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A Theory on Perturbations of the Dirac Operator

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We consider perturbations of a first-order differential operator with matrix coefficients known as the Dirac operator. These operators have one singular point which is allowed to be either zero or infinity. Unitary transformations are used to apply results for an operator with a singularity at infinity to one with a singularity at zero. After introducing notation and several preliminary results, we give necessary and sufficient conditions for perturbations to be relatively bounded or relatively compact with respect to the Dirac operator. These conditions involve explicit integral averages of the coefficients of the perturbation. Results are given for both limit point and limit circle type operators.

Keywords: Dirac operator; Perturbation theory; Relative bounded; Relative compact; Maximal and minimal operators

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1 INTRODUCTION

In this paper we develop a perturbation theory for the formal differential expression

$$\Gamma y(t) = W^{-1}(t)\{[Q_0(t)y(t)]' - Q_0^*(t)y'(t) + P_0(t)y(t)\}, \qquad (1)$$

where the functions y are defined on the interval $I=(a,b)$, $\infty \leq a < b \leq \infty$ and the coefficients W, P_0, and Q_0 are 2×2-matrix valued

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functions on I. Each coefficient is assumed to be locally Lebesgue integrable on I, $W(t)$ is positive definite, $P_0(t)$ is hermitian, i.e. $P_0 = P_0^*$, and $Q_0(t)$ is nonsingular. In our considerations $W(t)$ is a diagonal matrix and $Q_0(t)$ is frequently the constant matrix $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. An operator of the form (1) is called a *D*irac operator.

The expression Γ is *formally self-adjoint*, i.e., for "sufficiently smooth" functions $y, z : I \to \mathbb{C}^2$ with compact support

$$\int_I \langle W(t)\Gamma y(t), z(t)\rangle \, \mathrm{d}t = \int_I \langle W(t)y(t), \Gamma z(t)\rangle \, \mathrm{d}t$$

where $\langle ., . \rangle$ denotes the usual inner product in $\mathbb{C}^2$. Hence, Γ generates a hermitian operator which is densely defined (i.e. symmetric) in the separable, weighted Hilbert space $\mathcal{L}^2_W(I)$. This Hilbert space is the space of (equivalence classes of) Lebesgue measurable functions $y : I \to \mathbb{C}^2$ such that $\int_I (W_1|y_1|^2 + W_2|y_2|^2) < \infty$, where $y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$, and W_1 and W_2 are positive measurable functions on the interval. For $y, z \in \mathcal{L}^2_W(I)$, define

$$\langle y, z\rangle_W = \int_I z^*(t) \begin{pmatrix} W_1(t) & 0 \\ 0 & W_2(t) \end{pmatrix} y(t) \, \mathrm{d}t$$

and $\|y\|^2_W = \langle y, y\rangle_W$. We will omit the subscript W when there is no ambiguity.

Associated with the formal differential expressions Γ are maximal and minimal operators (Γ_1 and Γ_0, respectively) on the Hilbert space $\mathcal{L}^2_W(I)$. The *maximal operator* Γ_1 is the differential operator defined by Γ with the largest possible domain in $\mathcal{L}^2_W(I)$ which is mapped into $\mathcal{L}^2_W(I)$, i.e.,

$$D(\Gamma_1) = \{y \in \mathcal{L}^2_W(I) : \; y \in AC_{\mathrm{loc}}(I), \; \Gamma y \in \mathcal{L}^2_W(I)\},$$

where $AC_{\mathrm{loc}}(I)$ is the set of functions which are absolutely continuous on compact subsets of I. We define the *minimal unclosed operator* Γ_0' to be the restriction of Γ to the functions with compact support in the interior of I. The *minimal operator* Γ_0 is defined to be the closure of Γ_0'. Similar definitions may be made for the differential operators of arbitrary order.

We note several well-known facts [12, pp. 41, 46] for formally self-adjoint Γ:

1. $D(\Gamma_1)$ is dense in $\mathcal{L}^2_W(I)$ and Γ_1 is closed.
2. $\Gamma_0^* = \Gamma_1$ and $\Gamma_1^* = \Gamma_0$.
3. Any self-adjoint extension A of Γ_0 satisfies $\Gamma_0 \subset A \subset \Gamma_1$.

We say that Γ is *regular at a* if $a > -\infty$ and the assumptions on the coefficients are satisfied on $[a, b)$ instead of (a, b). We define *regular at b* similarly. If Γ is regular at a and regular at b, then we say that Γ is *regular*. Otherwise, Γ is *singular*.

Since $\Gamma_0 : D(\Gamma_0) \subset \mathcal{L}^2_W(I) \to \mathcal{L}^2_W(I)$ is a closed, symmetric operator,

$$D(\Gamma_0^*) = D(\Gamma_0) \oplus N(iE - \Gamma_0^*) \oplus N(-iE - \Gamma_0^*), \tag{2}$$

where $N(\pm iE - \Gamma_0^*) = \{y \in D(\Gamma_0^*) : \ \pm iEy - \Gamma y = 0\}$ and E denotes the identity operator. Equation (2) is referred to as the *first formula of Von Neumann* and can be rewritten as

$$D(\Gamma_1) = D(\Gamma_0) \oplus N(iE - \Gamma_1) \oplus N(-iE - \Gamma_1), \tag{3}$$

since $\Gamma_0^* = \Gamma_1$.

Deficiency indices play an important role in the study of self-adjoint operators associated with the differential expression Γ in that they determine the number of boundary conditions necessary to construct a self-adjoint operator [12, Chapter 4]. The *deficiency indices* of Γ_0, denoted by $d_\pm(\Gamma_0)$, are defined by

$$d_+(\Gamma_0) = \dim R(iE - \Gamma_0)^\perp = \dim N(-iE - \Gamma_1)$$

and

$$d_-(\Gamma_0) = \dim R(-iE - \Gamma_0)^\perp = \dim N(iE - \Gamma_1).$$

If the coefficients of Γ_0 are real, we have $d_+(\Gamma_0) = d_-(\Gamma_0)$. It is well-known [12, pp. 52, 55] that

$$d_\pm(\Gamma_0) \leq 2, \tag{4}$$

and if Γ is regular at a and singular at b, then

$$d_+(\Gamma_0) + d_-(\Gamma_0) \geq 2. \tag{5}$$

If Γ is regular at a and singular at b, we say that Γ is *limit circle at b* if $d_\pm(\Gamma_0) = 2$ and Γ is *limit point at b* if $d_+(\Gamma_0) + d_-(\Gamma_0) = 2$. This notation stems from the geometric method of Weyl for the second-order equation. (See [5, Chapter 9].)

As an example, we compute the deficiency indices for the minimal operator Γ_0 corresponding to the differential expression (1) with

$$W(t)=\begin{pmatrix} t^{\gamma} & 0 \\ 0 & t^{\gamma}\end{pmatrix},\quad P_0(t)=0,\quad \text{and}\quad Q_0(t)=\frac{1}{2}\begin{pmatrix} 0 & -t^{\alpha} \\ t^{\alpha} & 0\end{pmatrix}$$

for some constants γ and α. Since the coefficients are real, $d_+(\Gamma_0)=d_-(\Gamma_0)$.

We know that if there exists a $\lambda\in\mathbb{C}$ such that each solution of $\Gamma y(t)=\lambda y(t)$, $t\in I$, is square integrable, i.e., $\int_I y^*(t)W(t)y(t)\,\mathrm{d}t<\infty$, then for every λ each solution of $\Gamma y(t)=\lambda y(t)$, $t\in I$, is square integrable [3, Theorem 9.11.2]. Thus, to determine $d_+(\Gamma_0)$ and $d_-(\Gamma_0)$ it is enough to consider $\dim N(\Gamma_1)$. Notice that inequalities (4) and (5) imply that the deficiency indices are either one or two since $d_+(\Gamma_0)=d_-(\Gamma_0)$. Now, we determine conditions on γ and α for which we have $\mathcal{L}^2_W(I)$-solutions to $\Gamma y(t)=0$.

Two linearly independent solutions to the equation $\Gamma y(t)=0$ are

$$Y_1(t)=\begin{pmatrix} t^{-\alpha/2} \\ 0\end{pmatrix}\quad \text{and}\quad Y_2(t)=\begin{pmatrix} 0 \\ t^{-\alpha/2}\end{pmatrix}.$$

If we take the interval $I=[a,\infty)$, $a>0$, then $Y_1,Y_2\in D(\Gamma_1)$ iff $\gamma-\alpha<-1$. Therefore, $d_+(\Gamma_0)=d_-(\Gamma_0)=2$ (implying that Γ is limit circle at ∞) iff $\gamma-\alpha<-1$. We also have, via [3, Theorem 9.11.2], that Γ is limit point at ∞ iff $\gamma-\alpha\geq -1$.

On the other hand, if we take the interval $I=(0,a]$, $a>0$, then $Y_1,Y_2\in D(\Gamma_1)$ iff $\gamma-\alpha>-1$. Therefore, $d_+(\Gamma_0)=d_-(\Gamma_0)=2$ (implying that Γ is limit circle at 0) iff $\gamma-\alpha>-1$. Moreover, Γ is limit point at 0 iff $\gamma-\alpha\leq -1$ [3, Theorem 9.11.2].

One importance of perturbation theory is that it allows the decomposition of an operator into the sum of a simple operator and a complicated operator which is, in some sense, small with respect to the simple operator. Since some properties are preserved under certain types of perturbations, knowledge about the simple operator is often enough to gain some knowledge about the sum. For example, the essential spectrum is preserved under a relatively compact perturbation. Also, a relatively bounded, symmetric perturbation with relative bound less than one preserves self-adjointness. (See [7–9].)

In the case of a limit point operator, there is no difference in the perturbation theory of minimal and maximal operators. This fact has been proved in the case of a scalar operator by Anderson and Hinton ([2, Theorem 2.2]). In Theorem 2.6 we considerably simplify their proof and extend it to differential operators with matrix coefficients. (The proof does not depend on the order of the operator.) In Section 3 we prove perturbation theorems for several operators of the form (1) which are in the limit point case. In the simplest case of Theorem 3.1 with unity weights, a result is obtained which is analogous to the Schrodinger operator $-y''$ result which states that the perturbing term $V(x)y$ is a relatively bounded (relatively compact) perturbation of $-y''$ if and only if

$$\limsup_{x\to\infty} \int_x^{x+\varepsilon} V^2(t)\,\mathrm{d}t < \infty, \quad \left(\lim_{x\to\infty} \int_x^{x+\varepsilon} V^2(t)\,\mathrm{d}t = 0\right)$$

for some $\varepsilon > 0$. (See [11, p. 53].) The perturbation theorems of this paper may also be applied repeatedly to decompose an operator as is done following Theorem 3.4.

For limit circle operators the results for perturbations of minimal and maximal operators are quite different. In Section 4 we consider the limit circle operator of the form (1) with power coefficients. For the minimal operator the results are somewhat analogous to the limit point case. However, for the maximal operator we have the surprising result that the concepts of relative boundedness and relative compactness coincide.

2 PRELIMINARIES

The purpose of this section is to introduce notation and theorems which will be used throughout this work. We use definitions given by Goldberg [7] and Weidmann [12].

Let X and Y be Banach spaces and let G and F be linear operators, each having domain in X and range in Y. Denote the domains of G and F by $D(G)$ and $D(F)$, respectively. By definition the *graph norm* of F on $D(F)$, denoted $\|\cdot\|_F$, is given by $\|y\|_F = \|y\| + \|Fy\|$. We say that G is *relatively bounded with respect to F* (or *F-bounded*) if $D(F) \subseteq D(G)$ and G is bounded on $D(F)$ with respect to $\|\cdot\|_F$, i.e., there exist constants $\alpha, \beta > 0$

such that $\|Gy\| \leq \alpha\|y\| + \beta\|Fy\|$ for all $y \in D(F)$. The infimum of all such β is called the *relative bound of G with respect to F* (or the *F-bound of G*). A sequence $\{y_n\}_{n=1}^{\infty}$ is *F-bounded* if there exists a constant $C > 0$ such that $\|y_n\|_F < C$ for each n. We say that G is *relatively compact with respect to F* (or *F-compact*) if $D(F) \subseteq D(G)$ and G is compact on $D(F)$ with respect to $\|\cdot\|_F$, i.e., if $\{y_n\}_{n=1}^{\infty}$ is an F-bounded sequence, then $\{Gy_n\}_{n=1}^{\infty}$ contains a convergent subsequence.

Now, we establish some general properties of operators. As before the subscript 0 denotes a minimal operator, and the subscript 1 denotes a maximal operator.

