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Extensions of Heinz–Kato–Furuta Inequality, II

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In our preceding note, we discussed a generalized Schwarz inequality, an improvement of the Heinz–Kato–Furuta inequality and an inequality related to the Furuta inequality. In succession, we give further discussions on them from the viewpoint of the covariance–variance inequality and the chaotic order. Finally, we consider a relation between our improvement and Wielandt's theorem.

Keywords: Heinz–Kato–Furuta inequality; Furuta inequality; Chaotic order; Wielandt theorem; Covariance–variance inequality

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1. INTRODUCTION

This is in continuation of our preceding note [6]. An operator T means a bounded linear operator acting on a Hilbert space. After Lin's interesting improvement of a generalized Schwarz inequality [12], we showed the following inequality which is a further improvement of a generalized Schwarz inequality:

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THEOREM 1 [6, Theorem 1] *Let T be an operator on H and $0 \neq y \in H$. For $z \in H$ satisfying $T|T|^{\alpha+\beta-1}z \neq 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$,*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y) \tag{1}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and $x, y \in H$. In the case $\alpha, \beta > 0$, the equality in (1) *holds if and only if $T|T|^{\alpha+\beta-1}y$ and $|T|^{2\alpha}(x - [(|T|^{2\alpha}x, z)/(|T|^{2\alpha}z, z)]z)$ are proportional, or equivalently, $|T^*|^{2\beta}y$ and $T|T|^{\alpha+\beta-1}(x - [(|T|^{2\alpha}x, z)/(|T|^{2\alpha}z, z)]z)$ are proportional.*

This gives us improvements of the Heinz–Kato–Furuta inequality [9,10] and moreover a theorem due to Furuta [9] as follows:

THEOREM 2 [6, Theorem 3] *Let T be an operator on H. If A and B are positive operators such that $T^*T \leq A^2$ and $TT^* \leq B^2$. Then for each $r, s \geq 0$*

$$\begin{aligned} |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2 &+ \frac{|(|T|^{2(1+2r)\alpha}x, z)|^2(|T^*|^{2(1+2s)\beta}y, y)}{(|T|^{2(1+2r)\alpha}z, z)} \\ &\leq ((|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x, x)((|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}y, y) \end{aligned} \tag{2}$$

for all $p, q \geq 1$, $\alpha, \beta \in [0, 1]$ with $(1+2r)\alpha + (1+2s)\beta \geq 1$ and $x, y, z \in H$ such that $T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z \neq 0$ and $(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z, y) = 0$. In the case $\alpha, \beta > 0$, the equality in (2) *holds if and only if*

$$|T|^{2(1+2r)\alpha}x = (|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha}x,$$
$$|T^*|^{2(1+2s)\beta}y = (|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta}y$$

and

$$|T|^{2(1+2r)\alpha}\left(x - \frac{(|T|^{2(1+2r)\alpha}x, z)}{(|T|^{2(1+2r)\alpha}z, z)}z\right) \quad \text{and} \quad |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*y$$

are proportional; the latter is equivalent to that

$$T|T|^{2(1+2r)\alpha+2(1+2s)\beta-1}\left(x - \frac{(|T|^{2(1+2r)\alpha}x, z)}{(|T|^{2(1+2r)\alpha}z, z)}z\right) \quad \text{and} \quad |T^*|^{2(1+2s)\beta}y$$

are proportional.

We note that Theorem 2 is an alternative expression of the Furuta inequality [7,8].

In this note, we give further discussions to Theorems 1 and 2. In our recent note [2], we considered the covariance $\mathrm{Cov}_x(A, B)$ of operators A and B on H for a unit vector $x \in H$, and obtained the covariance–variance inequality. From this viewpoint, we give an interpretation to Theorem 1, in which the equality condition is clarified. On the other hand, we introduced the chaotic order among positive invertible operators by $A \gg B$ if $\log A \geq \log B$, and obtained a characterization of the chaotic order in terms of Furuta's type operator inequality [3–5]. Based on this, we give a chaotic version of Theorem 2. Furthermore we interpolate between it and Theorem 2. Finally we discuss Wielandt's theorem; it follows from Theorem 1 and the Kantorovich inequality easily.

2. THE COVARIANCE–VARIANCE INEQUALITY

Recently, we [2] discussed the covariance of operators in the frame of noncommutative probability established by Umegaki [13]. The covariance $\mathrm{Cov}(A, B)$ of operators A and B at a state u is defined by

$$\mathrm{Cov}_u(A, B) = (Au, Bu) - (Au, u)(B^*u, u)$$

and the variance of A at u is defined

$$\mathrm{Var}_u(A) = \mathrm{Cov}(A, A) = \|Au\|^2 - |(Au, u)|^2.$$

In [2], we point out the following covariance–variance inequality. Here we cite it with proof in order to consider the condition satisfying the equality.

