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Inequalities of Furuta and Mond–Pečarić on the Hadamard Product

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As a continuation of (J. Mičić, Y. Seo, S.-E. Takahasi and M. Tominaga, Inequalities of Furuta and Mond-Pečarić, *Math. Ineq. Appl.*, **2** (1999), 83–111), we shall discuss complementary results to Jensen's type inequalities on the Hadamard product of positive operators, which is based on the idea due to Furuta and Mond–Pečarić. We shall show Hadamard product versions of operator inequalities associated with extensions of Hölder–McCarthy and Kantorovich inequalities established by Furuta, Ky Fan and Mond–Pečarić.

Keywords: Hadamard product; Jensen's type inequalities

1. INTRODUCTION

According to [3] and [14], the Hadamard product of operators on a Hilbert space H is defined as follows: If U is an isometry of H into $H \otimes H$ such that $Ue_n = e_n \otimes e_n$ where $\{e_n\}$ is a fixed orthonormal basis of H, then

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the Hadamard product $A*B$ of (bounded) operators A and B on H for $\{e_n\}$ is expressed as

$$A*B=U^*(A\otimes B)U. \tag{1}$$

As an effect of (1), Fujii [3, Theorem 3] showed the following Jensen's type inequalities on the Hadamard product of positive operators, which is an operator version for the Aujla–Vasudeva theorem [2, Theorem 3.2]: If f is a submultiplicative operator convex (resp. supermultiplicative operator concave) function on $(0,\infty)$, then

$$f(A*B)\le f(A)*f(B) \quad (\text{resp. } f(A*B)\ge f(A)*f(B)) \tag{2}$$

for all $A,B\ge 0$.

Also, Liu and Neudecker [9] showed several matrix Kantorovich-type inequalities on the Hadamard product, and Mond–Pečarić moreover extended them in [13], also see [7]: If A and B are positive semidefinite Hermitian matrices such that $0<m\le A\otimes B\le M$, then

$$A^2*B^2-(A*B)^2\le \tfrac{1}{4}(M-m)^2, \tag{3}$$

$$(A^2*B^2)^{1/2}\le \frac{M+m}{2\sqrt{Mm}}(A*B). \tag{4}$$

We note that (3) and (4) are complementary to (2) for $f(t)=t^2$ and $\sqrt{t}$ respectively.

On the other hand, Furuta [5,6], Ky Fan [8] and Mond–Pečarić [11,12] showed several operator inequalities associated with extensions of Hölder–McCarthy and Kantorovich inequalities. For instance, Furuta showed that if A is a positive operator on H such that $0<m\le A\le M$, f is a real valued continuous convex function on $[m,M]$ and q is a real number such that $q>1$ or $q<0$, then there exists a constant $C(q)$ such that the following inequality

$$(f(A)x,x)\le C(q)(Ax,x)^q \tag{5}$$

holds for every unit vector $x\in H$ under some conditions. Also, one of the authors and *et al.* [15] showed a complementary result to Jensen inequality for convex functions: If f is a real valued continuous strictly convex function on $[m,M]$ such that $f(t)>0$, then for a given $\alpha>0$ there exists the most suitable constant β such that

$$(f(A)x,x) \le \alpha f((Ax,x)) + \beta \tag{6}$$

holds for every unit vector $x \in H$.

Moreover, in the previous paper [10], by combinig (5) with (6), we showed the following complementary results to operator inequalities associated with extensions of Hölder–McCarthy and Kantorovich inequalities established by Furuta, Ky Fan and Mond–Pečarić: If f is a real valued continuous convex function on $[m, M]$ and g is a real valued continuous function on $[m, M]$, then for a given real number α there exists the most suitable constant β such that

$$(f(A)x,x) \le \alpha g((Ax,x)) + \beta \tag{7}$$

holds for every unit vector $x \in H$ under some conditions.

In this note, we shall discuss complementary results to Jensen's type inequalities on the Hadamard product of positive operators corresponding to (7), which is based on the idea due to Furuta and Mond–Pečarić. We shall show Hadamard product versions of operator inequalities associated with extensions of Hölder–McCarthy and Kantorovich inequalities established by Furuta, Ky Fan and Mond–Pečarić.

2. JENSEN'S TYPE INEQUALITIES

Ando [1, Theorems 10 and 11] showed Jensen's type inequalities on the Hadamard product of positive definite matrices by applying concavity and convexity theorems. Also, Furuta [4, Theorem 2], Aujla–Vasudeva [2, Corollary 3.4], Fujii [3, Corollary 4] and Mond–Pečarić [12, Theorem 2.1] showed another Jensen's type inequalities on the Hadamard product.

First, we shall show that these inequalities are mutually equivalent.

THEOREM 1 *Let A and B be positive operators on a Hilbert space H. Then the following inequalities hold and follow from each other.*

(i)

$$A * B \le (A^p * B^p)^{1/p} \quad \textit{if } 1 \le p,$$
$$A * B \ge (A^p * B^p)^{1/p} \quad \textit{if either } \tfrac{1}{2} \le p \le 1 \textit{ or } p \le -1.$$

(ii)

$$A^p * B^p \leq (A * B)^p \quad \textit{if } 0 \leq p \leq 1,$$
$$A^p * B^p \geq (A * B)^p \quad \textit{if either } 1 \leq p \leq 2 \textit{ or } -1 \leq p \leq 0.$$

(iii)

$$(A^r * B^r)^{1/r} \leq (A^s * B^s)^{1/s} \quad \textit{if either } r \leq s \textit{ with } r, s \notin (-1, 1),$$
$$\textit{or } \tfrac{1}{2} \leq r \leq 1 \leq s \textit{ or } r \leq -1 \leq s \leq -\tfrac{1}{2}.$$

Proof (i) $\Rightarrow$ (iii) For the case of $1 \leq r \leq s$, put $p = s/r \geq 1$. Then we have $A * B \leq (A^{s/r} * B^{s/r})^{r/s}$ by (i). Replacing A by A^r and B by B^r and raising to the power $0 < 1/r \leq 1$, we have $(A^r * B^r)^{1/r} \leq (A^s * B^s)^{1/s}$ by the Löwner–Heinz inequality. Similarly for the other cases.

