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Iterative Methods for the Darboux Problem for Partial Functional Differential Equations

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We consider the Darboux problem for the hyperbolic partial functional differential equations

(1) $D_{xy}z(x,y) = f(x,y,z_{(x,y)}),\ (x,y) \in [0,a] \times [0,b]$,
(2) $z(x,y) = \phi(x,y),\ (x,y) \in [-a_0,a] \times [-b_0,b] \backslash (0,a] \times (0,b]$,

where $z_{(x,y)} : [-a_0,0] \times [-b_0,0] \to X$ is a function defined by $z_{(x,y)}(t,s) = z(x+t,y+s)$, $(t,s) \in [-a_0,0] \times [-b_0,0]$. If $X = \mathbb{R}$ then using the method of functional differential inequalities we prove, under suitable conditions, a theorem on the convergence of the Chaplyghin sequences to the solution of problem (1), (2). In case X is any Banach space we give analogous theorem on the convergence of the Newton method.

Keywords: Functional differential equation; Darboux problem; Classical solutions; Functional differential inequalities; Iterative method

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1. INTRODUCTION

Define $E = [0,a] \times [0,b]$, $E^0 = [-a_0,a] \times [-b_0,b] \backslash (0,a] \times (0,b]$, $E^* = E^0 \cup E$, where $a_0, b_0 \in \mathbb{R}_+$, $a, b > 0$. If X, Y are any Banach spaces then we denote by $C(X,Y)$ the class of all continuous functions from X to Y. If $X \subset E^*$ then we denote by $C^1(X,Y)$ the set of all functions $z \in C(X,Y)$ for which the derivatives D_xz, D_yz, exist and are continuous. Finally, let $C^{1,*}(E^*,Y)$ denote the subset of $C^1(E^*,Y)$ consisting of functions for which additionally the mixed derivative $D_{xy}z$ exists and is continuous on E.

For any function $z:[-a_0,a]\times[-b_0,b]\to\mathbb{R}$, and a point $(x,y)\in E$, we define $z_{(x,y)}:B\to\mathbb{R}$, where $B=[-a_0,0]\times[-b_0,0]$, by the formula

$$z_{(x,y)}(t,s)=z(x+t,y+s),\quad (t,s)\in B.$$

For given functions $\phi:E^0\to\mathbb{R}$, $f:E\times C^1(B;\mathbb{R})\to\mathbb{R}$, we consider the following Darboux problem:

$$D_{xy}z(x,y)=f\big(x,y,z_{(x,y)}\big),\quad (x,y)\in E, \tag{1}$$

$$z(x,y)=\phi(x,y),\quad (x,y)\in E^0. \tag{2}$$

In this paper we consider classical solutions of (1), (2). In other words a function $z\in C^{1,*}(E^*;\mathbb{R})$ is said to be a solution of (1), (2) if it satisfies Eq. (1) on E and fulfills Darboux condition (2) on E^0.

Remark 1 Equation (1) contains as special cases equations in which the right-hand side depends on the first order derivatives of an unknown function even though in f there is no explicit dependence on D_xz and D_yz. This is due to the fact that the last variable of f is a functional space $C^1(B,\mathbb{R})$. As an example consider the Darboux problem for the equation with deviated argument

$$\begin{aligned} D_{xy}z(x,y)&=\tilde{f}\big(x,y,z(\alpha_0(x,y)),\\ &\qquad D_xz(\alpha_1(x,y)),\ D_yz(\alpha_2(x,y))\big),\ (x,y)\in E,\\ z(x,y)&=\phi(x,y),\quad (x,y)\in E^0,\end{aligned}$$

where $\tilde{f}:E\times\mathbb{R}^3\to\mathbb{R}$, $\alpha_i:E\to E^*$, and $\alpha_i(x,y)-(x,y)\in B$ for $(x,y)\in E$, $i=0,1,2$. This problem is a special case of (1), (2) if we define f by the formula

$$\begin{aligned} f(x,y,w)=&\tilde{f}\big(x,y,w(\alpha_0(x,y)-(x,y)),\\ &D_xw(\alpha_1(x,y)-(x,y)),D_yw(\alpha_2(x,y)-(x,y))\big).\end{aligned}$$

In this paper we give sufficient conditions for the existence of two monotone sequences $\{u^{(m)}\},\{v^{(m)}\}$ such that if z is a solution of (1), (2) then $u^{(m)}\le z\le v^{(m)}$ on E and $\{u^{(m)}\},\{v^{(m)}\}$ are uniformly convergent to z on E. The convergence that we get is of the Newton type, which means that

$$0\le z(x,y)-u^{(m)}(x,y)\le\frac{2A}{2^{2^m}},\quad 0\le v^{(m)}(x,y)-z(x,y)\le\frac{2A}{2^{2^m}},\quad \text{on } E,$$

where A is some constant not dependent on m. The functions $u^{(m)}$, $v^{(m)}$ are defined as solutions of some linear functional differential equations obtained by the linearization of (1). The monotonicity of sequences $\{u^{(m)}\}, \{v^{(m)}\}$ is proved by a theorem on functional differential inequalities.

In the last section of this paper we also prove a theorem on the convergence of the Newton method for problem (1), (2) in a Banach space. Also in this case we get convergence of the Newton type.

The method of approximating solutions of differential equations by their linearization was introduced by Chaplyghin in [4]. In the original Chaplyghin method only one approximating sequence was defined (cf. [11,16]). This method has been applied by Mlak and Szechter [12] to the system of the first order semilinear partial differential equations and has been extended to functional differential equations in [6,8].

Monotone iterative scheme for functional differential Darboux problem, but without linearization and consequently with slower convergence has been studied by Brzychczy and Janus [5]. The authors used the ideas presented in the monograph by Ladde *et al.* [9] (cf. also [10]). From other important results concerning monotone iterative methods for non-functional hyperbolic problems we mention those by Blakley and Pandit [3] and Pandit [14]. In the last paper with help of the quasilinearization technique a quadratically converging successive approximations scheme is obtained under the monotonicity and convexity condition on the right-hand side. These methods have been also applied to higher order hyperbolic equations by Agarwal [1] and Agarwal and Sheng [2], where in the latter paper the periodic solutions were investigated.

Note that the iterative (not necessarily monotone) method is often used in the theory of hyperbolic functional differential equations to prove the existence of solutions [7,13,15].

2. FUNCTIONAL DIFFERENTIAL INEQUALITIES

We endow the space $C^1(E, \mathbb{R})$ with the norm

$$\|z\|_* = \|z\|_0 + \|D_x z\|_0 + \|D_y z\|_0,$$

where $\|\cdot\|_0$ denotes the usual supremum norm. The notation $u \leq_* v$ on E means that we have $u(x,y) \leq v(x,y)$, $D_x u(x,y) \leq D_x v(x,y)$ and $D_y u(x,y) \leq D_y v(x,y)$ for $(x,y) \in E$, and analogously we may define the relation $u <_* v$.