THEOREM 2.1 *Let F and G be closed linear operators in a Banach space $\mathcal{B}$ with $D(F) \subseteq D(G)$. Then G is F-bounded.*

Proof Let $\hat{G}$: graph $F \to \mathcal{B}$ be defined by $\hat{G}(y, Fy) = Gy$. Then $\hat{G}$ is linear since G is linear. Let $\{(y_n, Fy_n)\}_{n=1}^{\infty} \subset D(\hat{G})$ be such that $(y_n, Fy_n) \to (y, z)$ and $\hat{G}(y_n, Fy_n) \to \zeta$ as $n \to \infty$. Now, we show that $\hat{G}$ is closed, i.e., $(y, z) \in$ graph F and $\hat{G}(y, z) = \zeta$. Since F is closed, graph F is closed. Thus, $(y, z) \in$ graph F where $z = Fy$; hence, $(y_n, Fy_n) \to (y, Fy)$ in $D(\hat{G})$. Since G is closed, $y_n \to y$ and $Gy_n = \hat{G}(y_n, Fy_n) \to \zeta$ imply that $Gy = \zeta$. So, we have $\zeta = Gy = \hat{G}(y, Fy)$. Thus, $\hat{G}$ is closed. By the Closed Graph Theorem, G is F-bounded.

THEOREM 2.2 *Let F and G be formal differential expressions on an interval I where F is symmetric, the order of G is less than the order of F, and the coefficients of F and G are sufficiently smooth so that $D(F_0') \subseteq D(G_0')$.*

4. *If G_0' is F_0'-bounded, then G_0 is F_0-bounded.*
5. *If G_0' is F_0'-compact, then G_0 is F_0-compact.*

Proof (i) Let $y \in D(F_0)$. Then there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset D(F_0')$ such that $y_n \to y$ and $F_0'y_n \to F_0y$ as $n \to \infty$. Since G_0' is F_0'-bounded, $\{y_n\}_{n=1}^{\infty} \subset D(G_0')$ and there exists a constant $C_1 > 0$ such that

$$\|G_0'y_n - G_0'y_m\| = \|G_0'(y_n - y_m)\| \leq C_1(\|y_n - y_m\| + \|F_0'(y_n - y_m)\|). \tag{6}$$

Since $\{y_n\}_{n=1}^{\infty}$ and $\{F_0'y_n\}_{n=1}^{\infty}$ are convergent sequences, they are Cauchy. Hence, by inequality (6), $\{G_0'y_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the complete space $\mathcal{L}_W^2(a, b)$ and, therefore, converges. By definition of G_0, $y \in D(G_0)$.

Since $y \in D(F_0)$ is arbitrary, we have $D(F_0) \subseteq D(G_0)$. By applying Theorem 2.1 we conclude G_0 is F_0-bounded.

(ii) Since G_0' is F_0'-compact, G_0' is F_0'-bounded. By part (i), G_0 is F_0-bounded. Let $\{y_n\}_{n=0}^{\infty} \subset D(F_0)$ be an F_0-bounded sequence, i.e., there exists a constant $C_2 > 0$ such that for each n

$$\|y_n\| + \|F_0 y_n\| \leq C_2. \tag{7}$$

Since F_0 is the closure of F_0', there exists a $z_n \in D(F_0')$ such that for each n

$$\|y_n - z_n\| + \|F_0 y_n - F_0' z_n\| < \frac{1}{n}. \tag{8}$$

Then $\{z_n\}_{n=0}^{\infty} \subset D(F_0')$ is an F_0'-bounded sequence since, via the triangle inequality and inequalities (7) and (8),

$$\begin{aligned}\|z_n\|_{F_0'} &= \|z_n\| + \|F_0' z_n\| \\ &\leq (\|z_n - y_n\| + \|F_0' z_n - F_0 y_n\|) + (\|y_n\| + \|F_0 y_n\|) \\ &\leq \frac{1}{n} + C_2 \leq 1 + C_2.\end{aligned}$$

Since G_0' is F_0'-compact, there exists a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ of $\{z_n\}_{n=0}^{\infty}$ such that $\{G_0 z_{n_k}\}_{k=1}^{\infty}$ converges as $k \to \infty$, say to ζ. Thus, via the triangle inequality, the F_0-boundedness of G_0, and inequality (8), we have that $\{y_n\}_{n=1}^{\infty} \subset D(G_0)$ and for some constant $C_3 > 0$

$$\begin{aligned}\|G_0 y_{n_k} - \zeta\| &\leq \|G_0 y_{n_k} - G_0 z_{n_k}\| + \|G_0 z_{n_k} - \zeta\| \\ &= \|G_0(y_{n_k} - z_{n_k})\| + \|G_0 z_{n_k} - \zeta\| \\ &\leq C_3(\|y_{n_k} - z_{n_k}\| + \|F_0 y_{n_k} - F_0 z_{n_k}\|) + \|G_0 z_{n_k} - \zeta\| \\ &\leq \frac{C_3}{n_k} + \|G_0 z_{n_k} - \zeta\| \to 0 \text{ as } k \to \infty.\end{aligned}$$

Therefore, $\{G_0 y_n\}_{n=1}^{\infty}$ contains a convergent subsequence. By definition, G_0 is F_0-compact.

THEOREM 2.3 *Let F and G be as in Theorem 2.2 and let $D(F_1) \subseteq D(G_1)$. Then*

(i) *G_1 is F_1-bounded;*
(ii) *G_0 is F_0-bounded;*
(iii) *G_0 is F_0-compact if G_1 is F_1-compact.*

Proof

(i) Apply Theorem 2.1.
(ii) Let $y \in D(F_0')$.

Then via part (i), there exists a constant $C > 0$ such that

$$\|G_0'y\| = \|G_1y\| \leq C(\|y\| + \|F_1y\|) = C(\|y\| + \|F_0'y\|).$$

Since $y \in D(F_0')$ is arbitrary, we have that G_0' is F_0'-bounded. Apply Theorem 2.2 (i) to complete the proof.

(iii) Let $\{y_n\}_{n=1}^{\infty} \subset D(F_0)$ be an F_0-bounded sequence. Then $\{y_n\}_{n=1}^{\infty} \subset D(F_1)$ and is an F_1-bounded sequence. Since G_1 is F_1-compact, there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that the sequence $\{G_1y_{n_k}\}_{k=1}^{\infty}$ converges as $k \to \infty$. Now, part (ii) implies that $D(F_0) \subseteq D(G_0)$. Hence, $G_0y_n = G_1y_n$ for any n. Therefore, the sequence $\{G_0y_{n_k}\}_{k=1}^{\infty}$ converges as $k \to \infty$. By definition, G_0 is F_0-compact.

The following lemma is stated and proved in [2, Lemma 2.1]:

LEMMA 2.4 *Let X and Y be subspaces of a Banach space $\mathcal{B}$, where X is closed, Y is finite dimensional, and $X \cap Y = \{0\}$. Then there exists a constant $K > 0$ such that*

$$\|x + y\| \geq K\|y\| \quad \textit{for all } x \in X \textit{ and } y \in Y.$$

THEOREM 2.5 *Let F and G be as in Theorem 2.2, G_0 be F_0-compact, and $D(F_1) \subseteq D(G_1)$. Then G_1 is F_1-compact.*

Proof Let $\{y_n\}_{n=1}^{\infty} \subset D(F_1)$ be an F_1-bounded sequence, i.e., there exists a constant $C_1 > 0$ such that for each n

$$\|y_n\| + \|F_1y_n\| \leq C_1. \tag{9}$$

Since $D(F_1) = D(F_0) \oplus S$ where S is finite dimensional [see Eq. (3) and inequality (4)], y_n can be written as $y_n = y_{n,0} + y_{n,c}$ where $y_{n,0} \in D(F_0)$ and $y_{n,c} \in S$ for each n. Thus, we have

$$G_1y_n = G_1y_{n,0} + G_1y_{n,c} = G_0y_{n,0} + G_1y_{n,c}$$

for each n since $D(F_0) \subseteq D(G_0) \subseteq D(G_1)$. Since $F_0 \subset F_1$ and F_1 is bounded when acting upon a finite dimensional space, there exists a constant

$C_2 > 0$ such that for each n

$$\|F_1 y_{n,c}\| \le C_2 \|y_{n,c}\|. \tag{10}$$

Thus, via the triangle inequality and inequalities (9) and (10), we have for each n

$$\|F_1 y_{n,0}\| = \|F_1 y_n - F_1 y_{n,c}\| \le \|F_1 y_n\| + \|F_1 y_{n,c}\| \le C_1 + C_2 \|y_{n,c}\|. \tag{11}$$

Via Lemma 2.4,

$$\|y_{n,c}\| \le \|y_{n,c}\|_{F_1} \le \frac{1}{K} \|y_{n,0} + y_{n,c}\|_{F_1} = \frac{1}{K} \|y_n\|_{F_1}. \tag{12}$$

Thus, via inequalities (9), (11), and (12), we have that for each n,

$$\|F_0 y_{n,0}\| = \|F_1 y_{n,0}\| \le C_1 + \frac{C_2}{K} C_1, \tag{13}$$

i.e., $\{F_0 y_{n,0}\}_{n=1}^{\infty}$ is bounded in $\mathcal{L}_w^2(I)$.

Also, via the triangle inequality and inequalities (9) and (12), we have that for each n

$$\begin{aligned} \|y_{n,0}\| &= \|y_n - y_{n,c}\| \le \|y_n\| + \|y_{n,c}\| \\ &\le \|y_n\| + \frac{1}{K} \|y_n\|_{F_1} \le C_1 + \frac{1}{K} C_1. \end{aligned} \tag{14}$$

Thus, $\{y_{n,0}\}_{n=1}^{\infty}$ is bounded in $\mathcal{L}_w^2(I)$. Via inequalities (13) and (14), $\{y_{n,0}\}_{n=1}^{\infty}$ is an F_0-bounded sequence. Since G_0 is F_0-compact, $\{G_0 y_{n,0}\}_{n=1}^{\infty}$ contains a convergent subsequence, say $\{G_0 y_{n_k,0}\}_{k=1}^{\infty}$. When acting on a finite dimensional space, G_1 is bounded. Thus, there exists a constant $C_3 > 0$ such that for each k, $\|G_1 y_{n_k,c}\| \le C_3 \|y_{n_k,c}\|$. Via this inequality and inequalities (9) and (12), we have that $\|G_1 y_{n_k,c}\| \le (C_3/K) C_1$ for each k, i.e., $\{G_1 y_{n_k,c}\}_{k=1}^{\infty}$ is bounded in a finite dimensional subspace of $\mathcal{L}_w^2(I)$. Hence, $\{G_1 y_{n_k,c}\}_{k=1}^{\infty}$ contains a convergent subsequence. Therefore, $\{G_1 y_n\}_{n=1}^{\infty}$ contains a convergent subsequence since $G_1 y_n = G_0 y_{n,0} + G_1 y_{n,c}$. By definition, G_1 is F_1-compact.

THEOREM 2.6 *Let F and G be as in Theorem 2.2. If F is regular at a and limit point at b, then G_0 is F_0-bounded (G_0 is F_0-compact) if and only if G_1 is F_1-bounded (G_1 is F_1-compact).*

Proof Via Eq. (3) and inequality (4), we can write $D(F_1) = D(F_0) \oplus S$, where S is finite dimensional. Since F is regular at a and limit point at b, $\dim S = n \times m$. We know [12, p. 62] that there exists $n \times m$-functions which are in the domain of F_1, have compact support in (a, b), and are linearly independent modulo $D(F_0)$. Let us call this span of functions S_0. WLOG, we can take $S = S_0$. Since the order of G is less than the order of F, we have $S_0 \subseteq D(G_1)$. Thus, $D(F_0) \subseteq D(G_0)$ implies that $D(F_1) \subseteq D(G_1)$. Suppose that G_0 is F_0-bounded. Then $D(F_0) \subseteq D(G_0)$. Hence, G_1 is F_1-bounded via Theorem 2.3 (i). Suppose that G_0 is F_0-compact. Then $D(F_0) \subseteq D(G_0)$. Hence, G_1 is F_1-compact via Theorem 2.5. Moreover, if G_1 is F_1-bounded (G_1 is F_1-compact), then G_0 is F_0-bounded (G_0 is F_0-compact) via Theorem 2.3 (ii) (Theorem 2.3 (iii)).

The following theorem and lemma [4, pp. 570, 575, 576] are important in the proofs of later theorems.

THEOREM 2.7 *Let $I = [a, \infty)$ and let N, W, and P be positive measurable functions such that N, W^{-1}, and $P^{-1} \in \mathcal{L}_{\mathrm{loc}}(I)$. Suppose there exists an $\varepsilon_0 > 0$ and a positive continuous function $f = f(t)$ on I such that*

$$S_1(\varepsilon) := \sup_{t \in I} \left\{ f^2 \left[\frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} P^{-1} \right] \left[\frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} N \right] \right\} < \infty$$

and

$$S_2(\varepsilon) := \sup_{t \in I} \left\{ \left[\frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} W^{-1} \right] \left[\frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} N \right] \right\} < \infty$$

for all $\varepsilon \in (0, \varepsilon_0)$. Then there exists a constant $k > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $y \in D$,

$$\int_I N|y|^2 \le k \left\{ S_2(\varepsilon) \int_I W|y|^2 + \varepsilon^2 S_1(\varepsilon) \int_I P|y'|^2 \right\}$$

where

$$D = \left\{ y : y \in AC_{\mathrm{loc}}(I), \int_I W|y|^2 < \infty, \text{ and } \int_I P|y'|^2 < \infty \right\}.$$

Note that if $S_1(\varepsilon)$, $S_2(\varepsilon) < \infty$ for $\varepsilon = \varepsilon_1$, then $S_1(\varepsilon)$, $S_2(\varepsilon) < \infty$ for all $\varepsilon \in (0, \varepsilon_1]$.

