A Short Proof of the Best Possibility for the Grand Furuta Inequality

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In this note, we give a short proof to the best possibility for the grand Furuta inequality: for given p, $s \geq 1$, $t \in [0, 1]$, $r \geq t$ and $\alpha > 1$, there exist positive invertible operators S and T such that $S \geq T$ and

$$S^{(1-t+r)\alpha} \not\geq [S^{r/2}(S^{-t/2}T^pS^{-t/2})^sS^{r/2}]^{((1-t+r)/((p-t)s+r))\alpha}.$$

1. INTRODUCTION

Throughout this note, an operator T means a bounded linear operator acting on a Hilbert space H. An operator A is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$, and we denote $A > 0$ if $A \geq 0$ is invertible.

One of the most important inequalities is the Löwner–Heinz inequality:

$$A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha \quad \text{for } \alpha \in [0, 1]. \tag{1}$$

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Furthermore it is known that $[0, 1]$ is the best possible for (1). That is, for $\alpha > 1$ there exist $A, B > 0$ such that

$$A \geq B \geq 0 \quad \text{and} \quad A^\alpha \ngeq B^\alpha. \tag{2}$$

In 1987, Furuta established the following historical extension of (1), which is called the Furuta inequality now:
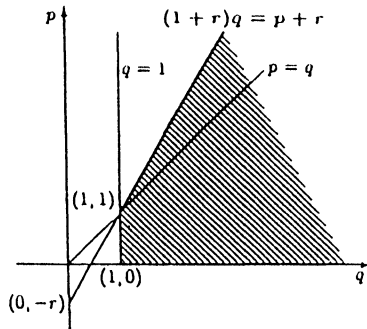
FURUTA INEQUALITY [11]   *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$A^{(p+r)/q} \geq (A^{r/2} B^p A^{r/2})^{1/q} \tag{3}$$

holds for all $p \geq 0$ and $q \geq 1$ such that

$$(1 + r)q \geq p + r. \tag{$*$}$$

The condition $(*)$ is expressed as in this figure.

See [12] for a one-page proof and also [4,21]. Recently the best possibility of the Furuta inequality was discussed by Tanahashi [22]. He proved that the condition $(*)$ is complete. More precisely,

THEOREM A   *Let $p > 0$ and $r \geq 0$ be given. If either $0 < q < 1$ or $(1 + r)q < p + r$, then there exist A and B such that $A \geq B > 0$ and*

$$A^{(p+r)/q} \ngeq (A^{r/2} B^p A^{r/2})^{1/q}.$$

In the case of $p \geq 1$, Theorem A is rephrased as follows:

THEOREM A'   *Let $p \geq 1$ and $r \geq 0$ be given. For $\alpha > 1$, there exist A and B such that $A \geq B > 0$ and*

$$A^{(1+r)\alpha} \not\geq (A^{r/2}B^pA^{r/2})^{((1+r)/(p+r))\alpha}.$$

2. GRAND FURUTA INEQUALITY

In 1995, Furuta [14] extended his inequality to an interpolational form combining with the Ando–Hiai inequality [2], which is called the grand Furuta inequality in [10]:

GRAND FURUTA INEQUALITY   *If $S \geq T \geq 0$ and $S > 0$, then for each $t \in [0, 1]$*

$$S^{1-t+r} \geq [S^{r/2}(S^{-t/2}T^pS^{-t/2})^sS^{r/2}]^{(1-t+r)/((p-t)s+r)} \tag{4}$$

holds for all $p, s \geq 1$ and $r \geq t$.

It was given a mean theoretic proof in [10] and very recently an elementary one-page proof in [15]. See also [16–20]. Now Tanahashi [23] considered the best possibility for the grand Furuta inequality:

THEOREM B   *Let $p, s \geq 1$, $t \in [0, 1]$, $r \geq t$. Then for each $\alpha > 1$ there exist $S, T > 0$ such that $S \geq T$ and*

$$S^{(1-t+r)\alpha} \not\geq [S^{r/2}(S^{-t/2}T^pS^{-t/2})^sS^{r/2}]^{((1-t+r)/((p-t)s+r))\alpha}. \tag{5}$$