A function $u \in C^{1,*}(E^*, \mathbb{R})$ is called a lower function of problem (1), (2) on E if

$$\begin{aligned} D_{xy}u(x,y) &\leq f\big(x,y,u_{(x,y)}\big) \quad \text{on } E, \\ u &\leq_* \phi \quad \text{on } E^0, \end{aligned} \tag{3}$$

and an upper function of problem (1), (2) on E if reverse inequalities hold.

Remark 2 In the second condition of (3) it would be sufficient assuming that we have $D_x u(x,y) \leq D_x v(x,y)$, $D_y u(x,y) \leq D_y v(x,y)$ *for* $(x,y) \in E$ and that $u(-a_0, -b_0) \leq v(-a_0, -b_0)$. We use the relation of (3) for simplicity of notation.

For $z \in C^1(E^*, \mathbb{R})$ put

$$f[z](x,y) = f\big(x,y,z_{(x,y)}\big), \quad (x,y) \in E.$$

We now state a theorem on functional differential inequalities which is analogous to the theorems in [9,10].

THEOREM 1 *Suppose that* $f\colon E \times C^1(B,\mathbb{R}) \to \mathbb{R}$ *of the variables* (x,y,w) *is nondecreasing with respect to* w *and that we have two functions* $u, v \in C^{1,*}(E^*, \mathbb{R})$ *satisfying the inequalities*

$$\begin{aligned} &D_{xy}u(x,y) - D_{xy}v(x,y) < f[u](x,y) - f[v](x,y) \quad \textit{for } (x,y) \in E, \\ &u <_* v \quad \textit{on } E^0. \end{aligned}$$

Then $u <_* v$ *on* E^*.

Proof It is sufficient to prove that

$$D_x u(x,y) < D_x v(x,y) \text{ and } D_y u(x,y) < D_y v(x,y) \quad \text{for } (x,y) \in E. \tag{4}$$

Suppose for a contradiction that (4) is false. Then there is a point $(x_0, y_0) \in E$ such that $D_x u(x_0,y_0) = D_x v(x_0,y_0)$ or $D_y u(x_0,y_0) = D_y v(x_0,y_0)$ but $D_x u(x,y) < D_x v(x,y)$ and $D_y u(x,y) < D_y v(x,y)$ for

$(x,y) \in [0,x_0] \times [0,y_0] \backslash \{(x_0,y_0)\}$. This yields that

$$u <_* v \quad \text{on } [0,x_0] \times [0,y_0] \setminus \{(x_0,y_0)\}. \tag{5}$$

Suppose that $D_x u(x_0,y_0) = D_x v(x_0,y_0)$ and notice that $y_0 > 0$. From the monotonicity of f and (5) we get

$$\begin{aligned} &D_x u(x_0,y_0) - D_x v(x_0,y_0) \\ &\quad < D_x u(x_0,0) - D_x v(x_0,0) + \int_0^{y_0} \{f[u](x_0,t) - f[v](x_0,t)\} \mathrm{d}t < 0, \end{aligned}$$

which is a contradiction.

Analogously we get a contradiction if $D_y u(x_0,y_0) = D_y v(x_0,y_0)$, which completes the proof of Theorem 1.

Assumption H_1 Suppose that

(1) $f \in C(E \times C^1(B,\mathbb{R}),\mathbb{R})$ of the variables (x,y,w) is nondecreasing with respect to w and there are $u^{(0)}$, $v^{(0)} \in C^{1,*}(E^*,\mathbb{R})$ such that

$$\begin{aligned} &D_{xy} u^{(0)}(x,y) < f[u^{(0)}](x,y), \; D_{xy} v^{(0)}(x,y) > f[v^{(0)}](x,y) \\ &\quad \text{for } (x,y) \in E, \\ &u^{(0)} <_* v^{(0)} \quad \text{on } E^0; \end{aligned}$$

(2) there is a continuous, bounded and nondecreasing function $\sigma : E \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\sigma(x,y,0) = 0$ for $(x,y) \in E$ and that the problem

$$\begin{aligned} &D_{xy} z(x,y) = \sigma(x,y,z(x,y)) + L\big[D_x z(x,y) + D_y z(x,y)\big], \\ &z(x,0) = 0 \;\text{ for } x \in [0,a], \quad z(0,y) = 0 \;\text{ for } y \in [0,b], \end{aligned}$$

has only trivial solution on E;

(3) the estimate

$$\begin{aligned} &f(x,y,w) - f(x,y,\bar{w}) \\ &\quad \le \sigma(x,y,\|w - \bar{w}\|_0) + L\big[\|D_x w - D_x \bar{w}\|_0 + \|D_y w - D_y \bar{w}\|_0\big] \end{aligned} \tag{6}$$

holds for $(x,y) \in E$, $w, \bar{w} \in C^1(B,\mathbb{R})$, where $u^{(0)}_{(x,y)} <_* \bar{w} \le_* w <_* v^{(0)}_{(x,y)}$.

Remark 3 The simplest example of σ satisfying condition (2) of Assumption H_1 is the linear function $\sigma(x,y,p) = Lp$, $(x,y) \in E$, $p \in \mathbb{R}_+$.

In that case f fulfills the Lipschitz condition with respect to the last variable with the norm $\|\cdot\|_*$. As an example of nonlinear σ we may take the function

$$\sigma(x,y,p) = \begin{cases} 0 & \text{for } (x,y) \in E,\ p = 0, \\ p|\ln p| & \text{for } (x,y) \in E,\ 0 < p < 1/e, \\ 1/e & \text{for } (x,y) \in E,\ p \geq 1/e. \end{cases}$$

Given $u^{(0)}, v^{(0)} \in C^{1,*}(E^*, \mathbb{R})$ satisfying condition (1) of Assumption H_1 we put

$$\Lambda(u^{(0)}, v^{(0)}) = \left\{ z \in C^{1,*}(E^*, \mathbb{R}):\ u^{(0)} <_* z <_* v^{(0)} \text{on } E^* \right\}.$$

Next we state a theorem on weak functional differential inequalities.

THEOREM 2 *Suppose that Assumption* H_1 *is satisfied and that we have two functions* $u, v \in \Lambda(u^{(0)}, v^{(0)})$ *satisfying the inequalities*

$$D_{xy}u(x,y) \leq \boldsymbol{f}[u](x,y), \quad D_{xy}v(x,y) \geq \boldsymbol{f}[v](x,y) \quad \textit{for } (x,y) \in E,$$
$$u \leq_* v \quad \textit{on } E^0.$$

Then $u \leq_* v$ *on* E^*.