LEMMA 3 *Let u be a unit vector in H. Then the covariance–variance inequality holds*:

$$|\mathrm{Cov}_u(A, B)|^2 \leq \mathrm{Var}_u(A)\ \mathrm{Var}_u(B) \tag{3}$$

for operators A and B on H. The equality holds in (3) *if and only if $(A - \bar{A})u$ and $(B - \bar{B})u$ are proportional, where $\bar{C} = (Cu, u)$ for an operator C.*

Proof By the Schwarz inequality, we have

$$\begin{aligned}|\mathrm{Cov}_u(A,B)|^2 &= |((A-\bar{A})u,(B-\bar{B}u)|^2\\ &\le \|(A-\bar{A})u\|^2\|(B-\bar{B})u\|^2\\ &= \mathrm{Var}(A)\cdot\mathrm{Var}(B).\end{aligned}$$

Moreover it implies that the equality holds in (3) if and only if $(A-\bar{A})u$ and $(B-\bar{B})u$ are proportional.

Proof of Theorem 1 Let $T=U|T|$ be the polar decomposition of T. Then we put

$$u=\frac{U|T|^{a}z}{\||T|^{\alpha}z\|};\qquad A=U|T|^{\alpha}x\otimes u,\quad B=|T^*|^{\beta}y\otimes u,$$

where $(x\otimes y)z=(z,y)x$ for $x,y,z\in H$. Since

$$\begin{aligned}B^*u=(u,|T^*|^{\beta}y)u&=\frac{1}{\||T|^{\alpha}z\|}(U|T|^{\alpha}z,|T^*|^{\beta}y)u\\ &=\frac{1}{\||T|^{\alpha}z\|}(T|T|^{\alpha+\beta-1}z,y)u=0\end{aligned}$$

by the assumption, we have

$$\begin{aligned}\mathrm{Cov}_u(A,B)&=(Au,Bu)-(Au,u)(B^*u,u)=(Au,Bu)\\ &=(U|T|^{\alpha}x,|T^*|^{\beta}y)=(T|T|^{\alpha+\beta-1}x,y),\\ \mathrm{Var}_u(A)&=\|Au\|^2-|(Au,u)|^2=\||T|^{\alpha}x\|^2-\frac{|(|T|^{2\alpha}x,z)|^2}{\||T|^{\alpha}z\|^2}\end{aligned}$$

and

$$\mathrm{Var}_u(B)=\|Bu\|^2-|(Bu,u)|^2=\|Bu\|^2=\||T^*|^{\beta}y\|^2.$$

Hence the covariance–variance inequality implies the desired inequality. Moreover Lemma 3 implies that the equality holds in (1) if and only if $(A-\bar{A})u$ and $(B-\bar{B})u$ are proportional, i.e., $U|T|^{\alpha}(x-[(|T|^{2\alpha}x,z)/\||T|^{\alpha}z\|^2]z)$ and $|T^*|^{\beta}y$ are proportional. Furthermore it is equivalent to the proportional conditions in Theorem 1 (cf. [3, Lemma]).

3. EXTENSIONS OF FURUTA'S TYPE INEQUALITY

Next we discuss the extensions of Theorem 2. Precisely, we give a chaotic version of Theorem 2 and moreover interpolate between the chaotic version and Theorem 2 itself. The chaotic order among positive invertible operators is meaningful in the discussion of Furuta's type operator inequalities. It is defined by $\log A \geq \log B$; in symbolic form, $A \gg B$. We use the following characterization of the chaotic order which is an extension of Ando's theorem [1,3,4] and cf. [5].

THEOREM A *For positive invertible operators A and B, $A \gg B$ if and only if*

$$(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q} \tag{†}$$

holds for $q \geq 1$, $p, r \geq 0$ with $2rq \geq p + 2r$.

Based on Theorem A, we have the following chaotic version of Theorem 2:

THEOREM 4 *Let T be an invertible operator on H. If A and B are positive invertible operators on H such that $A^2 \gg T^*T$ and $B^2 \gg TT^*$, then for each $r, s \geq 0$*

$$\begin{aligned} &|(T|T|^{r\alpha+s\beta-1}x, y)|^2 + \frac{|(|T|^{2r\alpha}x, z)|^2(|T^*|^{2s\beta}y, y)}{(|T|^{2r\alpha}z, z)} \\ &\quad \leq ((|T|^r A^p |T|^r)^{2r\alpha/(p+2r)}x, x)((|T^*|^s B^q |T^*|^s)^{2s\beta/(q+2s)}y, y) \end{aligned} \tag{4}$$

for all $p, q \geq 0$, $\alpha, \beta \in [0, 1]$ and $x, y, z \in H$ such that

$$T|T|^{r\alpha+s\beta-1}z \neq 0 \quad \text{and} \quad (T|T|^{r\alpha+s\beta-1}z, y) = 0.$$