His discussion is analogous to Theorem A by himself and so more complicated. Very recently Yamazaki [24] presents a simplified proof to Theorem B, which is based on the Furuta inequality, Theorem A and Yanagida's recent result [26] on the best possibility for a Furuta's type operator inequality equivalent to the chaotic order $\log A \geq \log B$ for $A, B > 0$, cited below:

THEOREM C   *Let $p > 0$ and $r > 0$ be given. For $\alpha > 1$, there exist $A, B > 0$ such that $\log A \geq \log B$ and*

$$A^{r\alpha} \not\geq (A^{r/2}B^pA^{r/2})^{(r/(p+r))\alpha}.$$

We note that Theorem C says the best possibility for the following characterization of the chaotic order, see [1,3,5–9,13,25]: for $A, B > 0$, $\log A \geq \log B$ if and only if

$$A^r \geq (A^{r/2} B^p A^{r/2})^{r/(p+r)}$$

holds for all $p, r \geq 0$.

Yamazaki's simplified proof in [24] was surprising to us because both Theorems A' and C were used very well. To prove Theorem B, he divides into two cases; $0 \leq t < 1$ and $t = 1$. The former needs Theorem A' and the latter does Theorem C. This striking contrast is the motivation of this note. We present a short proof to Theorem B with no use of Theorem C, in this note. Though our basic idea is essentially similar to Yamazaki's one, we use Theorem A' only, where (2) is regarded as the special case $p = 1$ and $r = 0$ in Theorem A'.

3. THE BEST POSSIBILITY OF GRAND FURUTA INEQUALITY

In this section, we give a straightforward proof to Theorem B.

Proof of Theorem B   Assume that $p \geq 1, s \geq 1, r \geq t, t \in (0, 1]$ and $\alpha > 1$ are given. Incidentally, the case $t = 0$ is just Theorem A' and so it can be omitted.

First of all, under the assumption $p > t$, we take $\beta = 1/(1 - t)$ if $0 < t < 1$ and β is sufficiently large if $t = 1$. Next we put

$$r_1 = r\beta, \quad \delta = \frac{t\beta}{2}, \quad p_1 = (p - t)s\beta \quad \text{and} \quad \alpha_1 = \frac{1 - t + r}{1 + r_1} \alpha\beta. \quad (6)$$

Then we have $r_1, \delta \geq 0$, $p_1 \geq 1$ and $\alpha_1 > 1$. Hence it follows from Theorem A' that there exist $A, B > 0$ such that $A \geq B > 0$ and

$$A^{(1+r_1)\alpha_1} \not\geq (A^{r_1/2} B^{p_1} A^{r_1/2})^{((1+r_1)/(p_1+r_1))\alpha_1}. \quad (7)$$

We here put

$$S = A^\beta \quad \text{and} \quad T = (A^\delta B^{p_1/s} A^\delta)^{1/p};$$

we have an example for Theorem B. As a matter of fact, $S \geq T$ is ensured by the Furuta inequality (3) because $p \geq 1$, $p_1/s \geq 0$, $\delta > 0$ and

$(1 + 2\delta)p \geq p_1/s + 2\delta$. On the other hand, it is easily checked that (5) is just the same as (7) by the set of (6).

Finally we give a counterexample for the case $p = t = 1$ (and $r \geq t = 1$, $s \geq 1$, $\alpha > 1$). For this, we apply (2), that is, there exist $A, B > 0$ such that $A \geq B$ and $A^\alpha \ngeq B^\alpha$. And we put

$$S = A^{1/r}$$

and

$$T = S^{1/2}(S^{-r/2}BS^{-r/2})^{1/s}S^{1/2} = A^{1/2r}(A^{-1/2}BA^{-1/2})^{1/s}A^{1/2r};$$

in other words,

$$A = S^r \quad \text{and} \quad B = S^{r/2}(S^{-1/2}TS^{-1/2})^s S^{r/2},$$

then S and T are as desired. Actually $S \geq T$ is shown as follows:

$$
\begin{aligned}
S \geq T &\iff 1 \geq (S^{-1/2}TS^{-1/2})^s \\
&\iff S^r \geq S^{r/2}(S^{-1/2}TS^{-1/2})^s S^{r/2} \\
&\iff A \geq B.
\end{aligned}
$$

Furthermore $A^\alpha \ngeq B^\alpha$ is an equivalent expression of (5) in this case $p = t = 1$. So the proof is complete.

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