Proof Consider the Darboux problem

$$D_{xy}z(x,y) = \sigma(x,y,z(x,y)) + \varepsilon + L\big[D_x z(x,y) + D_y z(x,y)\big],$$
$$z(x,0) = \varepsilon \mathrm{e}^x \ \text{ for } x \in [0,a], \quad z(0,y) = \varepsilon \mathrm{e}^y \ \text{ for } y \in [0,b].$$

There is $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ then there exists a solution $\omega^\varepsilon : E \to \mathbb{R}$ to this problem and

$$\lim_{\varepsilon \to 0} \omega^\varepsilon(x,y) = \lim_{\varepsilon \to 0} D_x\omega^\varepsilon(x,y) = \lim_{\varepsilon \to 0} D_y\omega^\varepsilon(x,y) = 0$$

uniformly on E. Furthermore, let $\tilde{\omega}^\varepsilon : E^* \to \mathbb{R}$ be an extension of ω^ε onto the set E^* such that on E^0 the function $\tilde{\omega}^\varepsilon$ and its first order derivatives are nondecreasing and $\tilde{\omega}^\varepsilon >_* 0$ on E^0. Given $\varepsilon < \varepsilon_0$ we define

$$u^\varepsilon(x,y) = u(x,y) - \tilde{\omega}^\varepsilon(x,y),$$
$$v^\varepsilon(x,y) = v(x,y) + \tilde{\omega}^\varepsilon(x,y) \quad \text{for } (x,y) \in E.$$

We may assume that $\varepsilon_0 > 0$ is sufficiently small in order that for $\varepsilon < \varepsilon_0$ we have $u^{(0)} <_* u^\varepsilon <_* u$ and $v <_* v^\varepsilon <_* v^{(0)}$ on E^*, and obviously $u^\varepsilon <_* v^\varepsilon$ on E^0.

Since

$$\begin{aligned} &f[v^\varepsilon](x,y) - f[v](x,y) \\ &\quad \le \sigma(x,y,\omega^\varepsilon(x,y)) + L\big[D_x\,\omega^\varepsilon(x,y) + D_y\,\omega^\varepsilon(x,y)\big] \end{aligned}$$

we get

$$\begin{aligned} D_{xy}v^\varepsilon(x,y) &= D_{xy}v(x,y) + D_{xy}\omega^\varepsilon(x,y) \ge f[v](x,y) + D_{xy}\omega^\varepsilon(x,y) \\ &\ge f[v^\varepsilon](x,y) + D_{xy}\,\omega^\varepsilon(x,y) - \sigma(x,y,\omega^\varepsilon(x,y)) \\ &\quad - L\big[D_x\,\omega^\varepsilon(x,y) + D_y\,\omega^\varepsilon(x,y)\big] \\ &= f[v^\varepsilon](x,y) + \varepsilon > f[v^\varepsilon](x,y). \end{aligned}$$

Analogously we may prove

$$D_{xy}u^\varepsilon(x,y) < f[u^\varepsilon](x,y).$$

From Theorem 1 we get $u^\varepsilon <_* v^\varepsilon$ on E. Letting $\varepsilon \to 0$ we get $u \le_* v$ on E, which completes the proof of Theorem 2.

Remark 4 Note that Theorems 1 and 2 include as special cases results for functional differential inequalities with right-hand sides explicitly dependent on first order derivatives of an unknown function. Indeed, suppose that $\tilde{f} \in C(E \times C(B,\mathbb{R})^3,\mathbb{R})$ is nondecreasing with respect to the last three variables. If we have $u, v \in C^{1,*}(E^*,\mathbb{R})$ such that

$$\begin{aligned} D_{xy}u(x,y) - D_{xy}v(x,y) &< \tilde{f}\Big(x,y,u_{(x,y)},(D_xu)_{(x,y)},(D_yu)_{(x,y)}\Big) \\ &\quad - \tilde{f}\Big(x,y,v_{(x,y)},(D_xv)_{(x,y)},(D_yv)_{(x,y)}\Big) \end{aligned}$$

for $(x,y) \in E$, and $u <_* v$ on E^0 then $u < v$ on E. This fact follows immediately from Theorem 1 if we take $f(x,y,w) = \tilde{f}(x,y,w,D_xw,D_yw)$. Analogously under suitable assumptions we may get a theorem on weak functional differential inequalities from Theorem 2.

ASSUMPTION H_2 Suppose that there exists the continuous derivative D_wf on $E \times C^1(B,\mathbb{R})$ such that for $(x,y) \in E$, $h, w, \bar{w} \in C^1(B,\mathbb{R})$, where

$u^{(0)}_{(x,y)} <_* w, \bar{w} <_* v^{(0)}_{(x,y)}$, we have

(i) $D_w f(x,y,w) \circ h \geq 0$ if $h \geq_* 0$,

(ii) $D_w f(x,y,\bar{w}) \circ h \geq D_w f(x,y,w) \circ h$ if $\bar{w} \geq_* w$, $h \geq_* 0$,

where $\circ$ denotes the composition of two functions.

Given $u, v \in \Lambda(u^{(0)}, v^{(0)})$ the lower and upper functions of problem (1), (2), respectively we define two functions $\mathcal{G}(\cdot\,; u)$, $\mathcal{H}(\cdot\,; u, v)$: $E \times C^1(B, \mathbb{R}) \to \mathbb{R}$ by

$$\mathcal{G}(x,y,w;u) = \boldsymbol{f}[u](x,y) + \boldsymbol{D_w f}[u](x,y) \circ (w - u_{(x,y)}),$$
$$\mathcal{H}(x,y,w;u,v) = \boldsymbol{f}[v](x,y) + \boldsymbol{D_w f}[u](x,y) \circ (w - v_{(x,y)})$$

for $(x,y,w) \in E \times C^1(B;\mathbb{R})$.

Now we prove a theorem which is essential in the definition of Chaplyghin sequences.

THEOREM 3 *Suppose that Assumptions* $\mathrm{H}_1, \mathrm{H}_2$ *are satisfied and*

(1) $u, v \in \Lambda(u^{(0)}, v^{(0)})$ *are the lower and upper functions of problem* (1), (2), *respectively and* $\tilde{z} \in C^{1,*}(E^*, \mathbb{R})$ *is a solution of* (1), (2);

(2) *the initial functions* $\alpha, \phi, \beta \in C^1(E^0, \mathbb{R})$ *satisfy the inequalities*

$$u \leq_* \alpha \leq_* \phi \leq_* \beta \leq_* v \quad on\ E^0; \tag{7}$$

(3) U, V *are solutions of the problems*

$$\begin{aligned} D_{xy}z(x,y) &= \mathcal{G}\big(x,y,z_{(x,y)};u\big) \quad on\ E, \\ z(x,y) &= \alpha(x,y) \quad on\ E^0, \end{aligned} \tag{8}$$

and

$$\begin{aligned} D_{xy}z(x,y) &= \mathcal{H}\big(x,y,z_{(x,y)};u,v\big) \quad on\ E, \\ z(x,y) &= \beta(x,y) \quad on\ E^0, \end{aligned} \tag{9}$$

respectively.