(iii) $\Rightarrow$ (ii) For $0 < p \leq 1$, put $s = 1/p \geq 1$ and $r = 1$ in (iii), then we have $A * B \leq (A^{1/p} * B^{1/p})^p$. We have only to replace A by A^p and B by B^p.

(ii) $\Rightarrow$ (i) Consider $1/p$ in (ii) for $1 \leq p < \infty$ and replacing A by A^p and B by B^p in (ii).

For the sake of convenience, we prepare some notations. Let A and B be positive operators on a Hilbert space H such that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$. Put $m = m_1 m_2$ and $M = M_1 M_2$. Also, let $f(t)$ be a real valued continuous function defined on an interval including $[m, M]$. Then f is called supermultiplicative (resp. submultiplicative) if $f(xy) \geq f(x)f(y)$ (resp. $f(xy) \leq f(x)f(y)$). We define:

$$a_f = \frac{f(M) - f(m)}{M - m}, \quad b_f = \frac{Mf(m) - mf(M)}{M - m}$$

and

$$X_f = [m_1, M_1] \cup [m_2, M_2] \cup [m, M].$$

Also, we introduce the following constant by Furuta [6]:

$$C_f(m, M; q) = \frac{mf(M) - Mf(m)}{(q-1)(M-m)} \left(\frac{(q-1)(f(M) - f(m))}{q(mf(M) - Mf(m))} \right)^q,$$

where q is a real number such that $q > 1$ or $q < 0$. It is denoted simply by $C(q)$.

We shall show an Hadamard product version corresponding to [10, Theorem 2] which is based on the idea to Furuta and Mond–Pečarić.

THEOREM 2 *Let A and B be positive operators on a Hilbert space H such that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$. Put $m = m_1 m_2$ and $M = M_1 M_2$. Suppose that either of the following conditions holds*

(i) *$f(t)$ is a real valued continuous supermultiplicative convex function defined on the interval X_f or;*
(ii) *$f(t)$ is a real valued continuous submultiplicative concave function defined on the interval X_f.*

Let $g(t)$ be a real valued continuous function on $[m, M]$, J_1 an interval including $\{f(t)f(s)\colon t \in [m_1, M_1], s \in [m_2, M_2]\}$ and J_2 an interval including $\{g(t)\colon t \in [m, M]\}$. If $F(u, v)$ is a real valued function defined on $J_1 \times J_2$, operator monotone in u, then the following inequalities hold

$$F(f(A) * f(B), g(A * B)) \leq \max_{t \in [m, M]} F(a_f t + b_f, g(t))I \tag{8}$$

in case (i), *or*

$$F(f(A) * f(B), g(A * B)) \geq \min_{t \in [m, M]} F(a_f t + b_f, g(t))I \tag{9}$$

in case (ii).

Proof Though the proof is quite similar to Mond and Pečarić [11, Theorem 4], we give a proof for the sake of completeness. Let the condition (i) be satisfied. Since f is convex, we have $f(t) \leq a_f t + b_f$ for any $t \in [m, M]$. Thus we obtain $f(A \otimes B) \leq a_f(A \otimes B) + b_f$ since $0 < m \leq A \otimes B \leq M$, so that it follows from the supermultiplicativity of f that

$$\begin{aligned} f(A) * f(B) &= U^*(f(A) \otimes f(B))U \leq U^* f(A \otimes B)U \\ &\leq U^*(a_f(A \otimes B) + b_f)U = a_f(A * B) + b_f I. \end{aligned}$$

By the monotonicity of F, we have

$$\begin{aligned} F(f(A) * f(B), g(A * B)) &\leq F(a_f(A * B) + b_f I, g(A * B)) \\ &\leq \max_{t \in [m, M]} F(a_f t + b_f, g(t))I. \end{aligned}$$

Thus we obtain the inequality (8). The proof in case (ii) is essentially the same.

Remark 3 Note that we do not assume the operator convexity or operator concavity of f.

As a complementary result, we cite the following theorem:

THEOREM 2′ *Let A, B, g, J_1 and J_2 be as in Theorem 2. Let $-F(u, v)$ is a real valued function defined on $J_1 \times J_2$, operator monotone in u.*

If $f(t)$ is a real valued continuous submultiplicative concave function defined on the interval X_f then

$$F(f(A) * f(B), g(A * B)) \leq \max_{t \in [m,M]} F(a_f t + b_f, g(t))I, \tag{10}$$

but if $f(t)$ is a real valued continuous supermultiplicative convex function defined on the interval X_f then

$$F(f(A) * f(B), g(A * B)) \geq \min_{t \in [m,M]} F(a_f t + b_f, g(t))I. \tag{11}$$

Remark 4 Note that the function $F(u, v) = u - \alpha v$ for a real number α and $F(u, v) = v^{-1/2} u v^{-1/2}$ $(v > 0)$ are both operator monotone in their first variables.

3. COMPLEMENTARY TO JENSEN'S TYPE INEQUALITIES

In this section, by virtue of Theorem 2, we shall show complementary results to Jensen's type inequalities on the Hadamard product of positive operator which corresponds to (7).

THEOREM 5 *Let A and B be positive operators on a Hilbert space H such that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$. Put $m = m_1 m_2$ and $M = M_1 M_2$. Let $f(t)$ be a real valued continuous supermultiplicative convex (resp. submultiplicative concave) function defined on the interval X_f and $g(t)$ a real valued continuous functions defined on $[m, M]$. Then for a given real number α*

$$f(A) * f(B) \leq \alpha g(A * B) + \beta I \quad (resp.\ f(A) * f(B) \geq \alpha g(A * B) + \beta I)$$

holds for

$$\beta = \max_{t\in[m,M]}\{a_f t + b_f - \alpha g(t)\} \quad (resp.\ \beta = \min_{t\in[m,M]}\{a_f t + b_f - \alpha g(t)\}).$$

Proof We only prove it for supermultiplicative concave case. Let us put $F(u, v) = u - \alpha v$ in Theorem 2. Then it follows from the inequality (8) that

$$\begin{aligned} f(A) * f(B) - \alpha g(A * B) &\leq \max_{t\in[m,M]} F(a_f t + b_f, g(t))I \\ &= \max_{t\in[m,M]}\{a_f t + b_f - \alpha g(t)\}I, \end{aligned}$$

which gives the desired inequality.