In the case $\alpha, \beta > 0$, *the equality holds if and only if*

$$|T|^{2r\alpha}x = (|T|^r A^p |T|^r)^{2r\alpha/(p+2r)}x, \qquad |T^*|^{2s\beta}y = (|T^*|^s B^q |T^*|^s)^{2s\beta/(q+2s)}y$$

and

$$|T|^{r\alpha+s\beta-1}T^*y \quad \text{and} \quad |T|^{2r\alpha}\left(x - \frac{(|T|^{2r\alpha}x, z)}{(|T|^{2r\alpha}z, z)}z\right)$$

are proportional; or equivalently,

$$|T^*|^{2s\beta}y \quad and \quad T|T|^{r\alpha+s\beta-1}\left(x - \frac{(|T|^{2r\alpha}x, z)}{(|T|^{2r\alpha}z, z)}z\right)$$

are proportional.

Proof The proof is similar to that of Theorem 2. By Theorem 1, we have

$$|(T|T|^{r\alpha+s\beta-1}x, y)|^2 + \frac{|(|T|^{2r\alpha}x, z)|^2(|T^*|^{2s\beta}y, y)}{(|T|^{2r\alpha}z, z)}$$
$$\leq (|T|^{2r\alpha}x, x)(|T^*|^{2s\beta}y, y).$$

Moreover Theorem A implies that

$$|T|^{2r\alpha} \leq (|T|^r A^p |T|^r)^{2r\alpha/(p+2r)} \quad \text{and} \quad |T^*|^{2s\beta} \leq (|T^*|^s B^q |T^*|^s)^{2s\beta/(q+2s)}.$$

Combining three inequalities above, we have

$$|(T|T|^{r\alpha+s\beta-1}x, y)|^2 + \frac{|(|T|^{2r\alpha}x, z)|^2(|T^*|^{2s\beta}y, y)}{(|T|^{2r\alpha}z, z)}$$
$$\leq (|T|^{2r\alpha}x, x)(|T^*|^{2s\beta}y, y)$$
$$\leq ((|T|^r A^p |T|^r)^{2r\alpha/(p+2r)}x, x)((|T^*|^s B^q |T^*|^s)^{2s\beta/(q+2s)}y, y).$$

As in the proof of [6, Theorem 3], the equality condition is easily checked.

Next we interpolate between Theorems 2 and 4. For this, we use the following Furuta's type operator inequality which interpolates the Furuta inequality and Theorem A as the cases $\delta = 1, 0$ respectively (see [4]).

THEOREM B *If $A^\delta \geq B^\delta$ for some $\delta \in [0, 1]$, then for each $r \geq 0$*

$$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q} \qquad (\dagger)$$

holds for $p \geq 0$ and $q \geq 1$ such that $(\delta + 2r)q \geq p + 2r$.

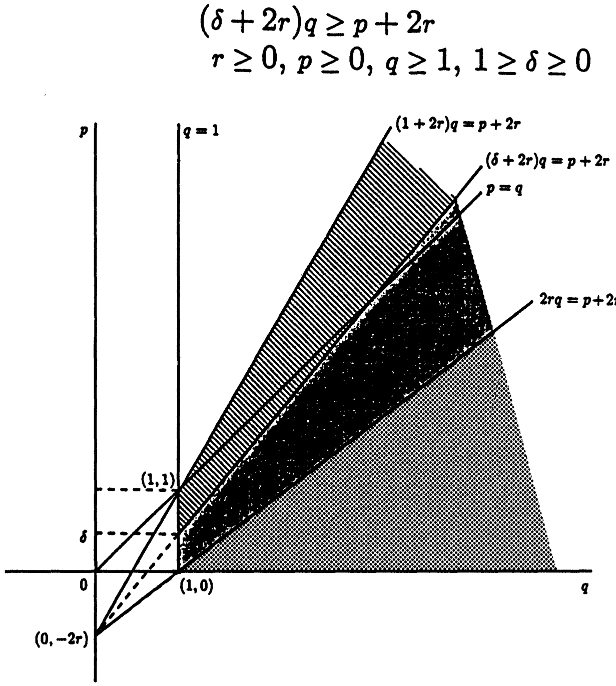

FIGURE 1

Figure 1 expresses the domain for which (†) holds, see [4].

As in the proofs of Theorems 2 and 4, we have the following theorem interpolating between them as $\delta = 1$ and $\delta = 0$ respectively.

