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The λ-function in the Space $\mathcal{P}(^2 l_2^2)$

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In this note, motivated by the question 1 in (Aron and Lohman, *Pacific J. Math.* **127** (1987), 209–231), we obtain an explicit formula for the λ-function in the real space $\mathcal{P}(^2 l_2^2)$. From this we see that the λ-function is continuous and attained at each point of the unit ball of $\mathcal{P}(^2 l_2^2)$, the space of real-valued continuous 2-homogeneous polynomials on l_2^2.

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Given a normed space E, B_E denotes its closed unit ball, $\text{ext}(B_E)$ the set of extreme points of B_E, and S_E the closed unit sphere of E. If $x \in B_E$, a triple (e, y, λ) is said to be amenable to x if $e \in \text{ext}(B_E)$, $y \in B_E$, $0 < \lambda \leq 1$, and $x = \lambda e + (1 - \lambda)y$. In this case, we define

$$\lambda(x) = \sup\{\lambda\colon (e, y, \lambda) \text{ is amenable to } x\}.$$

E is said to have the λ-property if each $x \in B_E$ admits an amenable triple. If, in addition, $\inf\{\lambda\colon x \in B_E\} > 0$, then E is said to have the uniform λ-property. For more details about λ-property and λ-functions in Banach spaces we refer to [1,2,4,5].

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Aron–Lohman [1] introduced the λ-function, and calculated explicitly the λ-function for the classical spaces $C_X(T), l_1(X), l_\infty(X)$ and $c(X)$. They showed that every finite dimensional normed space has the uniform λ-property.

Choi–Kim [3] obtained an explicit formula for the norm of the real space $\mathcal{P}(^2 l_2^2)$: *Let* $a, b, c \in \mathbf{R}$, $|a| \leq 1, |b| \leq 1$ *and* $|c| \leq 2$. *Suppose* $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2 l_2^2)$ *for the real Banach space* l_2^2. *Then*

$$\|P(x,y)\| = 1 \text{ if and only if } 4 - c^2 = 4(|a+b| - ab) \qquad (*)$$

Using $(*)$ we also classified the extreme points of the unit ball of $\mathcal{P}(^2 l_2^2)$: *For the real Banach space* l_2^2,

$$P(x,y) = ax^2 + by^2 + cxy \in \operatorname{ext}(B_{\mathcal{P}(^2 l_2^2)})$$

if and only if

$$|a| = |b| = 1 \text{ or } 0 \leq |a| < 1, \ a = -b, \ 4a^2 = 4 - c^2 \qquad (**)$$

In this note, motivated by the question 1 in [1], we obtain an explicit formula for the λ-function in the real space $\mathcal{P}(^2 l_2^2)$ using $(*)$ and $(**)$. From this we see that the λ-function is continuous and attained at each point of the unit ball of $\mathcal{P}(^2 l_2^2)$. Finally, we give an explicit formula for the norm and the λ-function in $\mathcal{P}(^2 l_2^2)$.

LEMMA 1 *Let* $P(x,y) = ax^2 + by^2 + cxy$ *in* $\mathcal{P}(^2 l_2^2)$, $\|P\| \leq 1$. *Then*

$$\begin{aligned}&\lambda(ax^2 + by^2 + cxy)\\&\quad = \lambda(\operatorname{sign}(ab)\min\{|a|,|b|\}x^2 + \max\{|a|,|b|\}y^2 + |c|xy).\end{aligned}$$

Proof It follows from the fact that the λ-function is invariant with respect to isometries.

THEOREM 2 *Let* $P(x,y) = ax^2 + by^2 + cxy$ *in* $\mathcal{P}(^2 l_2^2)$, $\|P\| \leq 1$. *Then*

$$\lambda(ax^2 + by^2 + cxy) = \frac{1}{2} + \frac{1}{4}\left| |a+b| - \sqrt{(a-b)^2 + c^2} \right|.$$

Therefore, the λ*-function is continuous and attained at each point of* $B_{\mathcal{P}(^2 l_2^2)}$.