Remark 6 If we put $\alpha = 1$ in Theorem 5 and $g(t)$ is a real valued strictly convex (resp. strictly concave) twice differentiable function defined on $[m, M]$, then we have the following:

$$f(A) * f(B) - g(A * B) \leq \beta I \quad (\text{resp.}\ f(A) * f(B) - g(A * B) \geq \beta I)$$

holds for $\beta = a_f \cdot t_0 + b_f - g(t_0)$ and

$$t_0 = \begin{cases} M & \text{if } g'(M) \leq a_f \\ m & \text{if } g'(m) \geq a_f \\ g'^{-1}(a_f) & \text{otherwise.} \end{cases}$$

Further if we choose α such that $\beta = 0$ in Theorem 5 then we have the following corollary.

COROLLARY 7 *Let A, B and f be as in Theorem 5 and $g(t)$ a strictly convex (resp. strictly concave) twice differentiable function defined on $[m, M]$.*
If $g(t) > 0$ on $[m, M]$ and $f(m) > 0, f(M) > 0$ then

$$f(A) * f(B) \leq \alpha_1 g(A * B) \quad (resp.\ f(A) * f(B) \geq \alpha_2 g(A * B)),$$

but, if $g(t) < 0$ on $[m, M]$ and $f(m) < 0, f(M) < 0$ then

$$f(A) * f(B) \leq \alpha_2 g(A * B) \quad (resp.\ f(A) * f(B) \geq \alpha_1 g(A * B)),$$

both hold for

$$\alpha_1 = \begin{cases} (a_f \cdot t_0 + b_f)/g(t_0) \\ \quad \text{if } [a_f g(M) - f(M)g'(M)][a_f g(m) - f(m)g'(m)] < 0 \\ \max\left\{\dfrac{f(m)}{g(m)}, \dfrac{f(M)}{g(M)}\right\} \\ \quad \text{otherwise} \end{cases}$$

and

$$\alpha_2 = \begin{cases} (a_f \cdot t_0 + b_f)/g(t_0) \\ \quad \text{if } [a_f g(M) - f(M)g'(M)][a_f g(m) - f(m)g'(m)] < 0 \\ \min\left\{\dfrac{f(m)}{g(m)}, \dfrac{f(M)}{g(M)}\right\} \\ \quad \text{otherwise.} \end{cases}$$

In the formulas of α we designed t_0 the unique solution of the equation

$$a_f g(t) - g'(t)(a_f \cdot t + b_f) = 0.$$

Proof This proof is quite similar to the one in [10, Theorem 6]. Indeed, for case $g(t) > 0$ apply Theorem 2 and for case $g(t) < 0$ apply Theorem 2′ both with $F(u, v) = v^{-1/2}uv^{-1/2}$ $(v > 0)$. We define $h(t) \equiv (a_f \cdot t + b_f)/g(t)$. If $h'(m)h'(M) < 0$ then the equation $h'(t) = 0$ has the unique solution $t_0 \in (m, M)$. If $h'(m)h'(M) \geq 0$ then it follows that $h(t)$ is monotone on $[m, M]$ and its extreme occurs at m or at M.

If $g(t) = t^q$, then we have the following theorem which is considered as the Hadamard product version of [6, Theorem 1.1].

THEOREM 8 *Let A and B be positive operators on a Hilbert space H such that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$. Put $m = m_1 m_2$, $M = M_1 M_2$ and q a real number. If $f(t)$ is a real valued continuous supermultiplicative convex function defined on the interval X_f, then for a given real number α*

$$f(A) * f(B) \leq \alpha(A * B)^q + \beta I \tag{12}$$

holds for

$$\beta = \begin{cases} \alpha(q-1)\left(\dfrac{a_f}{\alpha q}\right)^{q/(q-1)} + b_f \\ \quad \text{if } m \leq \left(\dfrac{a_f}{\alpha q}\right)^{1/(q-1)} \leq M \text{ and } \alpha q(q-1) > 0 \\ \max\{f(M) - \alpha M^q,\ f(m) - \alpha m^q\} \\ \quad \text{otherwise.} \end{cases}$$

But if $f(t)$ is a real valued continuous submultiplicative concave function defined on the interval X_f then for a given real number α

$$f(A)*f(B) \geq \alpha(A*B)^q + \beta I \tag{13}$$

holds for

$$\beta = \begin{cases} \alpha(q-1)\left(\dfrac{a_f}{\alpha q}\right)^{q/(q-1)} + b_f \\ \quad \text{if } m \leq \left(\dfrac{a_f}{\alpha q}\right)^{1/(q-1)} \leq M \text{ and } \alpha q(q-1) < 0 \\ \min\{f(M) - \alpha M^q, f(m) - \alpha m^q\} \\ \quad \text{otherwise.} \end{cases}$$

Proof We only prove it for the supermultiplicative convex case. Put $h(t) = a_f t + b_f - \alpha t^q$ $(t > 0)$ and suppose that $a_f/(\alpha q) > 0$. Put $t_1 = (a_f/(\alpha q))^{1/(q-1)}$ and hence we have that $h'(t) = 0$ if and only if $t = t_1$. If $m \leq (a_f/(\alpha q))^{1/(q-1)} \leq M$ and $\alpha q(q-1) > 0$, then $h''(t) = -\alpha q(q-1)t^{q-2} < 0$ $(t > 0)$ and so $\max_{t\in[m,M]} h(t) = h(t_1)$. If $m \leq (a_f/(\alpha q))^{1/(q-1)} \leq M$ and $\alpha q(q-1) < 0$, then $h''(t) > 0$ $(t > 0)$ and so $\max_{t\in[m,M]} h(t) = \max\{h(M), h(m)\}$. Also if either $(a_f/(\alpha q))^{1/(q-1)} < m$ or $M < (a_f/(\alpha q))^{1/(q-1)}$, then $h(t)$ is either nonincreasing or nondecreasing on $[m, M]$ and hence $\max_{t\in[m,M]} h(t) = \max\{h(M), h(m)\}$.

Next suppose that $a_f/(\alpha q) < 0$. Then $a_f/(\alpha q) - t^{q-1} < 0$ $(t > 0)$, hence $h'(t) = \alpha q(a_f/(\alpha q) - t^{q-1})$ has the same sign on $(0, \infty)$ and so $\max_{t\in[m,M]} h(t) = \max\{h(M), h(m)\}$.