LEMMA 2.8 *Let $f, g \in AC_{\mathrm{loc}}(I)$ be positive functions on an interval I satisfying $|f'(t)| \leq N_0$ and $|f(t)g'(t)| \leq M_0 g(t)$ a.e. on I for some constants N_0 and M_0. Then for fixed $t \in I$, $0 < \varepsilon < 1/N_0$, and $t \leq \tau \leq t + \varepsilon f(t)$, we have that*

$$(1 - \varepsilon N_0) f(t) \leq f(\tau) \leq (1 + \varepsilon N_0) f(t)$$

and

$$\mathrm{e}^{-M_0/N_0} g(t) \leq g(\tau) \leq \mathrm{e}^{M_0/N_0} g(t).$$

This lemma implies that both positive and negative powers of $f(\tau)$ and $g(\tau)$ are essentially constant for $t \leq \tau \leq t + \varepsilon f(t)$ and fixed t.

In the proofs of several results in this work, we will use unitary transformations. Here, we develop sufficient conditions for a unitary change of dependent and independent variables for a maximal (or minimal or self-adjoint extension of a minimal) operator of the form

$$Mz = \begin{pmatrix} W_1^{-1} & 0 \\ 0 & W_2^{-1} \end{pmatrix} \left\{ \frac{1}{2}[(\Theta z)^{\cdot} + \Theta \dot{z}] + \mathcal{N} z \right\}, \quad x \in X, \tag{15}$$

where

$$\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} \eta_1 & \eta \\ \eta & \eta_2 \end{pmatrix},$$

and $\dot{} = \mathrm{d}/\mathrm{d}x$.

Let $t = f(x)$ where $f: X \to T$ is a strictly increasing (decreasing) $C^{(1)}$-function, and let

$$Uz(t) = \begin{pmatrix} \mu_1(x) & 0 \\ 0 & \mu_2(x) \end{pmatrix} \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \quad \text{for } z = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \in \mathcal{L}^2_W(X),$$

where $\mu_i \in C(X)$, for $i = 1, 2$, are never zero on X.

If the weight functions w_i satisfy on T

$$w_i(t) = \frac{W_i(x)}{\mu_i^2(x)|\dot{f}(x)|} \quad \text{for } i = 1,2, \tag{16}$$

then after some calculations we have for $\tilde{z}, z \in \mathcal{L}^2_W(X)$

$$\begin{aligned}\langle U\tilde{z}, Uz\rangle_{\mathcal{L}^2_w(T)} &= \int_T [Uz(t)]^* \begin{pmatrix} w_1^{-1}(t) & 0 \\ 0 & w_2^{-1}(t) \end{pmatrix} [U\tilde{z}(t)]\,\mathrm{d}t \\ &= \langle \tilde{z}, z\rangle_{\mathcal{L}^2_W(X)}. \end{aligned} \tag{17}$$

Thus, U is a unitary map from $\mathcal{L}^2_W(X)$ onto $\mathcal{L}^2_W(T)$. Let $L = UMU^{-1}$, $y = Uz \in D(L)$ for $z \in D(M)$, and $' = \mathrm{d}/\mathrm{d}t$. Then

$$\begin{aligned} MU^{-1}y(x) = W^{-1}\bigg\{ \frac{1}{2}\{(\Theta\mu^{-1}y)'\dot{f}(x) + \Theta[(\mu^{-1})^{\cdot}y \\ + \mu^{-1}y'\dot{f}(x)]\} + \mathcal{N}\mu^{-1}y\bigg\}, \end{aligned}$$

where

$$W(x) = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix} \quad \text{and} \quad \mu(x) = \begin{pmatrix} \mu_1(x) & 0 \\ 0 & \mu_2(x) \end{pmatrix}.$$

Hence, suppressing independent variables and carrying out the indicated operations, we have that

$$\begin{aligned} Ly &= UMU^{-1}y \\ &= \mu W^{-1}\bigg\{ \frac{1}{2}\{\Theta\mu^{-1}y'\dot{f} + (\Theta\mu^{-1})'y\dot{f} - \Theta\mu^{-1}\dot{\mu}\mu^{-1}y + \Theta\mu^{-1}y'\dot{f}\} \\ &\quad + \mathcal{N}\mu^{-1}y\bigg\} = \mu^2\dot{f}W^{-1}\bigg\{\frac{1}{2}[\tilde{\Theta}y' + (\tilde{\Theta}y)'] + \tilde{\mathcal{N}}y\bigg\}, \end{aligned}$$

where

$$\tilde{\Theta}(t) = (\mu^{-1}\Theta\mu^{-1})(x) \tag{18}$$

and

$$\tilde{\mathcal{N}}(t) = \frac{1}{\dot{f}(x)} \left\{ \frac{1}{2} [\mu^{-1}\dot{\mu}(\mu^{-1}\Theta\mu^{-1}) - (\mu^{-1}\Theta\mu^{-1})\dot{\mu}\mu^{-1}] + \mu^{-1}\mathcal{N}\mu^{-1} \right\}(x). \tag{19}$$

Thus, via Eq. (16)

$$Ly(t) = (\operatorname{sign} \dot{f}) \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \left\{ \frac{1}{2} [(\tilde{\Theta}y)' + \tilde{\Theta}y'] + \tilde{\mathcal{N}}y \right\}, \quad t \in T. \tag{20}$$

Also, for $\tilde{y} = U\tilde{z}, \tilde{z} \in D(M)$ we have

$$\begin{aligned} \langle Ly, L\tilde{y} \rangle_{\mathcal{L}_w^2(T)} &= \langle UMU^{-1}y, UMU^{-1}\tilde{y} \rangle_{\mathcal{L}_w^2(T)} \\ &= \langle UMz, UM\tilde{z} \rangle_{\mathcal{L}_w^2(T)} \\ &= \langle Mz, M\tilde{z} \rangle_{\mathcal{L}_W^2(X)}, \end{aligned} \tag{21}$$

where the last equality follows by Eq. (17). Therefore, if we have the conditions on the weight functions (16) and on $\tilde{\Theta}$ and $\tilde{\mathcal{N}}$, which are given by (18) and (19), then

$$\|z\|_{\mathcal{L}_W^2(X)} = \|y\|_{\mathcal{L}_w^2(T)} \text{ and } \|Mz\|_{\mathcal{L}_W^2(X)} = \|Ly\|_{\mathcal{L}_w^2(T)}$$

via Eqs. (17) and (21).

3 PERTURBATIONS OF T: LIMIT POINT CASE

In this section, we consider perturbations B of a higher-ordered differential operator T. These operators T and B are defined on I by the equations

$$Ty = \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \tag{22}$$

and

$$By = \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \begin{pmatrix} q_1 & q_4 \\ q_3 & q_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{23}$$

where the coefficients q_1, q_2, q_3, and q_4 are assumed to be real, locally Lebesgue integrable functions. Since B is a multiplicative operator, $B_1 = \mathrm{B}_0$. If $q_3 \equiv q_4$, then B_1 is self-adjoint.

We develop necessary and sufficient conditions for the perturbations to be relatively bounded or relatively compact with respect to T. These conditions involve explicit integral averages of the coefficients of B. In Theorem 3.1 we consider the maximal operators T_1 and B_1 associated with the differential operators (22) and (23), respectively. The proof relies heavily upon Theorem 2.7 and Lemma 2.8. Corollaries 3.2 and 3.3 apply Theorem 3.1 to perturbations of an operator of the form (1) with

$$W(x) = \begin{pmatrix} x^\gamma & 0 \\ 0 & x^\gamma \end{pmatrix}, \quad P_0(x) = 0, \quad \text{and} \quad Q_0(x) = \frac{1}{2}\begin{pmatrix} 0 & -x^\alpha \\ x^\alpha & 0 \end{pmatrix}$$

for some constants γ and α. Each proof makes use of a unitary transformation. Corollary 3.2 considers the operators on the interval $[a, \infty)$, $a > 0$, with $\gamma - \alpha \geq -1$ so that the unperturbed operator is limit point at ∞; whereas, Corollary 3.3 considers the operators on the interval $(0, a]$, $a > 0$, with $\gamma - \alpha \leq -1$ so that the perturbed operator is limit point at 0.

THEOREM 3.1 *Let $I = [a, \infty)$ for some $a > 0$, let $q = \left(\sum_{i=1}^4 q_i^2\right)^{1/2}$, and let $w_1 = w_2 = f^{-1}$ where $f \in AC_{\mathrm{loc}}(I)$ is positive and $|f'(t)| \leq N_0$ a.e. on I for some constant N_0. Then the following statements hold:*

(i) *B_1 is T_1-bounded if and only if*

$$\sup_{t \in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau) q^2(\tau)\, \mathrm{d}\tau < \infty, \tag{24}$$

for some $\varepsilon \in (0, 1/2N_0)$. Further, when (24) holds, the relative bound of B_1 with respect to T_1 is zero.

(ii) *B_1 is T_1-compact if and only if*

$$\lim_{t \to \infty} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau) q^2(\tau)\, \mathrm{d}\tau = 0, \tag{25}$$

for some $\varepsilon \in (0, 1/2N_0)$.

Proof By applying Lemma 2.8 with $g \equiv 1$, we know that positive and negative powers of f are essentially constant on intervals of length εf. Thus, we may replace

$$\frac{1}{f(t)} \int_{t}^{t+\varepsilon f(t)} f^2(\tau) q^2(\tau)\, \mathrm{d}\tau$$

with either

$$f(t) \int_{t}^{t+\varepsilon f(t)} q^2(\tau)\, \mathrm{d}\tau \quad \text{or} \quad \int_{t}^{t+\varepsilon f(t)} f(\tau) q^2(\tau)\, \mathrm{d}\tau.$$

(i) *Sufficiency* Let us consider any $y \in D(T_1)$. Then we have, suppressing the independent variable,

$$\|y\|^2 = \int_I f^{-1}[|y_1|^2 + |y_2|^2], \tag{26}$$

$$\|Ty\|^2 = \int_I f\,[|y_1'|^2 + |y_2'|^2], \tag{27}$$

and

$$\begin{aligned} \|By\|^2 = \int_I\ & f\{[q_1^2 + q_3^2]|y_1|^2 + [q_1q_4 + q_2q_3]y_1\bar{y}_2 \\ &+ [q_1q_4 + q_2q_3]\bar{y}_1y_2 + [q_2^2 + q_4^2]|y_2|^2\} \\ \le \int_I\ & f\{[q_1^2 + q_3^2]|y_1|^2 + 2|q_1q_4 + q_2q_3||y_1y_2| + [q_2^2 + q_4^2]|y_2|^2\}. \end{aligned} \tag{28}$$

We make use of the inequalities $2|y_1y_2| \le |y_1|^2 + |y_2|^2$ and $|q_1q_4 + q_2q_3| \le 2q^2$ in inequality (28) to obtain

$$\|By\|^2 \le \int_I 3f(\tau)q^2(\tau)[|y_1(\tau)|^2 + |y_2(\tau)|^2]\, \mathrm{d}\tau. \tag{29}$$

Now, we show that the hypotheses of Theorem 2.7 hold for some $\varepsilon \in (0,1/2N_0)$ with $N=fq^2$, $W=f^{-1}$, and $P=f$. Since positive and negative powers of f are essentially constant on intervals of length εf, we have for some constants C_1, $C_2 > 0$

$$S_1 = \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} f^{-1}(\tau)\,\mathrm{d}\tau\right]\left[\int_t^{t+\varepsilon f(t)} f(\tau)q^2(\tau)\,\mathrm{d}\tau\right]\right\}$$
$$\leq \frac{C_1}{\varepsilon}\sup_{t\in I}\left\{\int_t^{t+\varepsilon f(t)} f(\tau)q^2(\tau)\,\mathrm{d}\tau\right\} \tag{30}$$

and

$$S_2 = \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} f(\tau)\,\mathrm{d}\tau\right]\left[\int_t^{t+\varepsilon f(t)} f(\tau)q^2(\tau)\,\mathrm{d}\tau\right]\right\}$$
$$\leq \frac{C_2}{\varepsilon}\sup_{t\in I}\left\{\int_t^{t+\varepsilon f(t)} f(\tau)q^2(\tau)\,\mathrm{d}\tau]\right\} \tag{31}$$

for some $\varepsilon \in (0, 1/2N_0)$. Inequalities (24), (30), and (31) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, 1/2N_0)$. Therefore (via Theorem 2.7), there exists a constant $C_3 > 0$ such that

$$\int_I f(\tau)q^2(\tau)[|y_1(\tau)|^2 + |y_2(\tau)|^2]\,\mathrm{d}\tau$$
$$\leq C_3\left\{\int_I f^{-1}(\tau)[|y_1(\tau)|^2 + |y_2(\tau)|^2]\,\mathrm{d}\tau\right.$$
$$\left. +\varepsilon^2\int_I f(\tau)[|y_1'(\tau)|^2 + |y_2'(\tau)|^2]\,\mathrm{d}\tau\right\}. \tag{32}$$

By substituting (26), (27), and (29) into (32), we obtain for some constant $C_4 > 0$

$$\|By\|^2 \leq C_4(\|y\|^2 + \varepsilon^2\|Ty\|^2).$$

Thus, $y \in D(B_1)$. Since y is arbitrary, the above inequality implies that B_1 is T_1-bounded and that the relative bound of B_1 with respect to T_1 is zero.