Proof By Lemma 1, may assume that $|a| \le |b| = b$ and $c \ge 0$.

Case 1 $\|P\| < 1$. First, $(*)$ shows that

$$4 - c^2 > 4(a+b) - 4ab. \tag{1}$$

(A) Suppose that $P(x,y) = \lambda(x^2+y^2) + (1-\lambda)Q(x,y)$ for some $0 < \lambda \le 1$ and $Q \in \mathcal{P}(^2 l_2^2)$, $\|Q\| \le 1$.

By Proposition 1.2(b) [1] we may assume $\|Q\| = 1$. Then

$$Q(x,y) = \left(\frac{a-\lambda}{1-\lambda}\right)x^2 + \left(\frac{b-\lambda}{1-\lambda}\right)y^2 + \left(\frac{c}{1-\lambda}\right)xy.$$

and $(*)$ shows that

$$\left|\frac{a-\lambda}{1-\lambda}\right| \le 1, \quad \left|\frac{b-\lambda}{1-\lambda}\right| \le 1, \quad \left|\frac{c}{1-\lambda}\right| \le 2$$

and

$$4 - \left(\frac{c}{1-\lambda}\right)^2 = 4\left|\frac{a+b-2\lambda}{1-\lambda}\right| - 4\left(\frac{a-\lambda}{1-\lambda}\right)\left(\frac{b-\lambda}{1-\lambda}\right). \tag{2}$$

If $a+b-2\lambda \ge 0$, then Eq. (2) is equivalent to $4 - c^2 = 4(a+b) - 4ab$, contrary to (1). Suppose $a+b-2\lambda < 0$. Solving Eq. (2), we get

$$\lambda = \frac{1}{2} + \frac{1}{4}\left(|a+b| \pm \sqrt{(a-b)^2 + c^2}\right).$$

Since $\lambda \le \min\{(1+a)/2, (1+b)/2\} = (1+a)/2$, we have

$$\lambda = \frac{1}{2} + \frac{1}{4}\left(|a+b| - \sqrt{(a-b)^2 + c^2}\right).$$

It is easy to check that

$$\frac{a+b}{2} < \frac{1}{2} + \frac{1}{4}\left(|a+b| - \sqrt{(a-b)^2 + c^2}\right) \le \frac{1+a}{2}.$$

Hence

$$\sup\{\lambda\colon (x^2+y^2, Q, \lambda) \text{ is amenable to } P\}$$

$$= \frac{1}{2} + \frac{1}{4}\left(|a+b| - \sqrt{(a-b)^2 + c^2}\right).$$

(B) Suppose that $P(x,y)=\lambda(-x^2-y^2)+(1-\lambda)Q(x,y)$ for some $0<\lambda\leq 1$ and $Q\in\mathcal{P}(^2l_2^2)$, $\|Q\|=1$.

Then

$$Q(x,y)=\left(\frac{a+\lambda}{1-\lambda}\right)x^2+\left(\frac{b+\lambda}{1-\lambda}\right)y^2+\left(\frac{c}{1-\lambda}\right)xy$$

and (∗) shows that

$$\left|\frac{a+\lambda}{1-\lambda}\right|\leq 1,\quad \left|\frac{b+\lambda}{1-\lambda}\right|\leq 1,\quad \left|\frac{c}{1-\lambda}\right|\leq 2$$

and

$$4-\left(\frac{c}{1-\lambda}\right)^2=4\left|\frac{a+b+2\lambda}{1-\lambda}\right|-4\left(\frac{a+\lambda}{1-\lambda}\right)\left(\frac{b+\lambda}{1-\lambda}\right). \tag{3}$$

Solving Eq. (3), we get

$$\lambda=\frac{1}{2}-\frac{1}{4}\left(|a+b|\pm\sqrt{(a-b)^2+c^2}\right).$$

Note that

$$\frac{1-b}{2}<\frac{1}{2}-\frac{1}{4}\left(|a+b|\pm\sqrt{(a-b)^2+c^2}\right).$$

Since $\lambda\leq\min\{(1-a)/2,(1-b)/2\}=(1-b)/2$, P does not admit an amenable triple $(-x^2-y^2, Q, \lambda)$.