Then $u \leq_* U \leq_* \tilde{z} \leq_* V \leq_* v$ *on* E^*, *and also* U, V *are the lower and upper functions of problem* (1), (2), *respectively.*

Proof Note that since u, v are the lower and upper functions of (1), (2) we have $v \geq_* u$ on E^*, by force of Theorem 2. This together with condition (i) of Assumption H_2 yields that the functions $\mathcal{G}(\cdot\,; u)$, $\mathcal{H}(\cdot\,; u, v)$ are nondecreasing in w. Since they are also Lipschitzean with respect to w we may use Theorem 2 for $\mathcal{G}(\cdot\,; u)$ and $\mathcal{H}(\cdot\,; u, v)$.

We first prove that $u \leq_* U$ on E^*. Since u is the lower function of (1), (2) we have

$$D_{xy}u(x,y) \leq f\big(x,y,u_{(x,y)}\big) = \mathcal{G}\big(x,y,u_{(x,y)}; u\big), \tag{10}$$

where $(x,y) \in E$. Comparing (10) with (8) and using (7) gives our claim. Analogously since v is the upper function of (1), (2) we have

$$D_{xy}v(x,y) \geq f\big(x,y,v_{(x,y)}\big) = \mathcal{H}\big(x,y,v_{(x,y)}; u, v\big), \tag{11}$$

where $(x,y) \in E$, and the inequality $V \leq_* v$ on E^* follows by (11), (9) and (7).

Next we prove $u \leq_* V$ on E^*. Since $u \geq_* u$ we have by condition (ii) of Assumption H_2 the inequality

$$\begin{aligned}
&\mathcal{G}\big(x,y,V_{(x,y)}; u\big) - D_{xy}V(x,y)\\
&\quad = \boldsymbol{f}[u](x,y) + \boldsymbol{D}_w\boldsymbol{f}[u](x,y) \circ \big(V_{(x,y)} - u_{(x,y)}\big)\\
&\qquad - \boldsymbol{f}[v](x,y) - \boldsymbol{D}_w\boldsymbol{f}[u](x,y) \circ \big(V_{(x,y)} - v_{(x,y)}\big)\\
&\quad = (\boldsymbol{D}_w\boldsymbol{f}[u + \theta(v-u)](x,y) - \boldsymbol{D}_w\boldsymbol{f}[u](x,y)) \circ \big(u_{(x,y)} - v_{(x,y)}\big) \leq 0,
\end{aligned} \tag{12}$$

where $\theta \in (0,1)$. Comparing (12) with (10) and (7) gives our claim.

Analogously as (12) we may prove

$$\mathcal{H}\big(x,y,U_{(x,y)}; u, v\big) \geq D_{xy}U(x,y)$$

which together with (10) and (7) yields $U \leq_* v$ on E^*.

Finally we prove that $U \leq_* V$ on E^*. Since $u \leq_* U, V \leq_* v$ and $u \leq_* v + \theta(V - v)$, $\theta \in (0,1)$, we get similarly as in the proof of (12) the estimates

$$\begin{aligned}
&f\big(x,y,U_{(x,y)}\big) - D_{xy}U(x,y)\\
&\quad = \boldsymbol{f}[U](x,y) - \boldsymbol{f}[u](x,y) - \boldsymbol{D}_w\boldsymbol{f}[u](x,y) \circ \big(U_{(x,y)} - u_{(x,y)}\big)\\
&\quad = (\boldsymbol{D}_w\boldsymbol{f}[u + \theta(U-u)](x,y) - \boldsymbol{D}_w\boldsymbol{f}[u](x,y)) \circ \big(U_{(x,y)} - v_{(x,y)}\big) \geq 0
\end{aligned} \tag{13}$$

and

$$\begin{aligned} & f(x,y,V_{(x,y)}) - D_{xy}V(x,y) \\ & \quad = \boldsymbol{f}[V](x,y) - \boldsymbol{f}[v](x,y) - \boldsymbol{D}_w\boldsymbol{f}[u](x,y) \circ \left(V_{(x,y)} - v_{(x,y)}\right) \\ & \quad = (\boldsymbol{D}_w\boldsymbol{f}[v+\theta(V-v)](x,y) - \boldsymbol{D}_w\boldsymbol{f}[u](x,y)) \circ \left(V_{(x,y)} - v_{(x,y)}\right) \le 0, \end{aligned} \tag{14}$$

on E. Comparing (13) with (14) and (7) gives our claim. The inequalities (13) and (14) mean that U, V are the lower and upper functions of (1),(2), respectively. The inequalities $U \le_* \tilde{z} \le_* V$ on E we get easily comparing (13) and (14) with (1) and using (7), which completes the proof of Theorem 3.

Note that in Theorem 3 we have assumed that there is a solution $\tilde{z}$ of problem (1), (2). To ensure the existence of this solution we should assume that the initial function $\phi \in C^1(E^0, \mathbb{R})$ satisfies the following consistency condition:

$$\begin{aligned} D_x\phi(0,y) &= D_x\phi(0,0) + \int_0^y \boldsymbol{f}[\phi](0,t)\,\mathrm{d}t \quad \text{for } y \in [0,b], \\ D_y\phi(x,0) &= D_y\phi(0,0) + \int_0^x \boldsymbol{f}[\phi](s,0)\,\mathrm{d}s \quad \text{for } x \in [0,a]. \end{aligned} \tag{15}$$

Consistency condition (15) is necessary in the existence theorem since we seek classical solutions of the functional Darboux problem, where the functional dependence is based on the use of the operator $(x,y) \mapsto z_{(x,y)}$, and consequently the initial set E^0 has a nonempty interior. However, there is a class of functional problems where we may omit such conditions.

Consider the Darboux problem

$$D_{xy}z(x,y) = F(x,y,z,D_xz,D_yz), \quad (x,y) \in E, \tag{16}$$

$$z(x,0) = \sigma(x), \ \ x \in [0,a], \quad z(0,y) = \tau(y), \ \ y \in [0,b], \tag{17}$$

where $F \in C(E \times C(E,\mathbb{R})^3, \mathbb{R})$, $\sigma \in C^1([0,a],\mathbb{R})$, $\tau \in C^1([0,b],\mathbb{R})$. Suppose that F satisfies the following Volterra condition:

If $(x,y) \in E$ and $z_1, z_2 \in C(E,\mathbb{R})$ are such that $z_1(s,t) = z_2(s,t)$ for $(s,t) \in [0,x] \times [0,y]$ then $F(x,y,z_1,D_xz_1,D_yz_1) = F(x,y,z_2,D_xz_2,D_yz_2)$.