Finally suppose that $a_f = 0$ or $\alpha q = 0$ or $q = 1$. Then $h(t)$ is evidently nonincreasing or nondecreasing on $(0, \infty)$ and hence $\max_{t\in[m,M]} h(t) = \max\{h(M), h(m)\}$. Note that

$$h(t_1) = \alpha(q-1)\left(\frac{a_f}{\alpha q}\right)^{q/(q-1)} + b_f$$

and

$$\max\{h(M), h(m)\} = \max\{f(M) - \alpha M^q, f(m) - \alpha m^q\}.$$

Therefore, we obtain the desired result by virtue of Theorem 5.

Remark 9 If we put $\alpha = 1$ in Theorem 8 then we have the following: If $f(t)$ is a real valued continuous supermultiplicative convex (resp. submultiplicative concave) function on X_f and q a real number then

$$f(A) * f(B) \leq (A * B)^q + \beta_1 I \quad (\text{resp.}\ f(A) * f(B) \geq (A * B)^q + \beta_2 I)$$

holds for

$$\beta_1 = \begin{cases} (q-1)(a_f/q)^{q/(q-1)} + b_f \\ \quad \text{if } m \leq (a_f/q)^{1/(q-1)} \leq M \text{ and } q \notin (0,1) \\ \max\{f(M) - M^q, f(m) - m^q\} \\ \quad \text{otherwise,} \end{cases}$$

(resp.

$$\beta_2 = \begin{cases} (q-1)(a_f/q)^{q/(q-1)} + b_f \\ \quad \text{if } m \leq (a_f/q)^{1/(q-1)} \leq M \text{ and } q \in (0,1) \\ \min\{f(M) - M^q, f(m) - m^q\} \\ \quad \text{otherwise).} \end{cases}$$

Further if we choose α such that $\beta = 0$ in Theorem 8 then we have the following corollary.

COROLLARY 10 *Let A, B, f and q be as Theorem 8. Then*

$$f(A) * f(B) \leq \alpha_1 (A * B)^q \quad (\textit{resp.}\ f(A) * f(B) \geq \alpha_2 (A * B)^q)$$

holds for

$$\alpha_1 = \begin{cases} \dfrac{a_f}{q}\left(\dfrac{b_f}{a_f}\dfrac{q}{1-q}\right)^{1-q} \\ \quad \textit{if } m \leq \dfrac{b_f}{a_f}\dfrac{q}{1-q} \leq M \textit{ and } a_f(q-1) > 0 \\ \max\{f(m)/m^q, f(M)/M^q\} \\ \quad \textit{otherwise.} \end{cases}$$

(*resp.*

$$\alpha_2 = \begin{cases} \dfrac{a_f}{q}\left(\dfrac{b_f}{a_f}\dfrac{q}{1-q}\right)^{1-q} \\ \quad \textit{if } m \leq \dfrac{b_f}{a_f}\dfrac{q}{1-q} \leq M \textit{ and } a_f(q-1) < 0 \\ \min\{f(m)/m^q, f(M)/M^q\} \\ \quad \textit{otherwise).} \end{cases}$$

Proof This proof is quite similar to the one in Theorem 8. Indeed, hence $h(t)=(a_f\cdot t+b_f)/t^q$ for $t>0$. We have $h'(t)=0$ for $t\equiv t_1=b_f/a_f\cdot q/(1-q)$ and $h''(t_1)<0$ if and only if $a_f(q-1)>0$ (resp. $h''(t_1)>0$ if and only if $a_f(q-1)<0$). Further if $m\le t_1\le M$ then $\alpha_1=h(t_1)$ (resp. $\alpha_2=h(t_1)$). Otherwise $h(t)$ is monotone on $[m,M]$ and its extreme occurs at m or at M.

The following corollary is complementary to (2), which is considered as an Hadamard product version of [15, Theorem 1].

COROLLARY 11 *Let A and B be positive operators on a Hilbert space H such that $0<m_1\le A\le M_1$ and $0<m_2\le B\le M_2$. Put $m=m_1m_2$ and $M=M_1M_2$. Let $f(t)$ be a real valued continuous supermultiplicative strictly convex (resp. submultiplicative strictly concave) twice differentiable function on X_f then for a given $\alpha>0$*

$$f(A)*f(B)\le \alpha f(A*B)+\beta I \quad (\textit{resp.}\, f(A)*f(B)\ge \alpha f(A*B)+\beta I)$$

holds for $\beta=-\alpha f(t_0)+a_ft_0+b_f$ and

$$t_0=\begin{cases} M & \text{if } M\le f'^{-1}\left(\frac{a_f}{\alpha}\right)\\ m & \text{if } f'^{-1}\left(\frac{a_f}{\alpha}\right)\le m\\ f'^{-1}\left(\frac{a_f}{\alpha}\right) & \text{otherwise.}\end{cases}$$

Proof By virtue of Theorem 5, it is sufficient to see that $\beta=-\alpha f(t_0)+a_ft_0+b_f$. Put $h(t)=a_ft+b_f-\alpha f(t)$. By an easy differentiable calculus, we have $h'(t_0)=0$ when $t_0=f'^{-1}(a_f/\alpha)$ and t_0 gives the upper bound of $h(t)$ on $[m,M]$ if $t_0\in(m,M)$. Also, if $t_0\notin(m,M)$, then $h(t)$ is either nonincreasing or nondecreasing on $[m,M]$.

Remark 12 If we put $\alpha=1$ in Corollary 11 then we have the following: If $f(t)$ is a real valued continuous supermultiplicative strictly convex (resp. submultiplicative strictly concave) twice differentiable function on X_f then

$$f(A)*f(B)-f(A*B)\le\beta I \quad (\text{resp.}\, f(A)*f(B)-f(A*B)\ge\beta I)$$

holds for $\beta = a_f \cdot t_0 + b_f - f(t_0)$ and

$$t_0 = \begin{cases} M & \text{if } f'(M) \leq a_f \\ m & \text{if } f'(m) \geq a_f \\ f'^{-1}(a_f) & \text{otherwise.} \end{cases}$$

Further if we choose α such that $\beta = 0$ in Corollary 11 then we have the following corollary.