Necessity Let ϕ be a function in $C_0^\infty(\mathbf{R})$ such that $\phi \equiv 1$ on $[0, 1]$ and $\mathrm{supp}(\phi) = [-2, 2]$. Fix $\varepsilon \in (0, 1/2N_0)$. For each $r \geq a$ we define

$$\phi_r(t) = \phi\left(\frac{t-r}{\varepsilon f(r)}\right), \quad \text{for } t \geq a.$$

Then, $\phi_r \equiv 1$ on $[r, r+\varepsilon f(r)]$ and $\mathrm{supp}(\phi_r) = [r - 2\varepsilon f(r),\ r + 2\varepsilon f(r)]$.

For each $r \geq a$ we define

$$\Phi_r(t) = \begin{pmatrix} \phi_r(t) \\ 0 \end{pmatrix} \text{ and } \Psi_r(t) = \begin{pmatrix} 0 \\ \phi_r(t) \end{pmatrix}, \text{ for } t \geq a. \tag{33}$$

Via Lemma 2.8, a change of variables, and the continuity of ϕ, there exist constants $C_1, C_2 > 0$ such that for each $r \geq a$

$$\|\Phi_r\|^2 = \int_I f^{-1}(t)\phi_r^2(t)\,\mathrm{d}t \leq C_1\varepsilon \int_{-2}^{2} \phi^2(u)\,\mathrm{d}u \leq C_2. \tag{34}$$

Similarly, for each $r \geq a$

$$\|\Psi_r\|^2 \leq C_2. \tag{35}$$

Via Lemma 2.8, a change of variables, and the continuity of ϕ', there exist constants $C_3, C_4 > 0$ such that for each $r \geq a$

$$\|T\Phi_r\|^2 = \int_I f(t)\left[\frac{\mathrm{d}}{\mathrm{d}t}\phi_r(t)\right]^2 \mathrm{d}t \leq C_3\varepsilon^{-1} \int_{-2}^{2} [\phi'(u)]^2\,\mathrm{d}u \leq C_4. \tag{36}$$

Similarly, for $r \geq a$

$$\|T\Psi_r\|^2 \leq C_4. \tag{37}$$

Since $\phi_r \equiv 1$ on $[r, r + \varepsilon f(r)]$ and $\mathrm{supp}(\phi_r) = [r - 2\varepsilon f(r), r + 2\varepsilon f(r)]$,

$$\begin{aligned}\int_r^{r+\varepsilon f(r)} f(t)[q_1^2(t) + q_3^2(t)]\,\mathrm{d}t &\leq \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} f(t)[q_1^2(t) + q_3^2(t)]\phi_r^2(t)\,\mathrm{d}t \\ &= \|B\Phi_r\|^2. \end{aligned} \tag{38}$$

Similarly,

$$\begin{aligned}\int_r^{r+\varepsilon f(r)} f(t)[q_2^2(t) + q_4^2(t)]\,\mathrm{d}t &\leq \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} f(t)[q_2^2(t) + q_4^2(t)]\phi_r^2(t)\,\mathrm{d}t \\ &= \|B\Psi_r\|^2. \end{aligned} \tag{39}$$

Thus, via the T_1-boundedness of B_1 and inequalities (34)–(39), there exists a constant $C_5 > 0$ such that

$$\int_r^{r+\varepsilon f(r)} f(t)q^2(t)\,\mathrm{d}t \le C_5.$$

Since the right-hand side of the above inequality is independent of r, inequality (24) holds.

Note that the proof of necessity shows that (24) holds for every $\varepsilon \in (0, 1/2N_0)$.

Sufficiency By the previous argument B_1 is T_1-bounded. Thus, $D(\mathrm{T}_1) \subseteq D(B_1)$. For every positive integer $N > a$, define B_N on $D(T_1)$ by

$$B_N y = \begin{cases} By & \text{on } [a, N], \\ 0 & \text{on } (N, \infty). \end{cases}$$

In order to simplify the proof, we break the argument into two claims.

CLAIM 3.1.1 *$B_N \to B_1$ in the space of bounded operators on $D(T_1)$ with the T_1-norm.*

Proof of Claim 3.1.1 By a note in Section 2, we know that T_1 is closed since it is a formally self-adjoint maximal operator of the form (1). Therefore, $D(T_1)$ is complete under the T_1-norm.

Supressing the independent variable and using the inequalities $2|y_1y_2| \le |y_1|^2 + |y_2|^2$ and $|q_1q_4 + q_2q_3| \le 2q^2$, we have for $y \in D(T_1)$

$$\begin{aligned} \|By - B_Ny\|^2 &= \int_N^\infty f\{[q_1^2 + q_3^2]|y_1|^2 + [q_1q_4 + q_2q_3]y_1\bar{y}_2 \\ &\quad + [q_1q_4 + q_2q_3]\bar{y}_1y_2 + [q_2^2 + q_4^2]|y_2|^2\} \\ &\le \int_N^\infty f\{[q_1^2 + q_3^2]|y_1|^2 + 2|q_1q_4 + q_2q_3||y_1y_2| \\ &\quad + [q_2^2 + q_4^2]|y_2|^2\} \\ &\le 3\int_N^\infty fq^2[|y_1|^2 + |y_2|^2]. \end{aligned} \tag{40}$$

We apply the sufficiency argument in part (i) with $I_N = [N, \infty)$ to the last inequality in (40). Via Theorem 2.7 and inequalities (30) and (31), there exist a constant $k_1 > 0$ such that

$$\|By - B_N y\|^2 \leq \frac{k_1}{\varepsilon} \sup_{t\in I_N}\left\{\int_t^{t+\varepsilon f(t)} fq^2\right\}\left\{\int_{I_N} f^{-1}[|y_1|^2 + |y_2|^2] + \varepsilon^2 \int_{I_N} f[|y_1'|^2 + |y_2'|^2]\right\}. \quad (41)$$

Via inequality (41) there exists a constant $k_2 > 0$ such that

$$\|By - B_N y\|^2 \leq k_2 \sup_{t\in I_N}\left\{\int_t^{t+\varepsilon f(t)} f(\tau)q^2(\tau)\mathrm{d}\tau\right\} \cdot \{\|y\| + \|Ty\|\}^2. \quad (42)$$

Therefore, inequality (42) implies that for $y \neq 0$

$$\frac{\|By - B_N y\|}{\|y\|_{T_1}} \leq k_2^{1/2}\left(\sup_{t\in I_N}\left\{\int_t^{t+\varepsilon f(t)} f(\tau)q^2(\tau)\,\mathrm{d}\tau\right\}\right)^{1/2}. \quad (43)$$

Since $I_N = [N, \infty)$, Eq. (25) implies that $\sup_{t\in I_N}\{\int_t^{t+\varepsilon f(t)} f(\tau)q^2(\tau)\,\mathrm{d}\tau\} \to 0$ as $N \to \infty$ so that, via inequality (43), we have

$$\|B - B_N\| \to 0 \text{ as } N \to \infty.$$

CLAIM 3.1.2 *Each B_N is T_1-compact.*

Proof of Claim 3.1.2 Let $\{y_n\}_{n=1}^\infty \subset D(T_1)$ be a T_1-bounded sequence where $y_n = \binom{y_{n,1}}{y_{n,2}}$ for each n. We need to show that for each N the sequence $\{B_N y_n\}_{n=1}^\infty$ has a convergent subsequence. We make use of the Arzela–Ascoli Theorem.

Let us consider the sequence of the first component functions $\{y_{n,1}\}_{n=1}^\infty$. Using the following well-known inequality, we show that this sequence is uniformly bounded on $[a, N]$.

If $g \in AC[a,b]$ and $h(t) > 0$ is Lebesgue measurable with $\int_a^b h^{-1}(t)\,\mathrm{d}t < \infty$, then for $t \in [a,b]$

$$|g(t)| \leq \left(\frac{\int_a^b h^{-1}(\tau)|g(\tau)|^2\mathrm{d}\tau}{\int_a^b h^{-1}(\tau)\,\mathrm{d}\tau}\right)^{1/2} + \left(\int_a^b h^{-1}(\tau)\,\mathrm{d}\tau\right)^{1/2}\left(\int_a^b h(\tau)|g'(\tau)|^2\,\mathrm{d}\tau\right)^{1/2}. \quad (44)$$

Since f is a positive, continuous function on $[a, \infty)$, f^{-1} is bounded above and below on $[a, N]$. This fact along with the above inequality implies that there exists a constant $k_3 > 0$ such that for $t \in [a, N]$

$$\begin{aligned} |y_{n,1}(t)| &\le k_3\left\{\left(\int_a^N f^{-1}(\tau)|y_{n,1}(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} + \left(\int_a^N f(\tau)|y'_{n,1}(\tau)|^2\,\mathrm{d}\tau\right)^{1/2}\right\} \\ &\le k_3(\|y_n\| + \|Ty_n\|) \end{aligned}$$

for each n. Since $\{y_n\}_{n=0}^{\infty}$ is a T_1-bounded sequence, the above inequality implies that the sequence $\{y_{n,1}\}_{n=1}^{\infty}$ is uniformly bounded on $[a, N]$.

Now, we show that $\{y_{n,1}\}_{n=1}^{\infty}$ is equicontinuous on $[a, N]$. Via the Cauchy–Schwarz inequality and the boundedness of f^{-1} on $[a, N]$, there exists a constant $k_4 > 0$ such that for $s, t \in [a, N]$

$$\begin{aligned} |y_{n,1}(s) - y_{n,1}(t)| &= \left|\int_t^s y'_{n,1}(\tau)\,\mathrm{d}\tau\right| \\ &\le \left|\int_t^s f^{-1}(\tau)\,\mathrm{d}\tau\right|^{1/2} \left|\int_t^s f(\tau)|y'_{n,1}(\tau)|^2\,\mathrm{d}\tau\right|^{1/2} \\ &\le k_4|s-t|^{1/2}\left|\int_t^s f(\tau)|y'_{n,1}(\tau)|^2\,\mathrm{d}\tau\right|^{1/2} \\ &\le k_4|s-t|^{1/2}\|y_n\|_{T_1}. \end{aligned}$$

Since $\{y_n\}_{n=1}^{\infty}$ is a T_1-bounded sequence, the above inequality implies that the sequence $\{y_{n,1}\}_{n=0}^{\infty}$ is equicontinuous on $[a, N]$.

Therefore, via the Arzela–Ascoli Theorem we conclude that there exists a subsequence $\{y_{n_k,1}\}_{k=1}^{\infty}$ of $\{y_{n,1}\}_{n=0}^{\infty}$ which converges uniformly on $[a, N]$. By repeating the above arguments on the subsequence $\{y_{n_k,2}\}_{k=1}^{\infty}$ of $\{y_{n,2}\}_{n=1}^{\infty}$, we conclude that there exists a subsequence of $\{y_{n_k,2}\}_{k=1}^{\infty}$ which converges uniformly on $[a, N]$. Hence, there exists a subsequence of $\{y_n\}_{n=1}^{\infty}$ which converges uniformly on $[a, N]$. WLOG, we assume the sequence $\{y_n\}_{n=1}^{\infty}$ converges uniformly on $[a, N]$.

Now, using an argument similar to the one in (40), we have for each N

$$\begin{aligned} \|B_N y_n - B_N y_m\|^2 &\le 3\int_a^N f(\tau)q^2(\tau)\|y_n - y_m\|_2^2\,\mathrm{d}\tau \\ &\le 3\sup_{a\le t\le N}\|y_n - y_m\|_2^2\left\{\int_a^N f(\tau)q^2(\tau)\,\mathrm{d}\tau\right\}, \end{aligned}$$

where

$$\|y_n - y_m\|_2^2 = [y_{n,1}(t) - y_{m,1}(t)]^2 + [y_{n,2}(t) - y_{m,2}(t)]^2.$$

Since the integral on the right-hand side is finite, there exists a constant $k_5 > 0$ such that

$$\|B_N y_n - B_N y_m\|^2 \le k_5 \sup_{a \le t \le N} \|y_n - y_m\|_2^2.$$

Since $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the uniform norm, the above inequality implies that the sequence $\{B_N y_n\}_{n=1}^{\infty}$ is Cauchy in $\mathcal{L}_w^2(I)$ for each N. Therefore, $\{B_N y_n\}_{n=1}^{\infty}$ converges for each N as $n \to \infty$ since $\mathcal{L}_w^2(I)$ is complete. Hence, each B_N is T_1-compact.