Some examples of differential functional equations which are special cases of (16) have been considered in [7,13,15].

We show that the model of functional dependence of (16) is included in our theory. For $a_0 = a, b_0 = b$ we define the operator $\tilde{I}_{xy} : C(B, \mathbb{R}) \to C(E, \mathbb{R})$ in the following way. If $(x, y) \in E$ and $w \in C(B, \mathbb{R})$ then

$$(I_{xy}w)(s,t) = w(s-x, t-y) \quad \text{for } (s,t) \in [0,x] \times [0,y],$$

and by $\tilde{I}_{xy}w$ we define any continuous extension of $I_{xy}w$ onto the set E. Thus we may define $f: E \times C^1(B, \mathbb{R}) \to \mathbb{R}$ by

$$f(x,y,w) = F\big(x, y, \tilde{I}_{xy}w, \tilde{I}_{xy}(D_x w), \tilde{I}_{xy}(D_y w)\big),$$

and since F satisfies the Volterra condition this definition does not depend on the extension $\tilde{I}_{xy}$. If F is Lipschitzean with respect to the last three arguments and $\phi \in C^1(E^0, \mathbb{R})$ is any function satisfying consistency conditions (15) such that $\phi(x,0) = \sigma(x)$ for $x \in [0,a]$ and $\phi(0,y) = \tau(y)$ for $y \in [0,b]$ then problem (16), (17) becomes a special case of (1), (2). The right-hand side of (16) depends on the unknown function only in $[0,x] \times [0,y] \subset E$ and therefore it does not depend on ϕ, which means that we may drop the consistency condition for (16), (17).

3. CHAPLYGHIN SEQUENCES

We are now able to define two Chaplyghin sequences approximating the solution of (1), (2).

Assumption H_3 Suppose that condition (1) of Assumption H_1 is satisfied and that furthermore there are $L, K > 0$ such that for $(x,y) \in E$, $w, \bar{w} \in C^1(B, \mathbb{R})$, where $u^{(0)}_{(x,y)} <_* w, \bar{w} <_* v^{(0)}_{(x,y)}$, we have

$$|f(x,y,w) - f(x,y,\bar{w})| \leq L\|w - \bar{w}\|_*, \tag{18}$$

$$|D_w f(x,y,w) - D_w f(x,y,\bar{w})|_* \leq K\|w - \bar{w}\|_*, \tag{19}$$

and $|\cdot|_*$ denotes the operator norm generated by the $\|\cdot\|_*$ norm in $C^1(B, \mathbb{R})$.

Assumtption H_4 Suppose that

(1) there are two sequences $\{\alpha^{(m)}\}, \{\beta^{(m)}\} \in C^1(E^0, \mathbb{R})$ of initial functions such that the inequalities

$$\alpha^{(m)} \leq_* \alpha^{(m+1)} \leq_* \phi \leq_* \beta^{(m+1)} \leq_* \beta^{(m)} \tag{20}$$

hold for $(x,y) \in E^0, m \in \mathbb{N}$, and that $u^{(0)} \leq_* \alpha^{(1)}, \beta^{(1)} <_* \gamma^{(0)}$ on $E^{(0)}$;

(2) the following consistency conditions,

$$D_x\alpha^{(m+1)}(0,y) = D_x\alpha^{(m+1)}(0,0) + \int_0^y \mathcal{G}\Big(0,t,\alpha^{(m+1)}_{(0,t)};\alpha^{(m)}\Big)\mathrm{d}t,$$
$$D_y\alpha^{(m+1)}(x,0) = D_y\alpha^{(m+1)}(0,0) + \int_0^x \mathcal{G}\Big(s,0,\alpha^{(m+1)}_{(s,0)};\alpha^{(m)}\Big)\mathrm{d}s, \tag{21}$$

and

$$D_x\beta^{(m+1)}(0,y) = D_x\beta^{(m+1)}(0,0) + \int_0^y \mathcal{H}\Big(0,t,\beta^{(m+1)}_{(0,t)};\alpha^{(m)},\beta^{(m)}\Big)\mathrm{d}t,$$
$$D_y\beta^{(m+1)}(x,0) = D_y\beta^{(m+1)}(0,0) + \int_0^x \mathcal{H}\Big(s,0,\beta^{(m+1)}_{(s,0)};\alpha^{(m)},\beta^{(m)}\Big)\mathrm{d}s, \tag{22}$$

hold true for $x\in[0,a]$, $y\in[0,b]$, $m\in\mathbb{N}\cup\{0\}$, and $\alpha^{(0)}$, $\beta^{(0)}$ denote the restrictions of $u^{(0)}$, $v^{(0)}$ to the set $E^{(0)}$.

Let $T_{\alpha\beta}\colon C^1(E^*;\mathbb{R})\times C^1(E^*;\mathbb{R})\to C^1(E^*;\mathbb{R})\times C^1(E^*;\mathbb{R})$ be the operator defined by $T_{\alpha\beta}[u,v]=[U,V]$, where U, V are solutions of the problems (8), (9), respectively. We consider the sequences $\{u^{(m)}\}$,$\{v^{(m)}\}$ defined as follows:

(1) let $u^{(0)},v^{(0)}\in C^{1,*}(E^*,\mathbb{R})$ be the functions from condition (1) of Assumption H_1;
(2) if $u^{(m)},v^{(m)}\in C^1(E^*;\mathbb{R})$ are already defined then

$$\Big[u^{(m+1)},v^{(m+1)}\Big] = T_{\alpha^{(m+1)}\beta^{(m+1)}}\Big[u^{(m)},v^{(m)}\Big],\quad m\in\mathbb{N}. \tag{23}$$

Assumptions H_2–H_4 are sufficient for existence of solutions of problems (8) and (9) with initial functions $\alpha^{(m)}$ and $\beta^{(m)}$, which means that sequences $\{u^{(m)}\}$ and $\{v^{(m)}\}$ are well defined. Note that $u^{(0)}$, $v^{(0)}$ fulfill strict differential inequalities in condition (1) of Assumption H_1. This means that if in the proof of Theorem 3 we use Theorem 1 instead of Theorem 2 then $u^{(0)}<_* u^{(1)}$ and $v^{(1)}<_* v^{(0)}$ on E^* and consequently $u^{(1)},v^{(1)}\in\Lambda(u^{(0)},v^{(0)})$. Thus for other terms of sequences $\{u^{(m)}\}$ and $\{v^{(m)}\}$ we may use Theorem 3 without any modifications which for all $m \in \mathbb{N}$ gives

$$u^{(m)} \leq_* u^{(m+1)} \leq_* \tilde{z} \leq_* v^{(m+1)} \leq_* v^{(m)} \quad \text{on } E^*. \tag{24}$$