COROLLARY 13 *Let A, B and f be as Corollary* 11.
If $f(t) > 0$ on X_f then

$$f(A) * f(B) \leq \alpha f(A * B) \quad (\textit{resp. } f(A) * f(B) \geq \alpha f(A * B)),$$

but if $f(t) < 0$ on X_f then

$$f(A) * f(B) \geq \alpha f(A * B) \quad (\textit{resp. } f(A) * f(B) \leq \alpha f(A * B)),$$

all hold for $\alpha = (a_f \cdot t_0 + b_f)/f(t_0)$ and t_0 is the unique solution of the equation $a_f f(t) = f'(t)(a_f \cdot t + b_f)$.

Proof For case $f(t) > 0$ apply Theorem 2 and for case $f(t) < 0$ apply Theorem 2′ both with $F(u, v) = v^{-1/2} u v^{-1/2} (v > 0)$. The equation $h'(t) \equiv a_f - f'(t) = 0$ has the unique solution in (m, M). Since, if $f(t)$ is strictly convex (resp. strictly concave) differentiable function then $f(x) - f(y) > (x - y) f'(y)$ (resp. $f(x) - f(y) < (x - y) f'(y)$) for $x \in O(y)$ then for the continuous function $h'(t)$ we have $h'(m) h'(M) < 0$.

4. APPLICATIONS

In this section, we shall show applications of Theorem 8 for potential and exponential functions. Note that the potential function $f(t) = t^p$ is a real valued continuous supermultiplicative strictly convex (resp. submultiplicative strictly concave) differentiable if $p \notin [0, 1]$ (resp. $p \in (0, 1)$).

First, we state the following corollary obtained by applying $f(t) = t^p$ to Theorem 8.

COROLLARY 14 *Let A and B be positive operators on a Hilbert space H such that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$. Put $m = m_1 m_2$ and*

$M = M_1M_2$. If $p \notin [0,1]$ (resp. $p \in (0,1)$) and q a real number, then for a given real number α

$$A^p * B^p \leq \alpha(A*B)^q + \beta_1 I \quad (\textit{resp. } A^p * B^p \geq \alpha(A*B)^q + \beta_2 I)$$

holds for

$$\beta_1 = \begin{cases} \alpha(q-1)\left(\dfrac{1}{\alpha q}\bar{a}\right)^{q/(q-1)} + \bar{b} \\ \quad \textit{if } m \leq \left(\dfrac{1}{\alpha q}\bar{a}\right)^{1/(q-1)} \leq M \textit{ and } \alpha q(q-1) > 0 \\ \max\{m^p - \alpha m^q, M^p - \alpha M^q\} \\ \quad \textit{otherwise,} \end{cases}$$

(resp.

$$\beta_2 = \begin{cases} \alpha(q-1)\left(\dfrac{1}{\alpha q}\bar{a}\right)^{q/(q-1)} + \bar{b} \\ \quad \textit{if } m \leq \left(\dfrac{1}{\alpha q}\bar{a}\right)^{1/(q-1)} \leq M \textit{ and } \alpha q(q-1) < 0 \\ \min\{m^p - \alpha m^q, M^p - \alpha M^q\} \\ \quad \textit{otherwise}), \end{cases}$$

where $\bar{a} = (M^p - m^p)/(M-m)$ and $\bar{b} = (Mm^p - mM^p)/(M-m)$.

Further if we put $f(t) = t^p$ in Corollary 10 and Remark 9 then we have the following corollary.

COROLLARY 15 *Let A and B be positive operators on a Hilbert space H such that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$. Put $m = m_1m_2$ and $M = M_1M_2$. Then for $p, q \notin [0,1]$ and $p \cdot q > 0$*

$$A^p * B^p \leq (A*B)^q + \beta \tag{14}$$

holds for

$$\beta = \begin{cases} \bar{b} + (q-1)\left(\dfrac{1}{q}\bar{a}\right)^{q/(q-1)} & \textit{if } m \leq \left(\dfrac{\bar{a}}{q}\right)^{1/(q-1)} \leq M \\ \max\{m^p - m^q, M^p - M^q\} & \textit{otherwise} \end{cases}$$

and

$$A^p * B^p \leq \alpha \cdot (A * B)^q \tag{15}$$

for

$$\alpha = \begin{cases} \dfrac{\bar{b}}{1-q}\left(\dfrac{q-1}{q}\dfrac{\bar{a}}{\bar{b}}\right)^q & \text{if } m \leq \dfrac{\bar{b}}{\bar{a}}\dfrac{q}{1-q} \leq M \\ \max\{m^p/m^q, M^p/M^q\} & \text{otherwise.} \end{cases}$$

But for $p, q \in (0,1)$ *the following inequality*

$$A^p * B^p \geq (A * B)^q + \beta \tag{16}$$

holds for

$$\beta = \begin{cases} \bar{b} + (q-1)\left(\dfrac{1}{q}\bar{a}\right)^{q/(q-1)} & \text{if } m \leq \left(\dfrac{\bar{a}}{q}\right)^{1/(q-1)} \leq M \\ \min\{m^p - m^q, M^p - M^q\} & \text{otherwise} \end{cases}$$

and

$$A^p * B^p \geq \alpha \cdot (A * B)^q \tag{17}$$

for

$$\alpha = \begin{cases} \frac{\bar{b}}{1-q}\left(\dfrac{q-1}{q}\dfrac{\bar{a}}{\bar{b}}\right)^q & \text{if } m \leq \dfrac{\bar{b}}{\bar{a}}\dfrac{q}{1-q} \leq M \\ \min\{m^p/m^q, M^p/M^q\} & \text{otherwise,} \end{cases}$$

where $\bar{a} = (M^p - m^p)/(M - m)$ *and* $\bar{b} = (Mm^p - mM^p)/(M - m)$.

Proof If we put $\alpha = 1$ in Corollary 14 then we have the inequality (14) and if we choose α such that $\beta = 0$, then we have (15).

We have the following Hadamard product versions of inequalities of Furuta, Ky Fan and Mond–Pečarić which are extensions of the Liu–Neudecker inequalities in [9].