Since B_1 is the uniform limit of T_1-compact operators, B_1 is T_1-compact.

Necessity We use contradiction arguments to show that (25) must hold.

Suppose that for some $\varepsilon \in (0, 1/2N_0)$ there exists a $\rho > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers such that $r_n \to \infty$ as $n \to \infty$ and for each n

$$\int_{r_n}^{r_n + \varepsilon f(r_n)} f(t)[q_1^2(t) + q_3^2(t)]\,\mathrm{d}t \ge \rho.$$

Let $\{\Phi_r\}_{r \ge a}$ be defined by (33). Then via inequalities (34) and (36) there exist constants $C_2, C_4 > 0$ such that for each n

$$\|\Phi_{r_n}\|_{T_1}^2 = (\|\Phi_{r_n}\| + \|T\Phi_{r_n}\|)^2 \le (C_2^{1/2} + C_4^{1/2})^2.$$

Thus, $\{\Phi_{r_n}\}_{n=1}^{\infty}$ is a T_1-bounded sequence. Since B_1 is T_1-compact, $\{B\Phi_{r_n}\}_{n=1}^{\infty}$ has a convergent subsequence. WLOG, we assume $\{B\Phi_{r_n}\}$ converges, say to some y_0. Via the properties of ϕ_{r_n} we have for each n

$$\begin{aligned}
\rho &\le \int_{r_n}^{r_n + \varepsilon f(r_n)} f(t)[q_1^2(t) + q_3^2(t)]\,\mathrm{d}t \\
&\le \int_{r_n - 2\varepsilon f(r_n)}^{r_n + 2\varepsilon f(r_n)} f(t)[q_1^2(t) + q_3^2(t)]\phi_{r_n}^2(t)\,\mathrm{d}t \\
&= \|B\Phi_{r_n}\|^2.
\end{aligned}$$

Notice that a contradiction is reached if we show that $y_0 = 0$ a.e. in $[a, \infty)$. Let J_0 be a finite subinterval of $[a, \infty)$. Since $r_n \to \infty$ as $n \to \infty$ and $\operatorname{supp}(\phi_{r_n}) = [r_n - 2\varepsilon f(r_n), r_n + 2\varepsilon f(r_n)]$, we conclude that $\phi_{r_n} \equiv 0$ on J_0 for sufficiently large n. Hence, $\Phi_{r_n} \equiv 0$ and $B\Phi_{r_n} \equiv 0$ on J_0 for sufficiently large n. For such n

$$\|y_0\|_{J_0} = \|y_0 - B\Phi_{r_n}\|_{J_0} \leq \|y_0 - B\Phi_{r_n}\|.$$

Since $B\Phi_{r_n} \to y_0$ as $n \to \infty$, and the left-hand side of the above inequality is independent of n, we have $\|y_0\|_{J_0} = 0$. Thus, $y_0 = 0$ a.e. in $[a, \infty)$ since the interval J_0 is arbitrary. This contradiction implies that

$$\lim_{t \to \infty} \int_t^{t+\varepsilon f(t)} f(\tau)[q_1^2(\tau) + q_3^2(\tau)]\, \mathrm{d}\tau = 0, \tag{45}$$

for some $\varepsilon \in (0, 1/2N_0)$.

Moreover, by repeating the above argument with q_1 and q_3 replaced by q_2 and q_4, respectively, and Φ_r replaced by Ψ_r (as defined by (33)), we conclude that

$$\lim_{t \to \infty} \int_t^{t+\varepsilon f(t)} f(\tau)[q_2^2(\tau) + q_4^2(\tau)]\, \mathrm{d}\tau = 0,$$

for some $\varepsilon \in (0, 1/2N_0)$. This equation along with Eq. (45) implies (25) holds.

Note that the proof of necessity shows that (25) holds for every $\varepsilon \in (0, 1/2N_0)$.

Next, we prove two corollaries of Theorem 3.1. We consider the maximal operators associated with the following differential expressions on the interval I:

$$\begin{aligned}\hat{T}z(x) = \frac{1}{2}\begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix} \Bigg\{ \left[\begin{pmatrix} 0 & -x^{\alpha} \\ x^{\alpha} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right]^{\bullet} \\ + \begin{pmatrix} 0 & -x^{\alpha} \\ x^{\alpha} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^{\bullet} \Bigg\}\end{aligned} \tag{46}$$

and

$$\hat{B}z(x) = \begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix} \begin{pmatrix} b_1 & b_4 \\ b_3 & b_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{47}$$

where the coefficients b_1, b_2, b_3, and b_4 are assumed to be real, locally Lebesgue integrable functions and $^{\bullet} = \mathrm{d}/\mathrm{d}x$.

COROLLARY 3.2 *Let* $I = [a, \infty)$ *for some* $a > 0$, *let* $b = \left(\sum_{i=1}^{4} b_i^2\right)^{1/2}$, *and let* $\gamma - \alpha \geq -1$ *and* $N_0 = |\alpha - \gamma| a^{\alpha-\gamma-1}$. *Then the following statements hold*:

(i) $\hat{B}_1$ *is* $\hat{T}_1$*-bounded if and only if*

$$\sup_{x \in I} x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{-2\gamma} b^2(u)\, \mathrm{d}u < \infty, \tag{48}$$

for some $\varepsilon \in (0, 1/2N_0)$. *Further, when* (48) *holds, the relative bound of* $\hat{B}_1$ *with respect to* $\hat{T}_1$ *is zero.*

(ii) $\hat{B}_1$ *is* $\hat{T}_1$*-compact if and only if*

$$\lim_{x \to \infty} x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{-2\gamma} b^2(u)\, \mathrm{d}u = 0, \tag{49}$$

for some $\varepsilon \in (0, 1/2N_0)$.

Proof We begin by applying an argument in Section 2 to transform the differential expressions unitarily. Notice that $\hat{T}$ is of the form (15) with $W_1(x) = W_2(x) = x^\gamma$, $\Theta = \left(\begin{smallmatrix} 0 & -x^\alpha \\ x^\alpha & 0 \end{smallmatrix}\right)$, and $\mathcal{N} = 0$. Let $\mu_1(x) = \mu_2(x) = x^{\alpha/2}$ so that $y(x) = x^{\alpha/2} z(x)$ and, via Eqs. (16), (18), (19), and (20),

$$Ty(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} \tag{50}$$

and

$$By(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix} \begin{pmatrix} x^{-\alpha} b_1 & x^{-\alpha} b_4 \\ x^{-\alpha} b_3 & x^{-\alpha} b_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{51}$$

Now, we apply Theorem 3.1 (with $f(x) = x^{\alpha-\gamma}$ and $q(x) = x^{-\alpha} b(x)$) to the maximal operators T_1 and B_1 associated with the transformed differential expressions (50) and (51), respectively. Notice that for $x \in I$, $|\dot{f}(x)| = |\alpha - \gamma| x^{\alpha-\gamma-1} \leq |\alpha - \gamma| a^{\alpha-\gamma-1}$ since $\alpha - \gamma - 1 \leq 0$. Thus, the sharpest constant N_0 in Theorem 3.1 is given by $N_0 = |\alpha - \gamma| a^{\alpha-\gamma-1}$.

Hence, B_1 is T_1-bounded if and only if

$$\sup_{x \in I} x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{2(\alpha-\gamma)} [u^{-\alpha}\, b(u)]^2\, \mathrm{d}u < \infty,$$

for some $\varepsilon \in (0, 1/2N_0)$, i.e., B_1 is T_1-bounded if and only if

$$\sup_{x \in I} x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{-2\gamma} b^2(u)\,du < \infty,$$

for some $\varepsilon \in (0, 1/2N_0)$. Since the transformation is unitary, inequality (48) holds if and only if $\hat{B}_1$ is $\hat{T}_1$-bounded.

Similarly, B_1 is T_1-compact if and only if

$$\lim_{x \to \infty} x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{-2\gamma} b^2(u)\,du = 0,$$

for some $\varepsilon \in (0, 1/2N_0)$. Thus, via the unitary transformation, Eq. (49) holds if and only in $\hat{B}_1$ is $\hat{T}_1$-compact.

COROLLARY 3.3 *Let $I = (0, \frac{1}{a}]$ for some $a > 0$, let $b = (\sum_{i=1}^4 b_i^2)^{1/2}$, and let $\gamma - \alpha \leq -1$ and $N_0 = |\gamma - \alpha + 2| a^{\gamma-\alpha+1}$. Then the following statements hold*:

(i) *$\hat{B}_1$ is $\hat{T}_1$-bounded if and only if*

$$\sup_{x \in I} x^{\gamma-\alpha+2} \int_{x-\varepsilon' x^{\alpha-\gamma}}^{x} u^{-2\gamma-2} b^2(u)\,du < \infty, \tag{52}$$

for some sufficiently small ε'. Further, when (52) holds, the relative bound of $\hat{B}_1$ with respect to $\hat{T}_1$ is zero.

(ii) *$\hat{B}_1$ is $\hat{T}_1$-compact if and only if*

$$\lim_{x \to \infty} x^{\gamma-\alpha+2} \int_{x-\varepsilon' x^{\alpha-\gamma}}^{x} u^{-2\gamma-2} b^2(u)\,du = 0, \tag{53}$$

for some sufficiently small ε'.

Proof We prove this result by using a unitary transformation to transform the singularity at 0 to a singularity at ∞ and then applying Theorem 3.1 to the new operators.

Again, we use the argument in Section 2 to transform the operators T and B unitarily. Let $\mu_1(x) = \mu_2(x) = x^{\alpha/2}$ so that $y(t) = x^{\alpha/2} z(x)$, $t = 1/x$ and, via Eqs. (16), (18), (19), and (20),

$$-Ty(t) = \begin{pmatrix} t^{\gamma-\alpha+2} & 0 \\ 0 & t^{\gamma-\alpha+2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \tag{54}$$

and

$$By(t) = \begin{pmatrix} t^{\gamma-\alpha+2} & 0 \\ 0 & t^{\gamma-\alpha+2} \end{pmatrix} \begin{pmatrix} t^{\alpha-2}b_1 & t^{\alpha-2}b_4 \\ t^{\alpha-2}b_3 & t^{\alpha-2}b_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{55}$$

Now, we apply Theorem 3.1 (with $f(t) = t^{\gamma-\alpha+2}$ and $q(t) = x^{2-\alpha}b(x)$, $t = 1/x$) to the maximal operators T_1 and B_1 associated with the transformed differential expressions (54) and (55), respectively. Notice that for $t \in [a, \infty)$, $|f'(t)| = |\gamma - \alpha + 2| t^{\gamma-\alpha+1} \leq |\gamma - \alpha + 2| a^{\gamma-\alpha+1}$ since $\gamma - \alpha + 1 \leq 0$. Thus, the sharpest constant N_0 in Theorem 3.1 is given by $N_0 = |\gamma - \alpha + 2| a^{\gamma-\alpha+1}$.

Hence, B_1 is T_1-bounded if and only if

$$\sup_{a \leq t \leq \infty} t^{\alpha-\gamma-2} \int_t^{t+\varepsilon t^{\gamma-\alpha+2}} \tau^{2(\gamma-\alpha+2)} \left[\tau^{\alpha-2}\, b\left(\frac{1}{\tau}\right)\right]^2 \mathrm{d}\tau < \infty,$$

for some $\varepsilon \in (0, 1/2N_0)$, i.e., B_1 is T_1-bounded if and only if

$$\sup_{a \leq t \leq \infty} t^{\alpha-\gamma-2} \int_t^{t+\varepsilon t^{\gamma-\alpha+2}} \tau^{2\gamma}\, b^2\left(\frac{1}{\tau}\right) \mathrm{d}\tau < \infty, \tag{56}$$

for some $\varepsilon \in (0, 1/2N_0)$.

Via a change of variables and some analysis (See [6, pp. 34–35]), we can show that (56) is equivalent to (52). Since the transformation is unitary, inequality (52) holds if and only if $\hat{B}_1$ is $\hat{T}_1$-bounded. Using a similar argument, we can show that Eq. (53) holds if and only if $\hat{B}_1$ is $\hat{T}_1$-compact.

The following theorem is a well-known result, e.g., see [1, p. 59] and [11, pp. 52, 53].

