$$\begin{aligned} (-w^*(s))' &\leq \frac{[(-w^*(s))]^{1/p}}{nC_n^{1/n} s^{1-1/n}} \left(\int_0^s \psi^*(\tau) (kw^*(\tau) + 1)^{p-1} d\tau \right)^{1/p} \\ &\quad + \frac{kw^*(s) + 1}{\alpha^{p'/p} n C_n^{1/n} s^{1-1/n}} (F_w(s))^{1/p}, \quad \text{a.e. in } (0, |\Omega|), \end{aligned} \tag{3.6}$$

where ψ^* is the decreasing rearrangement of $\psi = \alpha^{-1} p' g + (\lambda p' / \alpha^{p'+1}) |f|^{p'}$ and F_w is a pseudo-rearrangement (or the relative rearrangement) of $|f|^{p'}$ with respect to w .

Proof Let us define two real functions $\phi_1(z), \phi_2(z), z \in \mathbb{R}$, as follows:

$$\begin{cases} \phi_1(z) = e^{k(p-1)|z|} \text{sign}(z), \\ \phi_2(z) = (e^{kz} - 1)/k, \end{cases} \tag{3.7}$$

where k is as in (3.5). We observe that $\phi_2(0) = 0$ and, for $z \neq 0$, $\phi_1'(z) > 0, \phi_2'(z) > 0$,

$$\phi_1(z) \phi_2'(|z|) \text{sign}(z) = |\phi_2'(|z|)|^p, \tag{3.8}$$

$$\phi_1'(z) - \frac{\lambda p'}{\alpha} |\phi_1(z)| = 0. \tag{3.9}$$

Furthermore, for $t > 0, h > 0$, let us put

$$S_{t,h}(z) = \begin{cases} \text{sign}(z) & \text{if } |z| > t + h, \\ ((|z| - t)/h) \text{sign}(z) & \text{if } t < |z| \leq t + h, \\ 0 & \text{if } |z| \leq t. \end{cases} \tag{3.10}$$

We use in (3.1) the test function $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ defined by

$$v = \phi_1(u) S_{t,h}(w) = \phi_1(u) S_{t,h}(\phi_2(|u|)),$$

where w is given in (3.5). Using (3.10) we get

$$\begin{aligned} & \frac{1}{h} \int_{t < w \leq t+h} a(x, u, \nabla u) \nabla u \phi_1(u) \phi_2'(|u|) \operatorname{sign}(u) \, dx \\ &= \int_{w > t} (H(x, u, \nabla u) \phi_1(u) - a(x, u, \nabla u) \nabla u \phi_1'(u)) S_{t,h}(w) \, dx \\ & \quad + \int_{w > t} f \phi_1'(u) S_{t,h}(w) \nabla u \, dx \\ & \quad + \frac{1}{h} \int_{t < w \leq t+h} f \phi_1(u) \phi_2'(|u|) \operatorname{sign}(u) \nabla u \, dx. \end{aligned} \tag{3.11}$$

Taking into account (3.8) and Young's inequality, the last two terms in (3.11) can be estimated as follows:

$$\begin{aligned} & \int_{w > t} f \phi_1'(u) S_{t,h}(w) \nabla u \, dx + \frac{1}{h} \int_{t < w \leq t+h} f \phi_1(u) \phi_2'(|u|) \operatorname{sign}(u) \nabla u \, dx \\ & \leq \frac{\alpha^{-p'/p}}{p'} \int_{w > t} |f|^{p'} \phi_1'(u) S_{t,h}(w) \, dx + \frac{\alpha}{p} \int_{w > t} |\nabla u|^p \phi_1'(u) S_{t,h}(w) \, dx \\ & \quad + \frac{\alpha^{-p'/p}}{p'h} \int_{t < w \leq t+h} |f|^{p'} |\phi_2'(|u|)|^p \, dx + \frac{\alpha}{ph} \int_{t < w \leq t+h} |\nabla u|^p |\phi_2'(|u|)|^p \, dx. \end{aligned} \tag{3.12}$$

Now (3.11), (3.12), (3.3), (3.8) and the ellipticity condition in (3.2) imply

$$\begin{aligned} & \frac{\alpha}{p'h} \int_{t < w \leq t+h} |\nabla u|^p |\phi_2'(|u|)|^p \, dx \leq \int_{w > t} \left[\lambda |\phi_1(u)| - \frac{\alpha}{p'} \phi_1'(u) \right] |\nabla u|^p S_{t,h}(w) \, dx \\ & \quad + \int_{w > t} \left[g + \frac{\lambda}{\alpha^{p'}} |f|^{p'} \right] |\phi_1(u)| S_{t,h}(w) \, dx + \frac{\alpha^{-p'/p}}{p'h} \int_{t < w \leq t+h} |f|^{p'} |\phi_2'(|u|)|^p \, dx. \end{aligned}$$