THEOREM 16 *Let* A *and* B *be positive operators on a Hilbert space* H *such that* $0 < m_1 \leq A \leq M_1$ *and* $0 < m_2 \leq B \leq M_2$. *Put* $m = m_1 m_2$ *and*

$M = M_1 M_2$. Then for $p \notin [0, 1]$

(i)

$$A^p * B^p - (A * B)^p \leq (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{p/(p-1)} + \frac{Mm^p - mM^p}{M-m}$$

(ii)

$$A^p * B^p \leq \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p - m^p}{mM^p - Mm^p}\right)^p (A * B)^p.$$

If $p \in (0, 1)$ we have the opposite inequalities.

Proof (i) If we put $q = p$ and $\alpha = 1$ in Corollary 14, then we have the desired constant β since $m^{p-1}p \leq (M^p - m^p)/(M-m) \leq M^{p-1}p$.

(ii) If we put $q = p$ and choose α such that $\beta = 0$ in Corollary 14, then the constant α coincides with Furuta's constant $C(p)$.

Remark 17 If we put $p = 2$ in Theorem 16, then we have $A^2 * B^2 - (A * B)^2 \leq \frac{1}{4}(M-m)^2$ and $(A^2 * B^2) \leq (M+m)^2/(4Mm) \times (A * B)^2$. We directly can prove the second inequality by using Kijima's theorem in [7]. Since $(M - A \otimes B)(A \otimes B - m) \geq 0$ for $0 < m \leq A \otimes B \leq M$, we have

$$A^2 \otimes B^2 = (A \otimes B)^2 \leq (M+m)(A \otimes B) - Mm.$$

Since $(M+m)^2 X^2 - 4Mm(M+m)X + 4M^2m^2 = ((M+m)X - 2Mm)^2 \geq 0$ for any positive operator X, it follows from (1) that

$$A^2 * B^2 \leq (M+m)(A * B) - Mm \leq \frac{(M+m)^2}{4Mm}(A * B)^2.$$

By virtue of Theorem 16 (ii), we show an operator version for the Mond–Pečarić theorem [13, Theorem 2.3]:

THEOREM (Mond–Pečarić) 1 *Let A and B be positive operators on a Hilbert space H such that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$. Put $m = m_1 m_2$ and $M = M_1 M_2$. Also let r and s be nonzero real numbers such that $r < s$ and either $r \notin (-1, 1)$ or $s \notin (-1, 1)$. Then*

$$(A^s * B^s)^{1/s} \leq \tilde{\Delta}(A^r * B^r)^{1/r} \tag{18}$$

holds for

$$\tilde{\Delta} = \left\{\frac{r(K^s - K^r)}{(s-r)(K^r-1)}\right\}^{1/s}\left\{\frac{s(K^r - K^s)}{(r-s)(K^s-1)}\right\}^{-1/r}$$

and $K = M/m$.

Proof We first prove (18) for $s \geq 1$. In this case we put $p = s/r$. If $r > 0$ then $p > 1$ and $m_1^r \leq A^r \leq M_1^r, m_2^r \leq B^r \leq M_2^r$. In this case Theorem 16 (ii) gives

$$A^{s/r} * B^{s/r} \leq C_{t^{s/r}}\left(m, M; \frac{s}{r}\right)(A * B)^{s/r},$$

where $C_{t^{s/r}}(m, M; s/r)$ is Furuta's constant for $f(t) = t^{s/r}$. Then replacing A by A^r and B by B^r, we have

$$(A^s * B^s) \leq C_{t^{s/r}}\left(m^r, M^r; \frac{s}{r}\right)(A^r * B^r)^{s/r},$$

and

$$\begin{aligned}
&C_{t^{s/r}}\left(m^r, M^r; \frac{s}{r}\right)\\
&= \frac{m^r(M^r)^{s/r} - M^r(m^r)^{s/r}}{(s/r-1)(M^r - m^r)}\left(\frac{(s/r-1)((M^r)^{s/r} - (m^r)^{s/r})}{s/r(m^r(M^r)^{s/r} - M^r(m^r)^{s/r})}\right)^{s/r}\\
&= \frac{m^rM^s - M^rm^s}{(s/r-1)(M^r - m^r)}\left(\frac{(s/r-1)(M^s - m^s)}{s/r(m^rM^s - M^rm^s)}\right)^{s/r}\\
&= \frac{r(M^s - K^rm^s)}{(s-r)(K^r-1)}\left(\frac{(s-r)(K^s-1)}{s(m^rK^s - M^r)}\right)^{s/r}\\
&= \frac{r(K^s - K^r)}{(s-r)(K^r-1)}\left(\frac{s(K^r - K^s)}{(r-s)(K^s-1)}\right)^{-s/r}.
\end{aligned}$$

Therefore, it follows from the Löwner–Heinz inequality (because $0 < 1/s \leq 1$) that

$$(A^s * B^s)^{1/s} \leq C_{t^{s/r}}\left(m^r, M^r; \frac{s}{r}\right)^{1/s}(A^r * B^r)^{1/r},$$

and

$$\tilde{\Delta} = C_{t^{s/r}}\left(m^r, M^r; \frac{s}{r}\right)^{1/s} = \left\{\frac{r(K^s - K^r)}{(s-r)(K^r - 1)}\right\}^{1/s} \left\{\frac{s(K^r - K^s)}{(r-s)(K^s - 1)}\right\}^{-1/r}.$$

Similarly, if $r < 0$ then $p < 0$ and $M_1^r \le A^r \le m_1^r$, $M_2^r \le B^r \le m_2^r$. Then Theorem 16 (ii) and the Löwner–Heinz inequality give

$$(A^s * B^s)^{1/s} \le C_{t^{s/r}}\left(M^r, m^r; \frac{s}{r}\right)^{1/s} (A^r * B^r)^{1/r},$$

where

$$\begin{aligned} C_{t^{s/r}}\left(M^r, m^r; \frac{s}{r}\right)^{1/s} &= \left\{\frac{r(K^{-s} - K^{-r})}{(s-r)(K^{-r} - 1)}\right\}^{1/s} \left\{\frac{s(K^{-r} - K^{-s})}{(r-s)(K^{-s} - 1)}\right\}^{-1/r} \\ &= \left\{\frac{r(K^s - K^r)}{(s-r)(K^r - 1)}\right\}^{1/s} \frac{1}{K} \left\{\frac{s(K^r - K^s)}{(r-s)(K^s - 1)}\right\}^{-1/r} K \\ &= \tilde{\Delta}. \end{aligned}$$