Remark 5 In condition (1) of Assumption H_1 we suppose that two functions $u^{(0)}$, $v^{(0)}$ satisfy strict differential inequalities. The existence of such functions may be the main difficulty in our method. We show the example of $u^{(0)}$ and $v^{(0)}$ provided f satisfies the Lipschitz condition (18) on the whole set $E \times C^1(B, \mathbb{R})$. There are constants M, $N > 0$ such that

$$|f(x,y,0)| < M \text{ on } E \quad \text{and} \quad \|\phi\|_* < N \text{ on } E^0.$$

Then the function $v^{(0)}(x,y) = [M+N]\exp\{\tilde{L}(a_0+b_0+x+y)\} - M$, where $\tilde{L} = L + \sqrt{L^2+L}$, fulfills the estimate

$$\begin{aligned} D_{xy}v^{(0)}(x,y) &= M + L\Big[\|v^{(0)}_{(x,y)}\|_0 + \|(D_x v^{(0)})_{(x,y)}\|_0 + \|(D_y v^{(0)})_{(x,y)}\|_0\Big] \\ &> |f(x,y,0)| + |f(x,y,v^{(0)}_{(x,y)}) - f(x,y,0)| \geq f(x,y,v^{(0)}_{(x,y)}), \end{aligned}$$

and $v^{(0)} >_* \phi$ on E^0. Taking $u^{(0)}(x,y) = -v^{(0)}(x,y)$ we get the second function.

Note that $v^{(0)}$ is actually the solution of the equation

$$D_{xy}z(x,y) = M + L[z(x,y) + D_x z(x,y) + D_y z(x,y)].$$

If instead of (18) the function f fulfills only the nonlinear estimate (6) then as $v^{(0)}$ we should take a function such that $v^{(0)} >_* \phi$ on E, which is a nonnegative and nondecreasing (together with first order derivatives) solution of the equation

$$D_{xy}z(x,y) = \bar{M} + \sigma(x,y,z(x,y)) + L[D_x z(x,y) + D_y z(x,y)],$$

where $\bar{M} > 0$ is a constant such that $|\sigma(x,y,0)| < \bar{M}$ on E.

Remark 6 If the sequences $\{u^{(m)}\}$ and $\{v^{(m)}\}$ are defined with $\alpha^{(m)} = \beta^{(m)} = \phi$, then consistency conditions (21), (22) for $\mathcal{G}$ and $\mathcal{H}$ reduce to condition (15). In this case we may replace Assumption H_4 with the assumption on existence of one function $\phi \in C^1(E^0, \mathbb{R})$ satisfying consistency condition (15).

Now we prove that $\{u^{(m)}\}$ and $\{v^{(m)}\}$ are convergent to $\tilde{z}$ on E^*.

THEOREM 4 *Suppose that Assumptions* H_2–H_4 *are satisfied and*

(1) *the sequences* $\{u^{(m)}\}$, $\{v^{(m)}\}$ *are defined by* (23);
(2) *the functions* $u^{(0)}, v^{(0)}$ *satisfy the inequality*

$$\|v^{(0)} - u^{(0)}\|_* \le A \quad \text{on } E^*,$$

where $A = L[4K(1+2\tilde{L})\exp\{\tilde{L}(a_0+b_0+a+b)\}]^{-1}$ *and* $\tilde{L}$ *is as in Remark* 5;
(3) *the initial estimates*

$$\|\beta^{(m)} - \alpha^{(m)}\|_* \le \min\{1, \tilde{L}\}\frac{A_0}{2^{2^m}}, \quad m \in \mathbb{N}, \tag{25}$$

where $A_0 = A[(1+2\tilde{L})\exp\{\tilde{L}(a_0+b_0+a+b)\}]^{-1}$, *hold true on* E^0.

Then for any $m \in \mathbb{N}$ *we have*

$$\|v^{(m)} - u^{(m)}\|_* \le \frac{2A}{2^{2^m}} \quad \text{on } E^*. \tag{26}$$

Proof We prove (26) by induction. From condition (2) it follows that (26) is satisfied for $m=0$. Suppose that it holds for some $m \in \mathbb{N}$ and write $\tilde{w}^{(m)} = v^{(m)} - u^{(m)}$. Then for all $(x,y) \in E$ we have

$$\begin{aligned}
&D_{xy}\tilde{w}^{(m+1)}(x,y)\\
&= \mathcal{H}\Big(x,y,v^{(m+1)}_{(x,y)}; u^{(m)}, v^{(m)}\Big) - \mathcal{G}\Big(x,y,u^{(m+1)}_{(x,y)}; u^{(m)}\Big)\\
&= \boldsymbol{f}[v^{(m)}](x,y) - \boldsymbol{f}[u^{(m)}](x,y) + \boldsymbol{D_w f}[u^{(m)}](x,y) \circ \Big(\tilde{w}^{(m+1)}_{(x,y)} - \tilde{w}^{(m)}_{(x,y)}\Big)\\
&= \boldsymbol{D_w f}[u^{(m)}](x,y) \circ \tilde{w}^{(m+1)}_{(x,y)} + \Big(\boldsymbol{D_w f}[u^{(m)} + \theta(v^{(m)} - u^{(m)})](x,y)\\
&\quad - \boldsymbol{D_w f}[u^{(m)}](x,y)\Big) \circ \tilde{w}^{(m)}_{(x,y)},
\end{aligned}$$

from which we get

$$\begin{aligned}
&D_{xy}\tilde{w}^{(m+1)}(x,y)\\
&\le K\left(\frac{2A}{2^{2^m}}\right)^2 + L\Big[\|\tilde{w}^{(m+1)}_{(x,y)}\|_0 + \|(D_x\tilde{w}^{(m+1)})_{(x,y)}\|_0 + \|(D_y\tilde{w}^{(m+1)})_{(x,y)}\|_0\Big].
\end{aligned}$$

The function

$$\omega(x,y)=\left[\frac{K}{L}\left(\frac{2A}{2^{2^m}}\right)^2+\frac{A_0}{2^{2^{m+1}}}\right]\exp\{\tilde{L}(a_0+b_0+x+y)\}-\frac{K}{L}\left(\frac{2A}{2^{2^m}}\right)^2$$

is a solution of the equation

$$D_{xy}\omega(x,y)=K\left(\frac{2A}{2^{2^m}}\right)^2+L\Big[\|\omega_{(x,y)}\|_0+\|(D_x\omega)_{(x,y)}\|_0+\|(D_y\omega)_{(x,y)}\|_0\Big],$$

which fulfills the inequality $\omega\geq_*\min\{1,\tilde{L}\}A_0[2^{2^{m+1}}]^{-1}$ on E^0. Since $\tilde{w}^{(m)}\leq_*\min\{1,\tilde{L}\}A_0[2^{2^{m+1}}]^{-1}$ on E^0 we get

$$\tilde{w}^{(m+1)}\leq_*\omega\quad\text{on } E^*, \tag{27}$$

by Theorem 2 used for f given by

$$f(x,y,w)=K\left(\frac{2A}{2^{2^m}}\right)^2+L\big[\|w^+\|_0+\|(D_xw)^+\|_0+\|(D_yw)^+\|_0\big],$$

where w^+ denotes a nonnegative part of w.