Using (3.9) and the definition of ϕ_1, ϕ_2 in (3.7), the above inequality gives

$$\begin{aligned} & \frac{1}{h} \int_{t < w \leq t+h} |\nabla w|^p \, dx \leq \int_{w > t} \psi(kw + 1)^{p-1} S_{t,h}(w) \, dx \\ & \quad + \frac{1}{\alpha^{p'} h} \int_{t < w \leq t+h} |f|^{p'} (kw + 1)^p \, dx, \end{aligned} \tag{3.13}$$

where $\psi = \alpha^{-1}p'g + (\lambda p'/\alpha^{p'+1})|f|^{p'}$. Letting h go to 0 in a standard way we get

$$\begin{aligned}
 & -\frac{d}{dt} \int_{w>t} |\nabla w|^p dx \\
 & \leq \int_{w>t} \psi(kw + 1)^{p-1} dx + \frac{(kt + 1)^p}{\alpha^{p'}} \left(-\frac{d}{dt} \int_{w>t} |f|^{p'} dx \right).
 \end{aligned}$$

Using Hardy–Littlewood inequality and (2.2) it follows that

$$\begin{aligned}
 -\frac{d}{dt} \int_{w>t} |\nabla w|^p dx & \leq \int_0^{\mu_w(t)} \psi^*(s)(kw^*(s) + 1)^{p-1} ds \\
 & + \frac{(kt + 1)^p}{\alpha^{p'}} (-\mu'_w(t))F_w(\mu_w(t)), \quad (3.14)
 \end{aligned}$$

where F_w is a pseudo-rearrangement (or the relative rearrangement) of $|f|^{p'}$ with respect to w .

Inequalities (2.4) and (3.14) give

$$\begin{aligned}
 nC_n^{1/n} \mu_w(t)^{1-1/n} & \leq (-\mu'_w(t))^{1/p'} \left(\int_0^{\mu_w(t)} \psi^*(s)(kw^*(s) + 1)^{p-1} ds \right)^{1/p} \\
 & + \frac{kt + 1}{\alpha^{p'/p}} (-\mu'_w(t))(F_w(\mu_w(t)))^{1/p}
 \end{aligned}$$

and then, using the definition of $w^*(s)$ we have

$$\begin{aligned}
 (-w^*(s))' & \leq \frac{[(-w^*(s))']^{1/p}}{nC_n^{1/n} s^{1-1/n}} \left(\int_0^s \psi^*(\tau)(k w^*(\tau) + 1)^{p-1} d\tau \right)^{1/p} \\
 & + \frac{k w^*(s) + 1}{\alpha^{p'/p} n C_n^{1/n} s^{1-1/n}} (F_w(s))^{1/p},
 \end{aligned}$$

that is (3.6).

In the case $f \equiv 0$ in (3.1), Lemma 3.1 can be slightly improved to obtain:

LEMMA 3.2 *Let u be a solution of (3.1) under the assumptions (3.2)–(3.4) and $f \equiv 0$. Define*

$$w = \frac{e^{k|u|} - 1}{k}, \quad k = \frac{\lambda}{\alpha(p-1)}. \quad (3.15)$$

Then the decreasing rearrangement of w satisfies the following differential inequality:

$$(-w^*(s))' \leq \left(\frac{1}{\alpha(nC_n^{1/n} s^{1-1/n})^p} \int_0^s g^*(\tau)(kw^*(\tau) + 1)^{p-1} d\tau \right)^{p'/p}, \quad (3.16)$$

for a.e. $s \in (0, |\Omega|)$.