THEOREM 3.4 *Suppose A, C and D are linear operators such that D is C-bounded with relative bound less than one.*

- (i) *If A is C-bounded, then A is $(C + D)$-bounded. Moreover, if the relative bound of A with respect to C is zero, then the relative bound of A with respect to $(C + D)$ is zero.*
- (ii) *If A is C-compact, then A is $(C + D)$-compact.*

Theorem 3.4 may be applied to successive perturbations B and C of T. For example, if $-\hat{T}$ is given by (46) with $\alpha = \gamma = 2$ and $\hat{B}$ and $\hat{C}$

are given by

$$\hat{B}z(x) = \frac{1}{x^2}\begin{pmatrix} -x^2 & kx \\ kx & x^2 \end{pmatrix}\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \quad \text{and}$$
$$\hat{C}z(x) = \frac{1}{x^2}\begin{pmatrix} -\delta x & 0 \\ 0 & -\delta x \end{pmatrix}\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$$

for $0 < x < \infty$, then $\hat{T} + \hat{B} + \hat{C}$ is the energy operator of the relativistic hydrogen-like atom [10, pp. 218–21]. The δ-terms represent the Coulomb field. By an argument of [7, pp. 169, 170], it follows that $(\hat{T} + \hat{B})_1 = \hat{T}_1 + \hat{B}_1$ so that Theorem 3.4 combined with Corollary 3.2 yields that towards infinity, i.e., on $[a, \infty), a > 0$, $\hat{C}$ is a relatively compact perturbation of $(\hat{T} + \hat{B})_1$.

We conclude this chapter with an argument to show that the operator T of Theorem 3.1 is limit point at ∞. Since T is a formally self-adjoint operator of the form (1), T is limit point at ∞ iff $d_+(T_0) + \mathrm{d}_-(T_0) = 2$. Hence, T is limit point at ∞ iff $d_+(T_0) = \mathrm{d}_-(T_0) = 1$ since the coefficients of T are real. In order to determine $d_+(T_0)$ and $d_-(T_0)$, it is enough to consider dim $N(T_1)$ (see [3, Theorem 9.11.2]). Note that inequalities (4) and (5) imply that the deficiency indices are either one or two since $d_+(T_0) = d_-(T_0)$. Suppose that $d_+(T_0) = d_-(T_0) = 2$. Then every solution to $Ty = 0$ is a linear combination of

$$Y_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad Y_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and is in $\mathcal{L}^2_w(I)$. Let Y_3 be a nontrivial solution of $Ty = 0$. Then for some constant $C > 0$

$$\begin{aligned} \|Y_3\|^2 &= \int_a^\infty Y_3^*(t)\begin{pmatrix} f^{-1}(t) & 0 \\ 0 & f^{-1}(t) \end{pmatrix} Y_3(t)\,\mathrm{d}t \\ &= C\int_a^\infty f^{-1}(t)\,\mathrm{d}t. \end{aligned}$$

Via the hypothesis on f, we conclude that $f(t) \le N_0 t + b$ for some constant b. Hence,

$$\int_a^\infty f^{-1}(t)\,\mathrm{d}t \ge \int_a^\infty (N_0 t + b)^{-1}\,\mathrm{d}t = \infty$$

so that $Y_3 \notin D(T_1)$. Since Y_3 is arbitrary, there exist no nontrivial $\mathcal{L}^2_w(I)$-solutions to $Ty = 0$, i.e., $d_+(T_0) = d_-(T_0) \neq 2$. This contradiction implies that the deficiency indices are one. Therefore, T is limit point at ∞.

4 PERTURBATIONS OF T: LIMIT CIRCLE CASE

In this section, we consider perturbations $\hat{B}$ of the higher-ordered differential operator $\hat{T}$, where $\hat{T}$ and $\hat{B}$ are defined by (46) and (47), respectively. In Theorem 4.1 we consider the minimal operators $\hat{T}_0$ and $\hat{B}_0$ associated with the differential expressions (46) and (47), respectively, on the interval $[a, \infty)$, $a > 0$, with $\gamma - \alpha < -1$. The proof follows a similar argument as that of Theorem 3.1 and applies the Hardy inequality:

$$\int_a^b t^{\beta} |y(t)|^2 \mathrm{d}t \leq \frac{4}{(\beta+1)^2} \int_a^b t^{\beta+2} |y'(t)|^2 \mathrm{d}t, \tag{57}$$

where $-\infty < a < b < \infty$, $\beta \neq -1$, and $y(a) = 0 = y(b)$.

As an application of Theorem 4.1, Theorem 4.2 deals with the maximal operators $\hat{T}_1$ and $\hat{B}_1$ associated with the differential expressions (46) and (47), respectively, on the interval $[a, \infty)$, $a \geq 1$, with $\gamma - \alpha < -1$. In this section we let $' = \mathrm{d}/\mathrm{d}x$.

THEOREM 4.1 *Let* $I = [a, \infty)$ *for some* $a > 0$, $b = (\sum_{i=1}^4 b_i^2)^{1/2}$, *and* $\gamma - \alpha < -1$. *Then*

(i) $\hat{B}_0$ *is* $\hat{T}_0$*-bounded if and only if*

$$\sup_{x \in I} \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b^2(u)\, \mathrm{d}u < \infty, \tag{58}$$

for some $\varepsilon = (0, \frac{1}{2})$

(ii) $\hat{B}_0$ *is* $\hat{T}_0$*-compact if and only if*

$$\lim_{x \to \infty} \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b^2(u)\, \mathrm{d}u = 0, \tag{59}$$

for some $\varepsilon = (0, \frac{1}{2})$.

Proof (i) *Sufficiency* As in Corollary 3.2, we make a unitary transformation. Thus, we can consider $\hat{T}$ and $\hat{B}$ to be of the form (50) and (51), respectively. We begin by showing that $\hat{B}_0'$ is $\hat{T}_0'$-bounded if inequality (58) holds. Let us consider $y \in D(\hat{T}_0')$. Since y has compact support in the interior of I, there exists a $b < \infty$ such that the support of y_1 and the support of y_2 are contained in $[a, b]$. Then we have, suppressing the independent variable,

$$\|y\|^2 = \int_a^b u^{\gamma-\alpha}[|y_1|^2 + |y_2|^2], \tag{60}$$

$$\|\hat{T}y\|^2 = \int_a^b u^{\alpha-\gamma}[|y_1'|^2 + |y_2'|^2], \tag{61}$$

and

$$\begin{aligned}\|\hat{B}y\|^2 &= \int_a^b u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|y_1|^2 + [b_1b_4 + b_2b_3]y_1\bar{y}_2 \\ &\qquad + [b_1b_4 + b_2b_3]\bar{y}_1y_2 + [b_2^2 + b_4^2]|y_2|^2\} \\ &\le \int_a^b u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|y_1|^2 + 2|b_1b_4 + b_2b_3||y_1y_2| \\ &\qquad + [b_2^2 + b_4^2]|y_2|^2\}. \end{aligned} \tag{62}$$

We make use of the inequalities $2|y_1y_2| \le |y_1|^2 + |y_2|^2$ and $|b_1b_4 + b_2b_3| \le 2b^2$ in inequality (62) to obtain

$$\|\hat{B}y\|^2 \le 3\int_a^b u^{-\gamma-\alpha}b^2(u)[|y_1(u)|^2 + |y_2(u)|^2]\,\mathrm{d}u. \tag{63}$$

Now, we show that the hypotheses of Theorem 2.7 hold for some $\varepsilon \in (0, \frac{1}{2})$ with $N = x^{-\gamma-\alpha}b^2$, $W = x^{\alpha-\gamma-2}$, $P = x^{\alpha-\gamma}$, and $f = x$. By applying Lemma 2.8 with $g \equiv 1$ and $f = x$, we know that positive and negative powers of x are essentially constant on intervals of length εx. Thus, we have for some constants $C_1, C_2 > 0$

$$\begin{aligned} S_1 &= \sup_{x\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_x^{x+\varepsilon x} u^{\gamma-\alpha}\,\mathrm{d}u\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}b^2(u)\,\mathrm{d}u\right]\right\} \\ &\le \frac{C_1}{\varepsilon}\sup_{x\in I}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u\right\} \end{aligned} \tag{64}$$

and

$$S_2 = \sup_{x\in I}\left\{\frac{1}{\varepsilon^2 x^2}\left[\int_x^{x+\varepsilon x} u^{\gamma-\alpha+2}\,\mathrm{d}u\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}b^2(u)\,\mathrm{d}u\right]\right\}$$
$$\le \frac{C_2}{\varepsilon}\sup_{x\in I}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u]\right\} \tag{65}$$

for some $\varepsilon \in \left(0, \frac{1}{2}\right)$. Inequalities (58), (64), and (65) give us $S_1, S_2 < \infty$ for some $\varepsilon \in \left(0, \frac{1}{2}\right)$. Therefore (via Theorem 2.7), there exists a constant $C_3 > 0$ such that

$$\int_a^b u^{-\gamma-\alpha}b^2(u)[|y_1(u)|^2 + |y_2(u)|^2]\,\mathrm{d}u$$
$$\le C_3\left\{\int_a^b u^{\alpha-\gamma-2}[|y_1(u)|^2 + |y_2(u)|^2]\,\mathrm{d}u\right.$$
$$\left. +\varepsilon^2\int_a^b u^{\alpha-\gamma}[|y_1'(u)|^2 + |y_2'(u)|^2]\,\mathrm{d}u\right\}. \tag{66}$$

Applying the Hardy inequality (57) to the first integral on the right-hand side of the above inequality gives

$$\int_a^b u^{-\gamma-\alpha}b^2(u)[|y_1(u)|^2 + |y_2(u)|^2]\,\mathrm{d}u$$
$$\le C_3\left\{\frac{4}{(\alpha-\gamma-1)}\int_a^b u^{\alpha-\gamma}[|y_1'(u)|^2 + |y_2'(u)|^2]\,\mathrm{d}u\right.$$
$$\left. +\varepsilon^2\int_a^b u^{\alpha-\gamma}[|y_1'(u)|^2 + |y_2'(u)|^2]\,\mathrm{d}u\right\}.$$

By substituting (60), (61), and (62) into the above inequality, we obtain for some constant $C_4 > 0$

$$\|\hat{B}y\|^2 \le C_4\|\hat{T}y\|^2 \le C_4(\|y\| + \|\hat{T}y\|)^2. \tag{67}$$

Thus, $y \in D(\hat{B}_0')$. Since y is arbitrary, the inequality above implies that $\hat{B}_0'$ is $\hat{T}_0'$-bounded. Via Theorem 2.2, $\hat{B}_0$ is $\hat{T}_0$-bounded.

Necessity Fix $\varepsilon \in (0, \frac{1}{2})$. For each $r \ge a$ define Φ_r and Ψ_r to be the vector-valued functions with compact support given by (33), where $f(r) = r$. Then via Lemma 2.8, a change of variables, and the continuity of ϕ, there exist constants $C_1, C_2 > 0$ such that for each $r \geq a$

$$\|\Phi_r\|^2 = \int_I x^{\gamma-\alpha}\phi_r^2(x)\,\mathrm{d}x \leq \ C_1\varepsilon r^{\gamma-\alpha}\int_{-2}^{2}\phi^2(u)\,\mathrm{d}u \leq C_2 r^{\gamma-\alpha+1}. \quad (68)$$

Similarly, for each $r \geq a$

$$\|\Psi_r\|^2 \leq C_2 r^{\gamma-\alpha+1}. \quad (69)$$

Via Lemma 2.8, a change of variables, and the continuity of ϕ', there exist constants $C_3, C_4 > 0$ such that for each $r \geq a$

$$\begin{aligned} \|\hat{T}\Phi_r\|^2 &= \int_I x^{\alpha-\gamma}\left[\frac{\mathrm{d}}{\mathrm{d}x}\phi_r(x)\right]^2 \mathrm{d}x \\ &\leq C_3\varepsilon^{-1}r^{\alpha-\gamma-1}\int_{-2}^{2}[\phi'(u)]^2\,\mathrm{d}u \leq C_4 r^{\alpha-\gamma-1}. \end{aligned} \quad (70)$$

Similarly, for each $r \geq a$

$$\|\hat{T}\Phi_r\|^2 \leq C_4 r^{\alpha-\gamma-1}. \quad (71)$$

Since $\phi_r \equiv 1$ on $[r, r+\varepsilon r]$ and $\mathrm{supp}(\phi_r) = [r-2\varepsilon r, r+2\varepsilon r]$,

$$\begin{aligned} &\int_r^{r+\varepsilon r} x^{-\gamma-\alpha}[b_1^2(x) + b_3^2(x)]\,\mathrm{d}x \\ &\quad\leq \int_{r-2\varepsilon r}^{r+2\varepsilon r} x^{-\gamma-\alpha}[b_1^2(x) + b_3^2(x)]\phi_r^2(x)\,\mathrm{d}x = \|\hat{B}\Phi_r\|^2. \end{aligned} \quad (72)$$

Similarly,

$$\begin{aligned} &\int_r^{r+\varepsilon r} x^{-\gamma-\alpha}[b_2^2(x) + b_4^2(x)]\,\mathrm{d}x \\ &\quad\leq \int_{r-2\varepsilon r}^{r+2\varepsilon r} x^{-\gamma-\alpha}[b_2^2(x) + b_4^2(x)]\phi_r^2(x)\,\mathrm{d}x = \|\hat{B}\Psi_r\|^2. \end{aligned} \quad (73)$$

Thus, via the $\hat{T}_0$-boundedness of $\hat{B}_0$ and inequalities (68)–(73), there exists a constant $C_5 > 0$ such that

$$\int_r^{r+\varepsilon r} x^{-\gamma-\alpha}b^2(x)\,\mathrm{d}x \leq C_5(r^{\gamma-\alpha+1} + r^{\alpha-\gamma-1}).$$

After multiplying the above inequality by $r^{\gamma-\alpha+1}$, we apply Lemma 2.8 to the left-hand side and obtain a constant $C_6 > 0$ such that

$$\frac{1}{r}\int_r^{r+\varepsilon r} x^{2-2\alpha} b^2(x)\,\mathrm{d}x \le C_6(r^{2(\gamma-\alpha+1)} + 1).$$

Since $\gamma - \alpha + 1 < 0$, the right-hand side of the above inequality is bounded on I. Hence, inequality (58) holds.