(C) Suppose that $P(x,y)=\lambda(lx^2-ly^2\pm 2\sqrt{1-l^2}xy)+(1-\lambda)Q(x,y)$ for some $0<\lambda\leq 1$, $-1\leq l\leq 1$ and $Q\in\mathcal{P}(^2l_2^2)$, $\|Q\|=1$.

Then

$$Q(x,y)=\left(\frac{a-\lambda l}{1-\lambda}\right)x^2+\left(\frac{b+\lambda l}{1-\lambda}\right)y^2+\left(\frac{c\pm 2\lambda\sqrt{1-l^2}}{1-\lambda}\right)xy$$

and (∗) shows that

$$\left|\frac{a-\lambda l}{1-\lambda}\right|\leq 1,\quad \left|\frac{b+\lambda l}{1-\lambda}\right|\leq 1,\quad \left|\frac{c\pm 2\lambda\sqrt{1-l^2}}{1-\lambda}\right|\leq 2$$

and

$$\left(\frac{c \pm 2\lambda\sqrt{1-l^2}}{1-\lambda}\right)^2 = 4\left(1 - \frac{a-\lambda l}{1-\lambda}\right)\left(1 - \frac{b+\lambda l}{1-\lambda}\right). \tag{4}$$

Solving Eq. (4), we get

$$\lambda = \frac{4(1-a)(1-b) - c^2}{4((b-a)l + 2 - a - b \pm c\sqrt{1-l^2})}.$$

Computation shows that

$$\begin{aligned}
&\max_{-1\leq l\leq 1} \frac{4(1-a)(1-b) - c^2}{4((b-a)l + 2 - a - b \pm c\sqrt{1-l^2})} \\
&= \frac{4(1-a)(1-b) - c^2}{4\min_{-1\leq l\leq 1}(b-a)l + 2 - a - b \pm c\sqrt{1-l^2}} \qquad \text{(by (1))} \\
&= \frac{4(1-a)(1-b) - c^2}{4\left((2-a-b) - \sqrt{(a-b)^2 + c^2}\right)} \\
&= \frac{1}{2} - \frac{1}{4}\left(|a+b| - \sqrt{(a-b)^2 + c^2}\right)
\end{aligned}$$

at $l = (a-b)/\sqrt{(a-b)^2 + c^2}$. Thus we have

$$\lambda \leq \frac{1}{2} - \frac{1}{4}\left(|a+b| - \sqrt{(a-b)^2 + c^2}\right).$$

Computation shows that P admits an amenable triple

$$\left(\frac{a-b}{\sqrt{(a-b)^2+c^2}}x^2 + \frac{b-a}{\sqrt{(a-b)^2+c^2}}y^2 \right.$$
$$\left. + \frac{2|c|}{\sqrt{(a-b)^2+c^2}}xy, Q, \frac{1}{2} - \frac{1}{4}\left(|a+b| - \sqrt{(a-b)^2+c^2}\right)\right).$$

Hence

$$\sup\{\lambda\colon (lx^2 - ly^2 \pm 2\sqrt{1-l^2}xy, Q, \lambda) \text{ is amenable to } P,\ -1 \le l \le 1\}$$
$$= \frac{1}{2} - \frac{1}{4}\left(|a+b| - \sqrt{(a-b)^2 + c^2}\right).$$

By the cases (A)–(C), we have

$$\lambda(ax^2 + by^2 + cxy) = \max\left\{\frac{1}{2} \pm \frac{1}{4}\left(|a+b| - \sqrt{(a-b)^2 + c^2}\right)\right\}$$
$$= \frac{1}{2} + \frac{1}{4}\left||a+b| - \sqrt{(a-b)^2 + c^2}\right|.$$

Case 2 $\|P\| = 1$. First, $(*)$ shows that

$$4 - c^2 = 4(a+b) - 4ab. \tag{5}$$

(A′) Suppose that $P(x,y) = \lambda(x^2+y^2) + (1-\lambda)Q(x,y)$ for some $0 < \lambda \le 1$ and $Q \in \mathcal{P}(^2l_2^2)$, $\|Q\| = 1$.