THEOREM 5 *Let T be an operator on H. If A and B are positive operators such that $|T|^\delta \le A^\delta$ and $|T^*|^\delta \le B^\delta$ for some $\delta \in [0,1]$. Then for each $r, s \ge 0$*

$$|(T|T|^{(\delta+2r)\alpha+(\delta+2s)\beta-1}x,y)|^2 + \frac{|(|T|^{2(\delta+2r)\alpha}x,z)|^2(|T^*|^{2(\delta+2s)\beta}y,y)}{(|T|^{2(\delta+2r)\alpha}z,z)}$$
$$\le ((|T|^{2r}A^{2p}|T|^{2r})^{(\delta+2r)\alpha/(p+2r)}x,x)((|T^*|^{2s}B^{2q}|T^*|^{2s})^{(\delta+2s)\beta/(q+2s)}y,y)$$

for all $p, q \ge 1$, $\alpha, \beta \in [0,1]$ with $(\delta+2r)\alpha + (\delta+2s)\beta \ge 1$ and $x, y, z \in H$ such that

$$T|T|^{(\delta+2r)\alpha+(\delta+2s)\beta-1}z \ne 0 \quad \text{and} \quad (T|T|^{(\delta+2r)\alpha+(\delta+2s)\beta-1}z, y) = 0.$$

In the case $\alpha, \beta > 0$, *the equality holds if and only if*

$$(|T|^{2r}A^{2p}|T|^{2r})^{(\delta+2r)\alpha}x = |T|^{2(\delta+2r)\alpha}x,$$
$$(|T^*|^{2s}B^{2q}|T^*|^{2s})^{(\delta+2s)\beta}y = |T^*|^{2(\delta+2s)\beta}y$$

and

$$|T|^{2(\delta+2r)\alpha}\left(x - \frac{(|T|^{2(\delta+2r)\alpha}x, z)}{(|T|^{2(\delta+2r)\alpha}z, z)}z\right) \quad \textit{and} \quad |T|^{(\delta+2r)\alpha+(\delta+2s)\beta-1}T^*y$$

are proportional; the latter is equivalent to that

$$T|T|^{2(\delta+2r)\alpha+2(\delta+2s)\beta-1}\left(x - \frac{(|T|^{2(\delta+2r)\alpha}x, z)}{(|T|^{2(\delta+2r)\alpha}z, z)}z\right) \quad \textit{and} \quad |T^*|^{2(\delta+2s)\beta}y$$

are proportional.

Concluding this section, we remark that Theorem A (resp. B) is equivalent to Theorem 4 (resp. 5). As a matter of fact, suppose that $A \gg B$. If we take $T = B$, $x = y$, $r = s$ and $\alpha = \beta$ in Theorem 4, then we have

$$(B^{2r\alpha}x, x) \le ((B^rA^pB^r)^{2r\alpha/(p+2r)}x, x)$$

because $(B^{2r\alpha}z, x) = 0$. That is, we obtain Theorem A. Similarly we can show that Theorem 5 implies Theorem B.

4. A CONCLUDING REMARKS

As an improvement of the Cauchy–Schwarz inequality, the Wielandt theorem is well known [11, 7.4.32]:

THE WIELANDT THEOREM *If* $0 < m \le T \le M$, *then*

$$|(Tx, y)|^2 \le \left(\frac{M-m}{M+m}\right)^2 (Tx, x)(Ty, y) \tag{5}$$

for every orthogonal pair x *and* y.

Also it is well known that the Wielandt theorem implies the celebrated Kantorovich inequality;

THE KANTOROVICH INEQUALITY *If* $0 < m \le T \le M$, *then*

$$(Tx, x)(T^{-1}x, x) \le \frac{(M+m)^2}{4mM}$$

for every unit vector $x \in H$.

Roughly speaking, we shall show that the converse of the above statement holds, that is, the Kantorovich inequality implies the Wielandt theorem.

Now we pointed out that Theorem 1 is a generalized Schwarz inequality. From this viewpoint, Theorem 1 may be regarded as a generalization of the Wielandt theorem. As a matter of fact, for an orthogonal pair x, y if we put $z = T^{-1}x$, $\alpha = \beta = 1/2$ in Theorem 1, then

$$|(Tx, y)|^2 + \frac{(x, x)^2(Ty, y)}{(T^{-1}x, x)} \le (Tx, x)(Ty, y),$$

that is,

$$|(Tx, y)|^2 \le \left\{1 - \frac{\|x\|^4}{(Tx, x)(T^{-1}x, x)}\right\}(Tx, x)(Ty, y).$$

Hence we have by the Kantorovich inequality

$$\begin{aligned}|(Tx, y)|^2 &\le \left\{1 - \frac{\|x\|^4}{(Tx, x)(T^{-1}, x)}\right\}(Tx, x)(Ty, y)\\ &\le \left\{1 - \frac{4Mm}{(M+m)^2}\right\}(Tx, x)(Ty, y)\\ &= \left(\frac{M-m}{M+m}\right)^2 (Tx, x)(Ty, y).\end{aligned}$$

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