Therefore, by (27) it is easy to get

$$\|\tilde{w}^{(m+1)}\|_*\leq\frac{2A}{2^{2^{m+1}}}\quad\text{on } E^*,$$

and Theorem 4 follows by the induction.

Remark 7 Inequalities (20), (25) yield the uniform convergence of sequences $\{\alpha^{(m)}\}$ and $\{\beta^{(m)}\}$ (together with their derivatives) to the initial function ϕ. Therefore, we obtain consistency condition (15) by letting $m\to\infty$ in (21) or (22), and consequently assumptions of Theorem 4 are sufficient for the existence of the unique solution $\tilde{z}$ of problem (1), (2).

Thus inequalities (24), (26) imply the following error estimates of the differences between the terms of Chaplyghin sequences (together with their derivatives) and $\tilde{z}$:

$$0\leq_*\tilde{z}(x,y)-u^{(m)}(x,y)\leq_*\frac{2A}{2^{2^m}},$$
$$0\leq_*v^{(m)}(x,y)-\tilde{z}(x,y)\leq_*\frac{2A}{2^{2^m}}\quad\text{on } E.$$

4. THE NEWTON METHOD

Let X be a Banach space with the norm $|\cdot|$. In this section we consider the Darboux problem (1), (2) with given functions $\phi: E^0 \to X$, $f: E \times C^1(B, X) \to X$. Let the definition of the norm $\|\cdot\|_*$ in $C^1(E, X)$ be the same as in Section 2, as well as the definition of $\mathcal{G}(\cdot; u): E \times C^1(B, X) \to X$ for given $u \in C^1(E^*, X)$.

Assumption H_5 Suppose that

(1) $\phi, \alpha^{(m)} \in C^1(E^0, X)$, $m \in \mathbb{N}$, and the consistency condition (21) hold for $x \in [0, a]$, $y \in [0, b]$;
(2) $f \in C(E \times C^1(B, X), X)$, there is a continuous derivative $D_w f$ and the Lipschitz conditions (18), (19) hold on $E \times C^1(B, X)$.

We consider the sequence $\{u^{(m)}\}$ defined as follows:

(1) $u^{(0)} \in C^1(E^*, X)$ is any function with restriction to E^0 denoted by $\alpha^{(0)}$;
(2) if $u^{(m)} \in C^1(E^*, \mathbb{R})$ is already defined then $u^{(m+1)}$ is a solution of the problem

$$
\begin{aligned}
D_{xy} z(x, y) &= \mathcal{G}\left(x, y, z_{(x,y)}; u^{(m)}\right) \quad \text{on } E, \\
z(x, y) &= \alpha^{(m+1)}(x, y) \quad \text{on } E^0,
\end{aligned}
\tag{28}
$$

which exists if Assumption H_5 holds.

Remark 8 We call $\{u^{(m)}\}$ the Newton sequence since in case $\alpha^{(m)} = \phi$ it is a sequence generated by the classical Newton method starting from $u^{(0)}$ for the operator $F: C^1(E^*, X) \to C^1(E^*, X)$ defined by

$$
\begin{aligned}
F(u)(x, y) &= u(x, y) - \phi(x, y) \quad \text{if } (x, y) \in E^0, \\
F(u)(x, y) &= u(x, y) - \phi(x, 0) - \phi(0, y) + \phi(0, 0) \\
&\quad - \int_0^x \int_0^y f(s, t, u_{(s,t)})\, \mathrm{d}s\, \mathrm{d}t \quad \text{if } (x, y) \in E.
\end{aligned}
$$

Note that the definition of $\{u^{(m)}\}$ is the same as the definition of one of the Chaplyghin sequences.

THEOREM 5 *Suppose that Assumption* H_5 *holds,* $\tilde{z}$ *is the solution of the problem* (1), (2) *and*

(1) *the sequence* $\{u^{(m)}\}$ *is defined by* (28);
(2) $u^{(0)}$ *satisfies the inequality*

$$\|u^{(0)} - \tilde{z}\|_* \leq A \quad \text{on } E^*,$$

where

$$A = \left\{4K(a+b+ab)\left[1 + \frac{1+2\tilde{L}}{L}\exp\{\tilde{L}(a+b)\}\right]\right\}^{-1}$$

and $\tilde{L}$ *is as in Remark* 5;
(3) *the initial estimates hold on* E^0

$$\|\alpha^{(m)} - \phi\|_* \leq \frac{A_1}{2^{2^m}}, \quad m \in \mathbb{N},$$

where

$$A_1 = \frac{A}{5}\left[1 + \frac{1+2\tilde{L}}{L}\exp\{\tilde{L}(a+b)\}\right]^{-1}.$$

Then for any $m \in \mathbb{N}$ *we have*

$$\|u^{(m)} - \tilde{z}\|_* \leq \frac{2A}{2^{2^m}} \quad on\ E^*. \tag{29}$$

Proof As in the proof of Theorem 4 we use induction. Obviously (29) is satisfied for $m=0$. Suppose that it holds for some $m \in \mathbb{N}$ and write $\tilde{w}^{(m)} = u^{(m)} - \tilde{z}$, $\tilde{p}^{(m)} = D_x u^{(m)} - D_x \tilde{z}$, $\tilde{q}^{(m)} = D_y u^{(m)} - D_y \tilde{z}$. For all $(x,y) \in E$ we have

$$\begin{aligned} D_{xy}\tilde{w}^{(m+1)}(x,y) &= \mathcal{G}\left(x,y,u^{(m+1)}_{(x,y)};u^{(m)}\right) - f\left(x,y,\tilde{z}_{(x,y)}\right) \\ &= \boldsymbol{f}[u^{(m)}](x,y) - \boldsymbol{f}[\tilde{z}](x,y) + \boldsymbol{D_w f}[u^{(m)}](x,y) \circ \left(\tilde{w}^{(m+1)}_{(x,y)} - \tilde{w}^{(m)}_{(x,y)}\right). \end{aligned} \tag{30}$$