Then

$$Q(x,y) = \left(\frac{a-\lambda}{1-\lambda}\right)x^2 + \left(\frac{b-\lambda}{1-\lambda}\right)y^2 + \left(\frac{c}{1-\lambda}\right)xy$$

and $(*)$ shows that

$$\left|\frac{a-\lambda}{1-\lambda}\right| \le 1, \quad \left|\frac{b-\lambda}{1-\lambda}\right| \le 1, \quad \left|\frac{c}{1-\lambda}\right| \le 2$$

and

$$4 - \left(\frac{c}{1-\lambda}\right)^2 = 4\left|\frac{a+b-2\lambda}{1-\lambda}\right| - 4\left(\frac{a-\lambda}{1-\lambda}\right)\left(\frac{b-\lambda}{1-\lambda}\right). \tag{6}$$

If $a+b-2\lambda \ge 0$, then Eq. (6) is equivalent to

$$\lambda \le \min\left\{\frac{1+a}{2}, \frac{1+b}{2}, \frac{a+b}{2}\right\} = \frac{a+b}{2}.$$

If $a+b-2\lambda<0$, we have

$$\frac{a+b}{2}<\lambda\leq\min\left\{\frac{1+a}{2},\frac{1+b}{2}\right\}=\frac{1+a}{2}.$$

Solving Eq. (6), we get $\lambda=(a+b)/2$. Thus P does not admit an amenable triple if $a+b-2\lambda<0$. Hence

$$\sup\{\lambda\colon (x^2+y^2,Q,\lambda) \text{ is amenable to } P\}=\frac{a+b}{2}.$$

(B′) Suppose that $P(x,y)=\lambda(-x^2-y^2)+(1-\lambda)Q(x,y)$ for some $0<\lambda\leq 1$ and $Q\in\mathcal{P}(^2l_2^2)$, $\|Q\|=1$.

Then

$$Q(x,y)=\left(\frac{a+\lambda}{1-\lambda}\right)x^2+\left(\frac{b+\lambda}{1-\lambda}\right)y^2+\left(\frac{c}{1-\lambda}\right)xy$$

and $(*)$ shows that

$$\left|\frac{a+\lambda}{1-\lambda}\right|\leq 1,\quad \left|\frac{b+\lambda}{1-\lambda}\right|\leq 1,\quad \left|\frac{c}{1-\lambda}\right|\leq 2$$

and

$$4-\left(\frac{c}{1-\lambda}\right)^2=4\left|\frac{a+b+2\lambda}{1-\lambda}\right|-4\left(\frac{a+\lambda}{1-\lambda}\right)\left(\frac{b+\lambda}{1-\lambda}\right). \tag{7}$$

Solving Eq. (7), we get

$$\lambda=1-\frac{a+b}{2}.$$

Hence

$$\sup\{\lambda\colon (-x^2-y^2,Q,\lambda) \text{ is amenable to } P\}=\min\left\{1-\frac{a+b}{2},\frac{1-b}{2}\right\}.$$

(C′) Suppose that $P(x,y)=\lambda(lx^2-ly^2\pm 2\sqrt{1-l^2}xy)+(1-\lambda)Q(x,y)$ for some $0<\lambda\leq 1$, $-1\leq l\leq 1$ and $Q\in\mathcal{P}(^2l_2^2)$, $\|Q\|=1$.