Note that the proof of necessity shows that (58) holds for every $\varepsilon \in \left(0, \frac{1}{2}\right)$.

(ii) *Sufficiency* By the previous argument $\hat{B}_0$ is $\hat{T}_0$-bounded. Thus, $D(\hat{T}_0) \subseteq D(\hat{B}_0)$. For every positive integer $N > a$, define $\hat{B}_N$ on $D(\hat{T}_0)$ by

$$\hat{B}_N y = \begin{cases} \hat{B}y & \text{on } [a, N], \\ 0 & \text{on } (N, \infty). \end{cases}$$

Notice that each $\hat{B}_N$ is $\hat{T}_0$-bounded with the same norm as $\hat{B}_0$ since $\|\hat{B}_N y\| \le \|By\|$. In order to simplify the proof, we break the argument into two Claims.

CLAIM 4.1.1 *$\hat{B}_N \to \hat{B}_0$ in the space of bounded operators on $D(\hat{T}_0)$ with the $\hat{T}_0$-norm.*

Proof of Claim 4.1.1 By definition $\hat{T}_0$ is closed. Therefore, $D(\hat{T}_0)$ is complete under the $\hat{T}_0$-norm.

Let $y \in D(\hat{T}_0)$. Since $\hat{T}_0$ is the closure of $\hat{T}_0'$, for each integer $n \ge 1$ there exists a $y_n \in D(\hat{T}_0')$ such that

$$\|y - y_n\| + \|\hat{T}y - \hat{T}y_n\| < \frac{1}{n}, \tag{74}$$

where $y_n = \binom{y_{n,1}}{y_{n,2}}$ for each n.

For each $y_n \in \hat{T}_0'$ we have, suppressing the independent variable,

$$\begin{aligned} \|\hat{B}y_n - \hat{B}_N y_n\|^2 &= \int_N^\infty u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|y_{n,1}|^2 + [b_1b_4 + b_2b_3]y_{n,1}\overline{y_{n,2}} \\ &\qquad + [b_1b_4 + b_2b_3]\overline{y_{n,1}}y_{n,2} + [b_2^2 + b_4^2]|y_{n,2}|^2\} \\ &\le \int_N^\infty u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|y_{n,1}|^2 + 2|b_1b_4 + b_2b_3||y_{n,1}y_{n,2}| \\ &\qquad + [b_2^2 + b_4^2]|y_{n,2}|^2\} \\ &\le 3\int_N^\infty u^{-\gamma-\alpha} b^2[|y_{n,1}|^2 + |y_{n,2}|^2]. \end{aligned} \tag{75}$$

Since each y_n has compact support in the interior of I, there exists a $b < \infty$ such that the support of $y_{n,1}$ and the support of $y_{n,2}$ are contained in $[a, b]$. Thus, we can apply the sufficiency argument in part (i) with $I_N = [N, \infty)$ to the last inequality in (75).

Via Theorem 2.7 and inequalities (64)–(66), and (75), there exists a constant $k_1 > 0$ such that

$$\begin{aligned} \|\hat{B}y_n - \hat{B}_N y_n\|^2 \le k_1 \sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u\right\} \\ \times \left\{\int_a^b u^{\alpha-\gamma-2}[|y_{n,1}(u)|^2 + |y_{n,2}(u)|^2\,\mathrm{d}u \right. \\ \left. +\varepsilon^2\int_a^b u^{\alpha-\gamma}[|y'_{n,1}(u)|^2 + |y'_{n,2}(u)|^2]\,\mathrm{d}u\right\}. \end{aligned}$$

We apply the Hardy inequality (57), as before, to obtain a constant $k_2 > 0$ such that

$$\begin{aligned} \|\hat{B}y_n - \hat{B}_N y_n\|^2 &\le k_2 \sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u\right\} \\ &\quad\times \left\{\int_a^b u^{\alpha-\gamma}[|y'_{n,1}(u)|^2 + |y'_{n,2}(u)|^2]\,\mathrm{d}u\right\} \\ &\le k_2 \sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u\right\}\cdot \|y_n\|^2_{\hat{T}'_0}. \end{aligned}$$

Therefore, via the triangle inequality, the $\hat{T}_0$-boundedness of $\hat{B}_0$, and inequalities (74) and (76), we have for each n

$$\begin{aligned} \|\hat{B}y - \hat{B}_N y\| &\le \|\hat{B}y - \hat{B}y_n\| + \|\hat{B}y_n - \hat{B}_N y_n\| + \|\hat{B}_N y_n - \hat{B}_N y\| \\ &\le 2k_3(\|y - y_n\| + \|\hat{T}y - \hat{T}y_n\|) + \|\hat{B}y_n - \hat{B}_N y_n\| \\ &< \frac{2k_3}{n} + k_2^{1/2}\left(\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u\right\}\right)^{1/2}\|y_n\|_{\hat{T}'_0}. \end{aligned}$$

for some constant $k_3 > 0$. By applying the triangle inequality and inequality (74) again, we obtain for each n

$$\begin{aligned} \|\hat{B}y - \hat{B}_N y\| &\le \frac{2k_3}{n} + k_2^{1/2}\left(\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u\right\}\right)^{1/2} \\ &\quad\times(\|y_n - y\|_{\hat{T}_0} + \|y\|_{\hat{T}_0}) < \frac{2k_3}{n} \\ &\quad+ k_2^{1/2}\left(\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u\right\}\right)^{1/2}\left(\frac{1}{n} + \|y\|_{\hat{T}_0}\right). \end{aligned}$$

We let $n \to \infty$ to obtain for each N

$$\|\hat{B}y - \hat{B}_N y\| \leq k_2^{1/2}\left(\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u\right\}\right)^{1/2}\|y\|_{\hat{T}_0}.$$

Since $I_N = [N, \infty)$, Eq. (59) implies that

$$\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}b^2(u)\,\mathrm{d}u\right\} \to 0 \text{ as } N \to \infty.$$

so that, via the above inequality, we have for $y \neq 0$

$$\|\hat{B} - \hat{B}_N\| \to 0 \text{ as } N \to \infty.$$

CLAIM 4.1.2 *Each $\hat{B}_N$ is $\hat{T}_0$-compact.*

Proof of Claim 4.1.2 Let $\{y_n\}_{n=1}^{\infty} \subset D(\hat{T}_0)$ be a $\hat{T}_0$-bounded sequence where $y_n = \binom{y_{n,1}}{y_{n,2}}$ for each n. We need to show that for each N the sequence $\{\hat{B}_N y_n\}_{n=1}^{\infty}$ has a convergent subsequence. We make use of the Arzela–Ascoli Theorem.

Let us consider the sequence of the first component functions $\{y_{n,1}\}_{n=1}^{\infty}$. We show that this sequence is uniformly bounded on $[a, N]$. Via inequality (44) there exists a constant $k_4 > 0$ such that for each $x \in [a, N]$

$$\begin{aligned}|y_{n,1}(x)| &\leq k_4\left\{\left(\int_a^N u^{\gamma-\alpha}|y_{n,1}(u)|^2\,\mathrm{d}u\right)^{1/2} + \left(\int_a^N u^{\alpha-\gamma}|y'_{n,1}(u)|^2\,\mathrm{d}u\right)^{1/2}\right\}\\ &\leq k_4(\|y_n\| + \|\hat{T}y_n\|).\end{aligned}$$

Since $\{y_n\}_{n=1}^{\infty}$ is a $\hat{T}_0$-bounded sequence, the above inequality implies that the sequence $\{y_{n,1}\}_{n=1}^{\infty}$ is uniformly bounded on $[a, N]$.

Now, we show that $\{y_n\}_{n=1}^{\infty}$ is equicontinuous on $[a, N]$. Via the Cauchy–Schwarz inequality and the boundedness of $u^{\gamma-\alpha}$ on $[a, N]$, there exists a constant $k_5 > 0$ such that for $s, t \in [a, N]$

$$\begin{aligned}|y_{n,1}(s) - y_{n,1}(t)| &= \left|\int_t^s y'_{n,1}(u)\,\mathrm{d}u\right|\\ &\leq \left|\int_t^s u^{\gamma-\alpha}\,\mathrm{d}u\right|^{1/2}\left|\int_t^s u^{\alpha-\gamma}|y'_{n,1}(u)|^2\,\mathrm{d}u\right|^{1/2}\\ &\leq k_5|s-t|^{1/2}\left|\int_t^s u^{\alpha-\gamma}|y'_{n,1}(u)|^2\,\mathrm{d}u\right|^{1/2}\\ &\leq k_5|s-t|^{1/2}\|y_n\|_{\hat{T}_0}.\end{aligned}$$

for each n. Since $\{y_n\}_{n=1}^{\infty}$ is a $\hat{T}_0$-bounded sequence, the above inequality implies that the sequence $\{y_{n,1}\}_{n=1}^{\infty}$ is equicontinuous on $[a, N]$.

Therefore, via the Arzela–Ascoli Theorem we conclude that there exists a subsequence $\{y_{n_k,1}\}_{k=1}^{\infty}$ of $\{y_{n,1}\}_{n=1}^{\infty}$ which converges uniformly on $[a, N]$. By repeating the above arguments on the subsequence $\{y_{n_k,2}\}_{k=1}^{\infty}$ of $\{y_{n,2}\}_{n=1}^{\infty}$, we conclude that there exists a subsequence of $\{y_{n_k,2}\}_{k=1}^{\infty}$ which converges uniformly on $[a, N]$. Hence, there exists a subsequence of $\{y_n\}_{n=1}^{\infty}$ which converges uniformly on $[a, N]$. WLOG, we assume the sequence $\{y_n\}_{n=1}^{\infty}$ converges uniformly on $[a, N]$.

Now, using an argument similar to the one in (40), we have for each N

$$\begin{aligned}\|\hat{B}_N y_n - \hat{B}_N y_m\|^2 &\leq 3 \int_a^N u^{-\gamma-\alpha} b^2(u)[|y_{n,1}(u) - y_{m,1}(u)|^2 \\ &\quad + |y_{n,2}(u) - y_{m,2}(u)|^2]\,\mathrm{d}u \\ &\leq 3 \sup_{a \leq x \leq N} \|y_n - y_m\|_2^2 \int_a^N u^{-\gamma-\alpha} b^2(u)\,\mathrm{d}u.\end{aligned}$$

Since the integral on the right-hand side is finite on $[a, N]$, there exists a constant $C > 0$ such that

$$\|\hat{B}_N y_n - \hat{B}_N y_m\|^2 \leq C \sup_{a \leq x \leq N} \|y_n - y_m\|_2^2.$$

Since $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the uniform norm, the above inequality implies that the sequence $\{\hat{B}_N y_n\}_{n=1}^{\infty}$ is Cauchy in $\mathcal{L}_w^2(I)$ for each N. Therefore, $\{\hat{B}_N y_n\}_{n=1}^{\infty}$ converges for each N as $n \to \infty$ since $\mathcal{L}_w^2(I)$ is complete. Hence, each $\hat{B}_N$ is $\hat{T}_0$-compact.

Since $\hat{B}_0$ is the uniform limit of $\hat{T}_0$-compact operators, $\hat{B}_0$ is $\hat{T}_0$-compact.

Necessity We use contradiction arguments to show that Eq. (59) must hold.