Proof We use the same arguments of the proof of Lemma 3.1. The only difference is that now we take k as in (3.15). Instead of (3.11) we obtain

$$\begin{aligned} & \frac{1}{h} \int_{t < w \leq t+h} a(x, u, \nabla u) \nabla u \phi_1(u) \phi_2'(|u|) \operatorname{sign}(u) dx \\ &= \int_{w > t} (H(x, u, \nabla u) \phi_1(u) - a(x, u, \nabla u) \nabla u \phi_1'(u)) S_{t,h}(w) dx. \end{aligned}$$

By ellipticity condition in (3.2) and assumption (3.3) we get

$$\frac{\alpha}{h} \int_{t < w \leq t+h} |\nabla u|^p |\phi_2'(|u|)|^p dx \leq \int_{w > t} g |\phi_1(u)| S_{t,h}(w) dx.$$

Letting h go to 0 and then using (2.4) we have

$$nC_n^{1/n} \mu_w(t)^{1-1/n} \leq (-\mu_w'(t))^{1/p'} \left(\frac{1}{\alpha} \int_0^{\mu_w(t)} g^*(s)(kw^*(s) + 1)^{p-1} ds \right)^{1/p}.$$

The assertion follows easily.

An immediate consequence of the above results is the following uniform L^∞ -estimate for solutions of (3.1).

THEOREM 3.3 *Let u be a solution of (3.1) under the assumptions (3.2)–(3.4). If f and g satisfy the inequality*

$$\begin{aligned} & \frac{1}{(nC_n^{1/n})^{p'}} \frac{n\sigma(p-1)}{p\sigma-n} \left(\frac{p'}{\alpha} |\Omega|^{p/n-1/m} \|g\|_m + \frac{\lambda p'}{\alpha^{p'+1}} |\Omega|^{p/n-p'/q} \|f\|_q^{p'} \right)^{p'/p} \\ & + \frac{p'}{\alpha^{p'/p} n C_n^{1/n}} \left(\frac{n(q(p-1)-1)}{q(p-1)-n} \right)^{1-p'/(qp)} |\Omega|^{1/n-p'/(qp)} \|f\|_q^{p'/p} < \frac{\alpha(p-1)}{\lambda p'}, \end{aligned} \quad (3.17)$$

where $\sigma = \min(m, q/p')$, then there exists a constant M , which depends only on $n, p, q, m, |\Omega|, \|f\|_q, \|g\|_m$, such that

$$\|u\|_\infty \leq M. \quad (3.18)$$

Moreover, in the case where $f \equiv 0$, the estimate (3.18) holds if

$$\frac{1}{(nC_n^{1/n})^{p'}} \frac{nm}{mp - n} |\Omega|^{p'/n - p'/(mp)} \|g\|_m^{p'/p} < \frac{\alpha^{p'}}{\lambda}. \quad (3.19)$$

Proof Let us first prove (3.18) under assumption (3.17). By Young's inequality Lemma 3.1 implies

$$\begin{aligned} (-w^*(s))' &\leq \left(\frac{1}{(nC_n^{1/n} s^{1-1/n})^p} \int_0^s \psi^*(\tau) \left(\frac{\lambda p'}{\alpha(p-1)} w^*(\tau) + 1 \right)^{p-1} d\tau \right)^{p'/p} \\ &\quad + \frac{p'}{\alpha^{p'/p} n C_n^{1/n} s^{1-1/n}} \left(\frac{\lambda p'}{\alpha(p-1)} w^*(s) + 1 \right) (F_w(s))^{1/p}. \end{aligned}$$

Integrating between 0 and $|\Omega|$ we get

$$\|w\|_\infty \leq \frac{\lambda p'}{\alpha(p-1)} A \|w\|_\infty + A, \quad (3.20)$$

where

$$\begin{aligned} A &= \int_0^{|\Omega|} \left[\left(\frac{1}{(nC_n^{1/n} s^{1-1/n})^p} \int_0^s \psi^*(\tau) d\tau \right)^{p'/p} \right. \\ &\quad \left. + \frac{p'}{\alpha^{p'/p} n C_n^{1/n} s^{1-1/n}} (F_w(s))^{1/p} \right] ds. \end{aligned}$$

Now we observe that

$$\begin{aligned} \int_0^s \psi^*(\tau) d\tau &\leq \|\psi\|_\sigma s^{1-1/\sigma} \\ &\leq \left(\frac{p'}{\alpha} |\Omega|^{1/\sigma-1/m} \|g\|_m + \frac{\lambda p'}{\alpha^{p'+1}} |\Omega|^{1/\sigma-p'/q} \|f\|_q^{p'} \right) s^{1-1/\sigma}, \end{aligned} \quad (3.21)$$

where $\sigma = \min(m, q/p')$. Furthermore, taking into account the fact that $q > p'$, property (2.3) gives

$$\begin{aligned} & \int_0^{|\Omega|} \frac{1}{s^{1-1/n}} (F_w(s))^{1/p} ds \\ & \leq \left(\frac{n(q(p-1) - 1)}{q(p-1) - n} \right)^{1-p'/(qp)} |\Omega|^{1/n-p'/(qp)} \|F_w\|_{q/p'}^{1/p} \\ & \leq \left(\frac{n(q(p-1) - 1)}{q(p-1) - n} \right)^{1-p'/(qp)} |\Omega|^{1/n-p'/(qp)} \|f\|_q^{p'/p}. \end{aligned} \quad (3.22)$$