Next, we prove (18) for $s \ngeq 1$. In this case we put $p = r/s$. If $-1 < s < 1$ or $s \le -1$ then according to the assumptions we have $r \le -1$ or $r < -1$. If $0 < s < 1$ then $p < 0$ and $m_1^s \le A^s \le M_1^s$, $m_2^s \le B^s \le M_2^s$. Then Theorem 16 (ii) gives

$$A^r * B^r = (A^s)^{r/s} * (B^s)^{r/s} \le C_{t^{r/s}}\left(m^s, M^s; \frac{r}{s}\right)(A^s * B^s)^{r/s}.$$

Therefore since $-1 < 1/r < 0$ it follows that

$$(A^r * B^r)^{1/r} \ge C_{t^{r/s}}\left(m^s, M^s; \frac{r}{s}\right)^{1/r} (A^s * B^s)^{1/s}.$$

But a simple calculation implies that $\tilde{\Delta} C_{t^{r/s}}(m^s, M^s; r/s)^{1/r} = 1$, so that we obtain the desired inequality.

Finally, if $s < 0$ then $p > 1$ and $M_1^s \le A^s \le m_1^s$, $M_2^s \le B^s \le m_2^s$. Then Theorem 16 (ii) gives

$$A^r * B^r = (A^s)^{r/s} * (B^s)^{r/s} \le C_{t^{r/s}}\left(M^s, m^s; \frac{r}{s}\right)(A^s * B^s)^{r/s}.$$

Therefore since $-1 < 1/r < 0$ it follows that

$$(A^r * B^r)^{1/r} \geq C_{t^{r/s}}\left(M^s, m^s; \frac{r}{s}\right)^{1/r}(A^s * B^s)^{1/s}.$$

But a simple calculation implies that $C_{t^{r/s}}(M^s, m^s; r/s)^{1/r}\tilde{\Delta} = 1$, so that we have the desired inequality.

Further we show an operator version for the Mond–Pečarić theorem [13, Theorem 2.4].

THEOREM (Mond–Pečarić) 2 *Let A, B and r, s be as in Theorem (Mond–Pečarić) 1. Then*

$$(A^s * B^s)^{1/s} - (A^r * B^r)^{1/r} \leq \Delta I \tag{19}$$

holds for

$$\Delta = \max_{\theta \in [0,1]}\left\{[\theta M^s + (1-\theta)m^s]^{1/s} - [\theta M^r + (1-\theta)m^r]^{1/r}\right\}.$$

Proof We first prove (19) for $r \notin (-1, 1)$. If we put $\alpha = a_f$ and the function $g(t) = t$ in Theorem 5 then for a real valued continuous supermultiplicative convex (resp. submultiplicative concave) function f the inequality

$$f(A) * f(B) \leq a_f A * B + \beta I \quad (\text{resp. } f(A) * f(B) \geq a_f A * B + \beta I) \tag{20}$$

holds for

$$\beta = \max_{t \in [m,M]}\{a_f t + b_f - a_f t\} = b_f$$
$$(\text{resp. } \beta = \min_{t \in [m,M]}\{a_f t + b_f - a_f t\} = b_f).$$

Replacing A by A^s and B by B^s we have

$$\begin{aligned} f(A^s) * f(B^s) - (\tilde{a}_f A^s * B^s + \tilde{b}_f I) \leq 0 \\ (\text{resp. } f(A^s) * f(B^s) - (\tilde{a}_f A^s * B^s + \tilde{b}_f I) \geq 0), \end{aligned} \tag{21}$$

where

$$\tilde{a}_f = \frac{f(M^s) - f(m^s)}{M^s - m^s}, \quad \tilde{b}_f = \frac{M^s f(m^s) - m^s f(M^s)}{M^s - m^s}.$$

Therefore, the function $f(t) = t^{r/s}$ is supermultiplicative convex if $r < 0$, $r < s$ (resp. submultiplicative concave if $r > 0$, $r < s$). If we put it in inequalities (21) we have

$$\begin{gathered} A^r * B^r - (a A^s * B^s + bI) \leq 0 \\ (\text{resp. } A^r * B^r - (a A^s * B^s + bI) \geq 0), \end{gathered} \tag{22}$$

where

$$a = \frac{M^r - m^r}{M^s - m^s}, \quad b = \frac{M^s m^r - M^r m^s}{M^s - m^s}.$$

Thus, we have

$$(A^r * B^r)^{1/r} \geq [a A^s * B^s + bI]^{1/r}. \tag{23}$$

This inequality is a simple consequence of (22) and the fact that the function $f(t) = t^{1/r}$ (resp. $f(t) = t^{-1/r}$) is operator monotone if $r \geq 1$ (resp. $r \leq -1$). Therefore, it follows from the inequality (23), by the operational calculus that

$$\begin{aligned} (A^s * B^s)^{1/s} - (A^r * B^r)^{1/r} &\leq (A^s * B^s)^{1/s} - [a A^s * B^s + bI]^{1/r} \\ &\leq \max_{t \in \bar{T}} \{ t^{1/s} - (at + b)^{1/r} \} I, \end{aligned}$$

where T denotes the open interval joining m^s to M^s, and $\bar{T}$ is the closure of T, i.e. if $s > 0$ then $\bar{T} = [m^s, M^s]$, but if $s < 0$ then $\bar{T} = [M^s, m^s]$. We set $\theta = (t - m^s)/(M^s - m^s)$. Then a simple calculation implies $a \cdot t + b = \theta M^r + (1 - \theta) m^r$. Thus, if $r \notin (-1, 1)$, we have

$$\begin{aligned} &(A^s * B^s)^{1/s} - (A^r * B^r)^{1/r} \\ &\quad \leq \max_{\theta \in [0,1]} \left\{ [\theta M^s + (1 - \theta) m^s]^{1/s} - [\theta M^r + (1 - \theta) m^r]^{1/r} \right\} I. \end{aligned}$$

Then we obtain the inequality (19).