Integrating (30) over $[0, x] \times [0, y]$ and using the Hadamard mean value theorem we get

$$
\begin{aligned}
|\tilde{w}^{(m+1)}(x,y)| = \Big|&\tilde{w}^{(m+1)}(x,0) + \tilde{w}^{(m+1)}(0,y) - \tilde{w}^{(m+1)}(0,0) \\
&+ \int_0^x \int_0^y \Big\{ \boldsymbol{f}[u^{(m)}](s,t) - \boldsymbol{f}[\tilde{z}](s,t) \\
&\qquad + \boldsymbol{D_w f}[u^{(m)}](s,t) \circ \Big(\tilde{w}^{(m+1)}_{(s,t)} - \tilde{w}^{(m)}_{(s,t)} \Big) \Big\} \,\mathrm{d}s\,\mathrm{d}t \Big| \\
\le \frac{3A_1}{2^{2^{m+1}}} &+ \int_0^x \int_0^y \Big| \boldsymbol{D_w f}[u^{(m)}](s,t) \circ \tilde{w}^{(m+1)}_{(s,t)} \\
&+ \int_0^1 \Big\{ \boldsymbol{D_w f}[\tilde{z} + \tau(u^{(m)} - \tilde{z})](s,t) \\
&\qquad - \boldsymbol{D_w f}[u^{(m)}](s,t) \Big\} \circ \tilde{w}^{(m)}_{(s,t)} \mathrm{d}\tau \Big| \,\mathrm{d}s\,\mathrm{d}t \\
\le \frac{3A_1}{2^{2^{m+1}}} &+ \int_0^x \int_0^y \Big\{ |\boldsymbol{D_w f}[u^{(m)}](s,t) \circ \tilde{w}^{(m+1)}_{(s,t)}| \\
&\qquad + K \|\tilde{w}^{(m)}_{(s,t)}\|_0^2 \Big\} \,\mathrm{d}s\,\mathrm{d}t \\
\le \frac{3A_1}{2^{2^{m+1}}} &+ abK \left(\frac{2A}{2^{2^m}} \right)^2 \\
&+ \int_0^x \int_0^y L \Big[\|\tilde{w}^{(m+1)}_{(s,t)}\|_0 + \|\tilde{p}^{(m+1)}_{(s,t)}\|_0 + \|\tilde{q}^{(m+1)}_{(s,t)}\|_0 \Big] \mathrm{d}s\,\mathrm{d}t.
\end{aligned}
$$

Putting

$$
\begin{aligned}
\tilde{\tilde{w}}^{(m+1)}(x,y) &= \sup\{|\tilde{w}^{(m+1)}(s,t)|;\ \ (s,t) \in [-a_0, x] \times [-b_0, y]\}, \\
\tilde{\tilde{p}}^{(m+1)}(x,y) &= \sup\{|\tilde{p}^{(m+1)}(s,t)|;\ \ (s,t) \in [-a_0, x] \times [-b_0, y]\}, \\
\tilde{\tilde{q}}^{(m+1)}(x,y) &= \sup\{|\tilde{q}^{(m+1)}(s,t)|;\ \ (s,t) \in [-a_0, x] \times [-b_0, y]\},
\end{aligned}
$$

and $r^{(m+1)} = \tilde{\tilde{w}}^{(m+1)} + \tilde{\tilde{p}}^{(m+1)} + \tilde{\tilde{q}}^{(m+1)}$ we get the estimate

$$
\tilde{\tilde{w}}^{(m+1)}(x,y) \le \frac{3A_1}{2^{2^{m+1}}} + abK \left(\frac{2A}{2^{2^m}} \right)^2 + \int_0^x \int_0^y L r^{(m+1)}(x,t) \,\mathrm{d}s\,\mathrm{d}t. \quad (31)
$$

Analogously integrating (30) on $[0, y]$ and $[0, x]$ we obtain

$$\tilde{\tilde{p}}^{(m+1)}(x, y) \leq \frac{A_1}{2^{2^{m+1}}} + bK\left(\frac{2A}{2^{2^m}}\right)^2 + \int_0^y Lr^{(m+1)}(x, t)\, \mathrm{d}t \tag{32}$$

and

$$\tilde{\tilde{w}}^{(m+1)}(x, y) \leq \frac{A_1}{2^{2^{m+1}}} + aK\left(\frac{2A}{2^{2^m}}\right)^2 + \int_0^x Lr^{(m+1)}(s, y)\, \mathrm{d}s. \tag{33}$$

Adding the inequalities (31)–(33) we finally get

$$\begin{aligned} r^{(m+1)}(x, y) \leq & \frac{5A_1}{2^{2^{m+1}}} + (a + b + ab)K\left(\frac{2A}{2^{2^m}}\right)^2 \\ & + \int_0^x \int_0^y Lr^{(m+1)}(s, t)\, \mathrm{d}s\, \mathrm{d}t \\ & + \int_0^y Lr^{(m+1)}(x, t)\, \mathrm{d}t + \int_0^x Lr^{(m+1)}(s, y)\, \mathrm{d}s. \end{aligned}$$

If we substitute

$$\tilde{r}^{(m+1)}(x, y) = \int_0^x \int_0^y r^{(m+1)}(s, t)\, \mathrm{d}s\, \mathrm{d}t$$

in the above inequality then we get the differential inequality

$$\begin{aligned} D_{xy}\tilde{r}^{(m+1)}(x, y) \leq & \frac{5A_1}{2^{2^{m+1}}} + (a + b + ab)K\left(\frac{2A}{2^{2^m}}\right)^2 \\ & + L\left[\tilde{r}^{(m+1)}(x, y) + D_x\tilde{r}^{(m+1)}(x, y) + D_y\tilde{r}^{(m+1)}(x, y)\right], \end{aligned}$$

with initial values $\tilde{r}^{(m+1)}(x, 0) = \tilde{r}^{(m+1)}(0, y) = 0$. Analogously as (27) we get

$$\tilde{r}^{(m+1)} \leq_* \tilde{\omega} \quad \text{on } E^*,$$

where

$$\tilde{\omega}(x, y) = \frac{1}{L}\left[\frac{5A_1}{2^{2^{m+1}}} + (a + b + ab)K\left(\frac{2A}{2^{2^m}}\right)^2\right]\left[\exp\{\tilde{L}(x + y)\} - 1\right].$$

Thus

$$\begin{aligned} r^{(m+1)}(x,y) &\le \frac{5A_1}{2^{2^{m+1}}} + (a+b+ab)K\left(\frac{2A}{2^{2^m}}\right)^2 \\ &\quad + L\big[\tilde{\omega}(x,y) + D_x\tilde{\omega}(x,y) + D_y\tilde{\omega}(x,y)\big] \\ &\le \left[\frac{5A_1}{2^{2^{m+1}}} + (a+b+ab)K\left(\frac{2A}{2^{2^m}}\right)^2\right] \\ &\quad \times \left[1 + \frac{1+2\tilde{L}}{L}\exp\{\tilde{L}(a+b)\}\right] \\ &= \frac{2A}{2^{2^{m+1}}} \end{aligned}$$

on E^*, and we get (29) by induction, which completes the proof of Theorem 5.

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