Using (3.21) and (3.22) we can estimate the quantity A in (3.20), obtaining that under assumption (3.17) the following inequality holds:

$$\frac{\lambda p'}{\alpha(p-1)} A < 1.$$

Then (3.20) implies (3.18).

The proof of (3.18) under the hypotheses (3.19) and $f \equiv 0$ follows immediately from Lemma 3.2 and will be omitted.

Remark 3.1 It is easy to realize that the hypotheses of Theorem 3.3 can be given in terms of smallness assumption on the norms of g and f in suitable Lorentz spaces. For the sake of simplicity we will write it explicitly only in the case $f \equiv 0$. We have that (3.18) holds if (3.19) is replaced by

$$\frac{1}{(nC_n^{1/n})^{p'}} \int_0^{|\Omega|} \left(\frac{1}{s} \int_0^s g^*(\tau) d\tau \right)^{p'/p} s^{p'/n} \frac{ds}{s} < \frac{\alpha^{p'}(p-1)}{\lambda}. \quad (3.23)$$

The finiteness of the integral on the left hand side is equivalent to the fact that g belongs to the Lorentz space $L(n/p, p'/p)$. It is well known that such a space contains $L^m(\Omega)$, for every $m > n/p$. Finally we observe that, for $p = 2$, (3.23) reduces to the condition given in [14].

As we observed in the introduction the uniform estimate found in Theorem 3.3 can be used to prove an existence result for problem (3.1). In addition we have to assume the monotonicity condition

$$(a(x, \eta, \xi_1) - a(x, \eta, \xi_2))(\xi_1 - \xi_2) > 0 \quad \forall \xi_1 \neq \xi_2. \quad (3.24)$$

Indeed, using the arguments contained in [4,5,7,28] one can prove the following:

THEOREM 3.4 *Suppose (3.2)–(3.4), (3.24) hold. Under assumption (3.17) (or $f \equiv 0$ and (3.19)) at least one solution of (3.1) exists.*

Remark 3.2 We recall that, once a solution of (3.1) exists, then it is automatically Hölder-continuous. Such a statement is contained for example in [28,29], where existence results are obtained under a sign condition. However the proof of Hölder-continuity does not make use of such a hypothesis.

4. A CLASS OF VARIATIONAL INEQUALITIES

Let $\eta \in L^\infty(\Omega)$ and set $K(\eta) = \{v \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega), v \geq \eta \text{ a.e in } \Omega\}$. We consider the following variational problem:

$$\begin{cases} u \in K(\eta), \\ \int_{\Omega} a(x, u \nabla u) \nabla(v - u) \, dx \geq \int_{\Omega} H(x, u, \nabla u)(v - u) \, dx \\ \quad + \int_{\Omega} f \nabla(v - u) \, dx, \quad \forall v \in K(\eta). \end{cases} \tag{4.1}$$

If u is a solution of (4.1) we still denote the function given by (3.5) by w . Our aim is to derive for the function $\bar{w} = (w - \phi_2(\|\eta\|_\infty))_+$, where ϕ_2 is defined in (3.7), a result analogous to one proved for w in Lemma 3.1.

LEMMA 4.1 *Let u be a solution of (4.1) under the assumptions (3.2)–(3.4) and let $\bar{w} = (w - \phi_2(\|\eta\|_\infty))_+$ with w and ϕ_2 defined in (3.5) and (3.7). Then the decreasing rearrangement of \bar{w} satisfies the following differential inequality:*

$$\begin{aligned} -\bar{w}^{*'}(s) &\leq \frac{[-\bar{w}^{*'}(s)]^{1/p}}{nC_n^{1/n} s^{1-1/n}} \left(\int_0^s \psi^*(\tau)(k\bar{w}^*(\tau) + k\phi_2(\|\eta\|_\infty) + 1)^{p-1} \, d\tau \right)^{1/p} \\ &\quad + \frac{k\bar{w}^*(\tau) + k\phi_2(\|\eta\|_\infty) + 1}{\alpha^{p'/p} n C_n^{1/n} s^{1-1/n}} F_w^{1/p}(s), \end{aligned} \tag{4.2}$$

where k, ψ and F_w are as in the Lemma 3.1.