Next we prove it for $s \notin (-1, 1)$. In this case we replace s by r in the inequality (21). Because the function $f(t) = t^{s/r}$ is supermultiplicative convex if $s > 0$, $r < s$ (resp. submultiplicative concave if $s < 0$, $r < s$)

then we have

$$A^s * B^s - (\tilde{a} A^r * B^r + \tilde{b} I) \leq 0$$
$$(\text{resp. } A^s * B^s - (\tilde{a} A^r * B^r + \tilde{b} I) \geq 0), \tag{24}$$

where

$$\tilde{a} = \frac{M^s - m^s}{M^r - m^r} = \frac{1}{a}, \quad \tilde{b} = \frac{M^r m^s - M^s m^r}{M^r - m^r} = -\frac{b}{a}.$$

If $s \geq 1$ (resp. $s \leq -1$) then the function $f(t) = t^{1/s}$ (resp. $f(t) = t^{-1/s}$) is operator monotone and from (24) it follows

$$(A^s * B^s)^{1/s} \leq \left[\frac{1}{a} A^r * B^r - \frac{b}{a} I\right]^{1/s}.$$

Finally, if $s \notin (-1, 1)$, we have

$$(A^s * B^s)^{1/s} - (A^r * B^r)^{1/r} \leq \left[\frac{1}{a} A^r * B^r - \frac{b}{a} I\right]^{1/s} - (A^r * B^r)^{1/r}$$
$$\leq \max_{t \in \bar{T}_1} \left\{ \left(\frac{1}{a} t - \frac{b}{a}\right)^{1/s} - t^{1/r} \right\} I,$$

where T_1 denotes the open interval joining m^r to M^r, and $\bar{T}_1$ is the closure of T_1, i.e. if $r > 0$ then $\bar{T}_1 = [m^r, M^r]$ but if $r < 0$ then $\bar{T}_1 = [M^r, m^r]$. We set $\theta = (t - m^r)/(M^r - m^r)$. Then a simple calculation implies $1/a \cdot t - b/a = \theta M^s + (1 - \theta) m^s$. Thus, if $s \notin (-1, 1)$ then we have

$$(A^s * B^s)^{1/s} - (A^r * B^r)^{1/r}$$
$$\leq \max_{\theta \in [0,1]} \left\{ [\theta M^s + (1 - \theta) m^s]^{1/s} - [\theta M^r + (1 - \theta) m^r]^{1/r} \right\} I.$$

This is the desired inequality.

Now, we state the following corollary obtained by applying $g(\mathrm{t}) = e^{\lambda t}$ to Remark 6 and Corollary 7.

COROLLARY 18 *Let A and B be positive operators on a Hilbert space H such that* $0<m_1\leq A\leq M_1$ *and* $0<m_2\leq B\leq M_2$. *Put* $m=m_1m_2$ *and* $M=M_1M_2$. *Let* $f(t)$ *be a real valued continuous supermultiplicative strictly convex function on* X_f. *If* $\lambda\cdot a_f>0$ *then*

$$f(A)*f(B)-\exp\{\lambda(A*B)\}\leq\beta I$$

holds for

$$\beta=\begin{cases}\dfrac{a_f}{\lambda}\ln\left(\dfrac{a_f}{\lambda\mathrm{e}}\right)+b_f & \text{if } m<\lambda^{-1}\ln\left(\dfrac{a_f}{\lambda}\right)<M\\ \max\{f(m)-\mathrm{e}^{\lambda m},f(M)-\mathrm{e}^{\lambda M}\} & \text{otherwise}\end{cases}$$

and

$$f(A)*f(B)\leq\alpha\exp\{\lambda(A*B)\}$$

holds for

$$\alpha=\begin{cases}\dfrac{a_f}{\lambda\cdot\mathrm{e}}\exp\left\{\dfrac{\lambda b_f}{a_f}\right\} & \text{if } [a_f-\lambda f(m)][a_f-\lambda f(M)]<0\\ \max\{f(m)/\mathrm{e}^{\lambda m},f(M)/\mathrm{e}^{\lambda M}\} & \text{otherwise.}\end{cases}$$

Though the exponential function is neither supermultiplicative nor submultiplicative, we have the following corollary related to it:

COROLLARY 19 *Let A and B be positive operators on a Hilbert space H such that* $0<m_1\leq A\leq M_1$ *and* $0<m_2\leq B\leq M_2$. *Put* $m=m_1m_2$ *and* $M=M_1M_2$. *Let* $f(t)$ *be a real valued continuous strictly convex function on* X_f *such that* $f(xy)\geq f(x)+f(y)$. *If* $\lambda>0$ *and* $f(M)>f(m)$, *or* $\lambda<0$ *and* $f(M)<f(m)$, *then*

$$\exp\{f(A)\}*\exp\{f(B)\}-\exp\{\lambda(A*B)\}\leq\beta I$$

holds for

$$\beta=\begin{cases}\dfrac{\bar{a}}{\lambda}\ln\left(\dfrac{\bar{a}}{\lambda\mathrm{e}}\right)+\bar{b} & \text{if } m<\lambda^{-1}\ln\left(\dfrac{\bar{a}}{\lambda}\right)<M\\ \max\{\mathrm{e}^{f(m)}-\mathrm{e}^{\lambda m},\mathrm{e}^{f(M)}-\mathrm{e}^{\lambda M}\} & \text{otherwise}\end{cases}$$

and

$$\exp\{f(A)\} * \exp\{f(B)\} \le \alpha \exp\{\lambda(A * B)\}$$

holds for

$$\alpha = \begin{cases} \dfrac{\bar{a}}{\lambda \cdot \mathrm{e}} \exp\left\{\dfrac{\lambda \bar{b}}{\bar{a}}\right\} & \text{if } [\bar{a} - \lambda \mathrm{e}^{f(m)}][\bar{a} - \lambda \mathrm{e}^{f(M)}] < 0 \\ \max\{\mathrm{e}^{f(m)-\lambda m}, \mathrm{e}^{f(M)-\lambda M}\} & \text{otherwise,} \end{cases}$$

where $\bar{a} = (\mathrm{e}^{f(M)} - \mathrm{e}^{f(m)})/(M - m)$ *and* $\bar{b} = (M\mathrm{e}^{f(m)} - m\mathrm{e}^{f(M)})/(M - m)$.

Proof Since $f(t)$ is convex and $f(xy) \ge f(x) + f(y)$, it follows that $\exp\{f(x)\}$ is supermultiplicative convex. Replacing $f(x)$ by $\exp\{f(x)\}$ in Corollary 18 we obtain the desired inequalities.

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