Suppose that for some $\varepsilon \in (0, \frac{1}{2})$ there exists a $\rho > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers such that $r_n \to \infty$ as $n \to \infty$ and for each n

$$\frac{1}{r_n} \int_{r_n}^{r_n + \varepsilon r_n} x^{2-2\alpha}[b_1^2(x) + b_3^2(x)]\,\mathrm{d}x \geq \rho. \tag{76}$$

Let $\{\Phi_r\}_{r\geq a}$ be defined as in (33) with $f(r)=r$. Then via inequalities (69) and (71) there exist constants $C_2, C_4>0$ such that for each n

$$\begin{aligned}\|\Phi_{r_n}\|^2_{\hat{T}_0} &= (\|\Phi_{r_n}\| + \|\hat{T}\Phi_{r_n}\|)^2 \leq 2(\|\Phi_{r_n}\|^2 + \|\hat{T}\Phi_{r_n}\|^2)\\ &\leq 2C_2 r_n^{\gamma-\alpha+1} + 2C_4 r_n^{\alpha-\gamma-1}.\end{aligned} \tag{77}$$

For each $r\geq a$ define

$$\hat{\Phi}_r(x) = r^{\frac{1}{2}(\gamma-\alpha+1)}\Phi_r(x) \quad \text{for } x\geq a.$$

Via inequality (77) we have for each n

$$\|\hat{\Phi}_{r_n}\|^2_{\hat{T}_0} = r_n^{\gamma-\alpha+1}\|\Phi_{r_n}\|^2_{\hat{T}_0} \leq 2C_2 r_n^{2(\gamma-\alpha+1)} + 2C_4.$$

Since $\gamma-\alpha+1<0$, the above inequality implies that $\{\hat{\Phi}_{r_n}\}_{n=1}^{\infty}$ is a $\hat{T}_0$-bounded sequence. Via the $\hat{T}_0$-compactness of $\hat{B}_0$, $\{\hat{B}\hat{\Phi}_{r_n}\}_{n=1}^{\infty}$ has a convergent subsequence. WLOG, we assume $\{\hat{B}\hat{\Phi}_{r_n}\}_{n=1}^{\infty}$ converges, say to some y_0. Now, via inequality (76), properties of ϕ_{r_n}, and Lemma 2.8, there exists a constant $C>0$ such that for each n

$$\begin{aligned}\rho &\leq \frac{1}{r_n}\int_{r_n}^{r_n+\varepsilon r_n} x^{2-2\alpha}[b_1^2(x)+b_3^2(x)]\,\mathrm{d}x\\ &\leq \frac{1}{r_n}\int_{r_n-2\varepsilon r_n}^{r_n+2\varepsilon r_n} x^{2-2\alpha}[b_1^2(x)+b_3^2(x)]\phi_{r_n}^2(x)\,\mathrm{d}x\\ &\leq C r_n^{\gamma-\alpha+1}\int_{r_n-2\varepsilon r_n}^{r_n+2\varepsilon r_n} x^{-\gamma-\alpha}[b_1^2(x)+b_3^2(x)]\phi_{r_n}^2(x)\,\mathrm{d}x\\ &= C\|\hat{B}\hat{\Phi}_{r_n}\|^2.\end{aligned}$$

Hence, $\|\hat{B}\hat{\Phi}_{r_n}\| \geq (\rho/C)^{1/2} > 0$ for each n.

By an argument similar to the one used in the proof of Theorem 3.1 (ii), we conclude that $y_0=0$ a.e. in $[a,\infty)$. This contradiction implies that

$$\lim_{x\to\infty}\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_1^2(u)+b_3^2(u)]\,\mathrm{d}u = 0, \tag{78}$$

for some $\varepsilon\in(0,\frac{1}{2})$.

Moreover, by repeating the above argument with b_1 and b_3 replaced by b_2 and b_4, respectively, and Φ_r replaced by Ψ_r (as defined by (33)), we conclude that

$$\lim_{x\to\infty}\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_2^2(u)+b_4^2(u)]\,\mathrm{d}u = 0,$$

for some $\varepsilon \in (0, \frac{1}{2})$. This equation along with Eq. (78) implies that (59) holds.

Note that the proof of necessity shows that (59) holds for every $\varepsilon \in (0, \frac{1}{2})$.

THEOREM 4.2 *Let* $I=[a,\infty)$ *for some* $a \geq 1$, $b = (\sum_{i=1}^4 b_i^2)^{1/2}$, *and* $\gamma - \alpha < -1$. *Then the following three statements are equivalent*:

(i) $\int_I x^{-\gamma-\alpha} b^2(x)\,\mathrm{d}x < \infty$;
(ii) $\hat{B}_1$ *is* $\hat{T}_1$*-bounded*;
(iii) $\hat{B}_1$ *is* $\hat{T}_1$*-compact.*

Proof We break the proof into two cases.

Case I $\alpha = 0$.

We consider the maximal operators, $\hat{T}_1$ and $\hat{B}_1$, and the minimal operators, $\hat{T}_0$ and $\hat{B}_0$, associated with the following differential expressions on the interval I:

$$\hat{T}z(x) = \begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_1' \\ z_2' \end{pmatrix}$$

and

$$\hat{B}z(x) = \begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix}\begin{pmatrix} b_1 & b_4 \\ b_3 & b_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

(i) $\Rightarrow$ (iii): Since $\gamma < -1$ and $x \geq a \geq 1$, we have

$$\int_I x b^2(x)\,\mathrm{d}x \leq \int_I x^{-\gamma} b^2(x)\,\mathrm{d}x.$$

Hence, (i) implies that $\hat{B}_0$ is $\hat{T}_0$-compact via Theorem 4.1. Therefore, $D(\hat{T}_0) \subseteq D(\hat{B}_0)$.

Via Eq. (3) we can write

$$D(\hat{T}_1) = D(\hat{T}_0) \oplus S, \tag{79}$$

where S has dimension four since $\hat{T}$ is regular at a and limit circle at ∞.

CLAIM 4.2.1 $S \subseteq D(\hat{B}_1)$.

Proof of Claim 4.2.1 Let z_1 and z_2 be functions in $C^{(1)}(R)$ such that $z_1(a) = 1$, $\text{supp}(z_1) = [a-1, a+1]$, $z_2(x) = 0$ for $x \leq a$, and $z_2(x) = 1$ for $x \geq a+1$. We define

$$Z_1(x) = \begin{pmatrix} z_1(x) \\ 0 \end{pmatrix}, \quad Z_2(x) = \begin{pmatrix} 0 \\ z_1(x) \end{pmatrix},$$
$$Z_3(x) = \begin{pmatrix} z_2(x) \\ 0 \end{pmatrix}, \text{ and } Z_4(x) = \begin{pmatrix} 0 \\ z_2(x) \end{pmatrix}$$

for $x \geq a$. Notice that these four vector-valued functions are linearly independent. Since z_1 has compact support,

$$\|Z_1\|^2 = \|Z_2\|^2 = \int_a^{a+1} x^\gamma |z_1(x)|^2 \, dx,$$

which is finite since z_1 is continuous. Moreover,

$$\|Z_3\|^2 = \|Z_4\|^2 = \int_a^\infty x^\gamma |z_2(x)|^2 \, dx = \int_a^{a+1} x^\gamma |z_2(x)|^2 \, dx + \int_{a+1}^\infty x^\gamma \, dx,$$

which is finite since z_2 is continuous and $\gamma < -1$. Thus, each $Z_i \in \mathcal{L}_w^2(I)$, with weights x^γ.

We show that each $\hat{T}Z_i \in \mathcal{L}_w^2(I)$, with weights x^γ, so that each $Z_i \in D(\hat{T}_1)$. Since z_1 has compact support,

$$\|\hat{T}Z_1\| = \|\hat{T}Z_2\| < \infty.$$

Moreover, since $z_2'(x) = 0$ on $[a+1, \infty)$,

$$\|\hat{T}Z_3\|^2 = \|\hat{T}Z_4\|^2 = \int_a^{a+1} x^{-\gamma} |z_2'(x)|^2 \, dx,$$

which is finite since z_2' is continuous.

Define $S = \operatorname{span}\{Z_1, Z_2, Z_3, Z_4\}$. We now prove that

$$D(\hat{T}_1) = D(\hat{T}_0) \oplus S,$$

i.e., we show that no linear combination of the Z_i is in $D(\hat{T}_0)$. Suppose to the contrary that there exist constants c_1, c_2, c_3, c_4 (not all zero) such that

$$Z := c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4 \in D(\hat{T}_0).$$

Since $\hat{T}$ is regular at a and $Z \in D(\hat{T}_1)$, we have that $Z \in D(\hat{T}_0)$ if and only if $Z(a) = 0$ and $[Z, Y](x) \to 0$ as $x \to \infty$ for every $Y \in D(\hat{T}_1)$, where the *Lagrange identity* is defined by

$$[Y, \hat{Y}](x) = (y_1 \hat{y}_2 - y_2 \hat{y}_1)(x)$$

for real vector-valued functions Y and $\hat{Y}$ (see [12, Theorem 3.12]). Since $z_1(a) \neq 0$, we must take $c_1 = c_2 = 0$. Therefore, $Z = c_3 Z_3 + c_4 Z_4$.

In order to satisfy the second condition of [12, Theorem 3.12], we must have $[Z, Z_3](x) \to 0$ as $x \to \infty$ and $[Z, Z_4](x) \to 0$ as $x \to \infty$, where

$$[Z, Z_3](x) = c_3[Z_3, Z_3](x) + c_4[Z_4, Z_3](x) = c_4[Z_4, Z_3](x)$$

and

$$[Z, Z_4](x) = c_3[Z_3, Z_4](x) + c_4[Z_4, Z_4](x) = c_3[Z_3, Z_4](x).$$

Since

$$\lim_{x\to\infty} [Z_3, Z_4](x) = \lim_{x\to\infty} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = 1,$$

we must have $c_3 = 0$. Similarly, $c_4 = 0$. Hence, no linear combination of the Z_i is in $D(\hat{T}_0)$.

Since z_1 has compact support, $\|\hat{B}Z_1\| = \|\hat{B}Z_2\| < \infty$. Moreover,

$$\begin{aligned} \|\hat{B}Z_3\|^2 &= \int_a^\infty x^{-\gamma}[b_1^2(x) + b_3^2(x)]|z_2(x)|^2 \, \mathrm{d}x \\ &= \int_a^{a+1} x^{-\gamma}[b_1^2(x) + b_3^2(x)]|z_2(x)|^2 \, \mathrm{d}x \\ &\quad + \int_{a+1}^\infty x^{-\gamma}[b_1^2(x) + b_3^2(x)] \, \mathrm{d}x \end{aligned}$$

and

$$\begin{aligned}
\|\hat{B}Z_4\|^2 &= \int_a^\infty x^{-\gamma}[b_2^2(x)+b_4^2(x)]|z_2(x)|^2\,\mathrm{d}x \\
&= \int_a^{a+1} x^{-\gamma}[b_2^2(x)+b_4^2(x)]|z_2(x)|^2\,\mathrm{d}x \\
&\quad + \int_{a+1}^\infty x^{-\gamma}[b_2^2(x)+b_4^2(x)]\,\mathrm{d}x,
\end{aligned}$$

which are finite by (i). Therefore, each $Z_i \in D(\hat{B}_1)$.

Equation (79) and Claim 4.2.1 imply that $D(\hat{T}_1) \subseteq D(\hat{B}_1)$. Via Theorem 2.5 $\hat{B}_1$ is $\hat{T}_1$-compact.

(iii) $\Rightarrow$ (ii): Since $\hat{B}_1$ is $\hat{T}_1$-compact, $D(\hat{T}_1) \subseteq D(\hat{B}_1)$. Via Theorem 2.3 $\hat{B}_1$ is $\hat{T}_1$-bounded.

(ii) $\Rightarrow$ (i): Since $\hat{B}_1$ is $\hat{T}_1$-bounded, $D(\hat{T}_1) \subseteq D(\hat{B}_1)$. Via Eq. (79) $S \subseteq D(\hat{B}_1)$. Therefore, $\|BZ_i\| < \infty$ for each i, i.e.,

$$\int_I x^{-\gamma} b^2(x)\,\mathrm{d}x < \infty.$$

Case II $\alpha \neq 0$.

As in the proof of Corollary 3.2, we transform the differential expressions $\hat{T}$ and $\hat{B}$ unitarily into T and B, respectively, where

$$Ty(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$$

and

$$By(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix}\begin{pmatrix} x^{-\alpha}b_1 & x^{-\alpha}b_4 \\ x^{-\alpha}b_3 & x^{-\alpha}b_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Replacing γ with $\gamma - \alpha$, b with $x^{-\alpha}b$, and z with y in Case I, we have that the following three statements are equivalent:

(i′) $\int_I x^{-\gamma-\alpha} b^2(x)\,\mathrm{d}x < \infty$;
(ii′) B_1 is T_1-bounded;
(iii′) B_1 is T_1-compact.

Since the transformation is unitary, (ii′) holds iff $\hat{B}_1$ is $\hat{T}_1$-bounded, and (iii′) holds iff $\hat{B}_1$ is $\hat{T}_1$-compact.

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