Then

$$Q(x,y) = \left(\frac{a-\lambda l}{1-\lambda}\right)x^2 + \left(\frac{b+\lambda l}{1-\lambda}\right)y^2 + \left(\frac{c \pm 2\lambda\sqrt{1-l^2}}{1-\lambda}\right)xy$$

and (*) shows that

$$\left|\frac{a-\lambda l}{1-\lambda}\right| \leq 1, \quad \left|\frac{b+\lambda l}{1-\lambda}\right| \leq 1, \quad \left|\frac{c \pm 2\lambda\sqrt{1-l^2}}{1-\lambda}\right| \leq 2$$

and

$$\left(\frac{c \pm 2\lambda\sqrt{1-l^2}}{1-\lambda}\right)^2 = 4\left(1 - \frac{a-\lambda l}{1-\lambda}\right)\left(1 - \frac{b+\lambda l}{1-\lambda}\right). \tag{8}$$

Solving Eq. (8), we get

$$l = \frac{a-b}{2-a-b} \quad \text{and} \quad \lambda \leq \min\left\{\frac{1-a}{1-l}, \frac{1-b}{1+l}\right\} = 1 - \frac{a+b}{2}.$$

Computation shows that P admits an amenable triple

$$\left(\frac{a-b}{2-a-b}x^2 + \frac{b-a}{2-a-b}y^2 + \frac{2|c|}{2-a-b}xy, Q, 1 - \frac{a+b}{2}\right).$$

Hence

$$\sup\{\lambda\colon (lx^2 - ly^2 \pm 2\sqrt{1-l^2}xy, Q, \lambda) \text{ is amenable to } P,\ -1 \leq l \leq 1\}$$
$$= 1 - \frac{a+b}{2}.$$

By the cases (A′)–(C′), we have

$$\begin{aligned}\lambda(ax^2 + by^2 + cxy) &= \max\left\{\frac{a+b}{2}, 1 - \frac{a+b}{2}\right\} \\ &= \frac{1}{2} + \frac{1}{4}\left||a+b| - \sqrt{(a-b)^2 + c^2}\right| \quad \text{(by (5))}.\end{aligned}$$

By the cases 1 and 2, we have that

$$\lambda(ax^2 + by^2 + cxy) = \frac{1}{2} + \frac{1}{4}\left||a+b| - \sqrt{(a-b)^2 + c^2}\right|.$$

The above argument shows that the λ-function is continuous and attained at each point of the unit ball of $B_{\mathcal{P}(^2l_2^2)}$. This completes the proof.

Note that if E is a finite dimensional normed space, then $x \in \text{ext}(B_E)$ if and only if $\lambda(x) = 1$. From this fact and Theorem 2, we can reclassify the extreme points of the unit ball of $\mathcal{P}(^2l_2^2)$.

We can give an explicit relation between the norm and the λ-function in $\mathcal{P}(^2l_2^2)$.

THEOREM 3 *Let* $P(x,y) = ax^2 + by^2 + cxy$ *in* $\mathcal{P}(^2l_2^2)$, $\|P\| \le 1$. *Then*

$$\|P\| + 2\lambda(P) = 1 + \max\left\{|a+b|, \sqrt{(a-b)^2 + c^2}\right\}.$$

Proof By Lemma 1, we may assume that $|a| \le |b| = b$ and $c \ge 0$. From the proof of Lemma 2.1 [3] we get

$$\begin{aligned}\|P\| &= P\left(\sqrt{\frac{1}{2} - |a-b|\Big/2\sqrt{(a-b)^2 + c^2}},\right.\\ &\qquad\left.\sqrt{\frac{1}{2} + |a-b|\Big/2\sqrt{(a-b)^2 + c^2}}\right),\\ &= \left(|a+b| + \sqrt{(a-b)^2 + c^2}\right)\Big/2,\end{aligned}$$

which concludes the proof of the theorem combining Theorem 2.

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