In order to prove (4.2) we need a preliminary lemma.

LEMMA 4.2 *Let ϕ_2 be the function defined in (3.7) and let t be a positive number. If*

$$\phi_2(|z|) > \phi_2(\|\eta\|_\infty) + t, \tag{4.3}$$

then there exists $\tilde{\theta} > 0$ independent of z such that

$$|z| > \|\eta\|_\infty + \tilde{\theta}.$$

Proof Let t be a positive number. Then there exists $\tilde{\theta} > 0$ such that

$$e^{k(\|\eta\|_\infty + \tilde{\theta})} \tilde{\theta} \leq t,$$

with k defined in (3.5). If (4.3) holds one has $e^{k|z|} - e^{k\|\eta\|_\infty} > kt$, that is

$$e^{k(|z| - \|\eta\|_\infty)} > 1 + kte^{-k\|\eta\|_\infty} \geq 1 + k\tilde{\theta}e^{k\tilde{\theta}} \geq e^{k\tilde{\theta}}.$$

Proof of Lemma 4.1 As in the proof of Lemma 3.1 we will use a suitable test function in (4.1) making use of the functions defined in (3.5) and (3.7). Setting $\theta = t + \phi_2(\|\eta\|_\infty)$, for $\theta > \phi_2(\|\eta\|_\infty)$ and $h > 0$ we define

$$v = u - \tilde{\theta} \frac{\phi_1(u)}{\|\phi_1(u)\|_\infty} S_{\theta,h}(w), \tag{4.4}$$

where $u \in K(\eta)$ is solution of the problem (4.1) and $\tilde{\theta}$ is chosen as in Lemma 4.2. It is easy to verify that $v \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$; furthermore we claim that $v \geq \eta$ a.e. in Ω (see also [29]). Indeed we observe that in the set $\{w \leq \theta\}$ we have $S_{\theta,h}(w) = 0$ and then the claim is a consequence of the fact that $u \in K(\eta)$. On the other hand we have that in $\{w > \theta = t + \phi_2(\|\eta\|_\infty)\}$ the following inequality holds:

$$0 \leq \frac{\phi_1(u)}{\|\phi_1(u)\|_\infty} S_{\theta,h}(w) \leq 1,$$

which implies $v \geq u - \tilde{\theta}$. Recalling that $w = \phi_2(|u|)$ and applying Lemma 4.2 we have $v \geq \|\eta\|_\infty + \tilde{\theta}$ and then the claim is completely proved. We are now in a position to choose (4.4) as test function in the

problem (4.1) obtaining

$$\begin{aligned}
 & \frac{1}{h} \int_{\theta < w \leq \theta+h} a(x, u, \nabla u) \nabla u \phi_1(u) \phi_2'(|u|) \operatorname{sign}(u) \, dx \\
 & \leq \int_{w > \theta} (H(x, u, \nabla u) \phi_1(u) - a(x, u, \nabla u) \nabla u \phi_1'(u)) S_{\theta, h}(w) \, dx \\
 & \quad + \int_{w > \theta} f \phi_1'(u) S_{\theta, h}(w) \nabla u \, dx \\
 & \quad + \frac{1}{h} \int_{\theta < w \leq \theta+h} f \phi_1(u) \phi_2'(|u|) \operatorname{sign}(u) \nabla u \, dx. \tag{4.5}
 \end{aligned}$$

Proceeding as in Lemma 3.1 we get an inequality similar to (3.13), that is

$$\begin{aligned}
 \frac{1}{h} \int_{\theta < w \leq \theta+h} |\nabla w|^p \, dx & \leq \int_{w > \theta} \psi(kw + 1)^{p-1} S_{\theta, h}(w) \, dx \\
 & \quad + \frac{1}{h} \int_{\theta < w \leq \theta+h} \left(\frac{|f|}{\alpha} \right)^{p'} (kw + 1)^p \, dx, \tag{4.6}
 \end{aligned}$$

where $\psi = \alpha^{-1} p' g + (\lambda p' / \alpha^{p'+1}) |f|^{p'}$. We set $\bar{w} = (w - \phi_2(\|\eta\|_\infty))_+$ and we observe that since $\theta > \phi_2(\|\eta\|_\infty)$ we have $\{w > \theta\} = \{\bar{w} > t\}$ and $S_{\theta, h}(w) = S_{t, h}(\bar{w})$. Then (4.6) can be rewritten as

$$\begin{aligned}
 \frac{1}{h} \int_{t < \bar{w} \leq t+h} |\nabla \bar{w}|^p \, dx & \leq \int_{\bar{w} > t} \psi(k\bar{w} + \bar{k})^{p-1} S_{t, h}(\bar{w}) \, dx \\
 & \quad + \frac{1}{h} \int_{t < \bar{w} \leq t+h} \left(\frac{|f|}{\alpha} \right)^{p'} (k\bar{w} + \bar{k})^p \, dx,
 \end{aligned}$$

where $\bar{k} = k\phi_2(\|\eta\|_\infty) + 1$.

At this point by the same argument as used in Lemma 3.1 we obtain the assertion.

The previous lemma gives the following L^∞ -estimate.

THEOREM 4.3 *Let u be a solution of (4.1) under assumptions (3.2)–(3.4). If f and g satisfy the inequality*

$$\begin{aligned}
 & \frac{1}{(nC_n^{1/n})^{p'}} \frac{n\sigma(p-1)}{p\sigma-n} \left(\frac{p'}{\alpha} |\Omega|^{p/n-1/m} \|g\|_m + \frac{\lambda p'}{\alpha^{p'+1}} |\Omega|^{p/n-p'/q} \|f\|_q^{p'} \right)^{p'/p} \\
 & \quad + \frac{p'}{\alpha^{p'/p} n C_n^{1/n}} \left(\frac{n(q(p-1)-1)}{q(p-1)-n} \right)^{1-p'/(qp)} |\Omega|^{1/n-p'/(qp)} \|f\|_q^{p'/p} < \frac{\alpha(p-1)}{\lambda p'}, \tag{4.7}
 \end{aligned}$$

where $\sigma = \min(m, q/p')$, then there exists a constant M , which depends only on $n, p, q, m, |\Omega|, \|f\|_q, \|g\|_m, \phi_2(\|\eta\|_\infty)$ such that

$$\|u\|_\infty \leq M.$$

Proof Using Young's inequality in (4.2) and integrating between 0 and $|\Omega|$ we obtain

$$\|\bar{w}\|_\infty \leq \frac{\lambda p'}{\alpha(p-1)} A \|\bar{w}\|_\infty + \left(\frac{\lambda p'}{\alpha(p-1)} \phi_2(\|\eta\|_\infty) + 1 \right) A,$$

where

$$A = \int_0^{|\Omega|} \left[\left(\frac{1}{(nC_n^{1/n} s^{1-1/n})^p} \int_0^s \psi^*(\tau) \, d\tau \right)^{p'/p} + \frac{p'}{\alpha^{p'/p} n C_n^{1/n} s^{1-1/n}} (F_w(s))^{1/p} \right] ds.$$

We observe that the above quantity is the same A as appearing in (3.20). As in the proof of Theorem 3.3, assumption (4.7) implies

$$\frac{\lambda p'}{\alpha(p-1)} A < 1.$$

This means that

$$\|\bar{w}\|_\infty \leq C,$$

where C depends only on $n, p, q, m, |\Omega|, \|f\|_q, \|g\|_m, \phi_2(\|\eta\|_\infty)$. Recalling that

$$\bar{w} = \left(\frac{e^{k|u|} - 1}{k} - \phi_2(\|\eta\|_\infty) \right)_+$$

we obtain the assertion.

As in the previous section the arguments contained for example in [6,29], allow us to get the following:

THEOREM 4.4 *Suppose (3.2)–(3.4), (3.24) hold. Under assumption (4.7) at least one solution of (4.1) exists.*

Remark 4.1 In Remark 3.1 we have observed that the smallness assumptions on the norms of g and f in Theorem 3.3 can be given in terms of Lorentz norms. Also in the case of variational inequalities a similar remark holds.

Remark 4.2 As in the case of equations, Lemma 4.2 can be improved when $f \equiv 0$, in the sense that a version of Lemma 4.2 similar to Lemma 3.3 can be proved. In particular, one can show that, if $f \equiv 0$ and (3.19) is verified, then both Theorems 4.3 and 4.4 hold true.

Remark 4.3 As recalled in Remark 3.2 for the equations, any solution of (4.1) is Hölder-continuous under the additional assumption that the obstacle η belongs to $W_{\text{loc}}^{1,q}(\Omega)$ with $q > n$ (see [29]).

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