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On Powers of p-Hyponormal and Log-Hyponormal Operators

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Dedicated to the memory of Prof. Szökefalvi-Nagy, Béla in deep sorrow

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A bounded linear operator T on a Hilbert space H is said to be p-hyponormal for $p>0$ if $(T^*T)^p \geq (TT^*)^p$, and T is said to be log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. Firstly, we shall show the following extension of our previous result: If T is p-hyponormal for $p \in (0,1]$, then $(T^{n*}T^n)^{(p+1)/n} \geq \cdots \geq (T^{2*}T^2)^{(p+1)/2} \geq (T^*T)^{p+1}$ and $(TT^*)^{p+1} \geq (T^2T^{2*})^{(p+1)/2} \geq \cdots \geq (T^nT^{n*})^{(p+1)/n}$ hold for all positive integer n. Secondly, we shall discuss the best possibilities of the following parallel result for log-hyponormal operators by Yamazaki: If T is log-hyponormal, then $(T^{n*}T^n)^{1/n} \geq \cdots \geq (T^{2*}T^2)^{1/2} \geq T^*T$ and $TT^* \geq (T^2T^{2*})^{1/2} \geq \cdots \geq (T^nT^{n*})^{1/n}$ hold for all positive integer n.

Keywords: p-Hyponormal operator; Log-hyponormal operator; Furuta inequality

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1 INTRODUCTION

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

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An operator T is said to be p-hyponormal for $p>0$ if $(T^*T)^p \geq (TT^*)^p$ and an operator T is said to be log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. p-Hyponormal and log-hyponormal operators are defined as extensions of hyponormal one, i.e., $T^*T \geq TT^*$. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p \geq q > 0$ by the celebrated Löwner–Heinz theorem "$A \geq B \geq 0$ *ensures* $A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0,1]$," and every invertible p-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function. We remark that $(A^p - 1)/p \to \log A$ as $p \to +0$ for positive invertible operator $A > 0$, so that p-hyponormality of T approaches log-hyponormality of T as $p \to +0$. In this sense, log-hyponormal can be considered as 0-hyponormal.

Recently, Aluthge and Wang [2] showed the following results.

THEOREM A [2] *Let T be a p-hyponormal operator for $p \in (0,1]$. Then*

$$(T^{n^*}T^n)^{p/n} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n^*})^{p/n} \tag{1.1}$$

hold, that is, T^n is (p/n)-hyponormal for all positive integer n.

It is well known that even if T is hyponormal, T^2 is not hyponormal in general [9, Problem 209], but paranormal [4], i.e., $\|T^2x\| \geq \|Tx\|^2$ holds for every unit vector x. Now it turns out by Theorem A that T^2 is $(1/2)$-hyponormal for every hyponormal operator T, which is more precise since $(1/2)$-hyponormality ensures paranormality [1,7].

Very recently, in [8], we showed an extension of Theorem A as follows.

THEOREM B *Let T be a p-hyponormal operator for $p \in (0,1]$. Then*

$$(T^{n^*}T^n)^{(p+1)/n} \geq (T^*T)^{p+1} \tag{1.2}$$

and

$$(TT^*)^{p+1} \geq (T^nT^{n^*})^{(p+1)/n} \tag{1.3}$$

hold for all positive integer n.

We also discussed the best possibilities of Theorem 1 and Theorem A.

On the other hand, Yamazaki [12] showed another extension of Theorem A as follows.

THEOREM C [12] *Let T be a p-hyponormal operator for $p \in (0,1]$. Then*

$$(T^{n^*}T^n)^{1/n} \geq \cdots \geq (T^{2^*}T^2)^{1/2} \geq T^*T \tag{1.4}$$

and

$$TT^* \geq (T^2T^{2^*})^{1/2} \geq \cdots \geq (T^nT^{n^*})^{1/n} \tag{1.5}$$

hold for all positive integer n.

We remark that Theorem A follows from Theorem B (or Theorem C) obviously. In fact, the first and third inequalities of (1.1) hold by (1.2) and (1.3) of Theorem B (or (1.4) and (1.5) of Theorem C) and Löwner–Heinz theorem, and the second inequality of (1.1) holds since T is p-hyponormal.

Yamazaki [12] also showed the following Theorem D and Corollary E for log-hyponormal operators which are parallel results to Theorem C and Theorem A for p-hyponormal operators, respectively.

THEOREM D [12] *Let T be a log-hyponormal operator. Then*

$$(T^{n^*}T^n)^{1/n} \geq \cdots \geq (T^{2^*}T^2)^{1/2} \geq T^*T \tag{1.6}$$

and

$$TT^* \geq (T^2T^{2^*})^{1/2} \geq \cdots \geq (T^nT^{n^*})^{1/n} \tag{1.7}$$

hold for all positive integer n.

COROLLARY E [12] *Let T be a log-hyponormal operator. Then*

$$\log(T^{n^*}T^n)^{1/n} \geq \log T^*T \geq \log TT^* \geq \log(T^nT^{n^*})^{1/n} \tag{1.8}$$

hold, that is, T^n is also log-hyponormal for all positive integer n.

We remark that Corollary E is more general than the following result by Aluthge and Wang [1] "*If T is log-hyponormal, then T^{2^n} is log-hyponormal for any positive integer n.*"

In this paper, we shall show Theorem 1 stated below which is an extension of both Theorem B and Theorem C. We shall also discuss the best possibilities of Theorem D and Corollary E.

2 AN EXTENSION OF BOTH THEOREM B AND THEOREM C

THEOREM 1 *Let T be a p-hyponormal operator for $p \in (0, 1]$. Then*

$$(T^{n^*}T^n)^{(p+1)/n} \geq \cdots \geq (T^{2^*}T^2)^{(p+1)/2} \geq (T^*T)^{p+1} \tag{2.1}$$

and

$$(TT^*)^{p+1} \geq (T^2T^{2^*})^{(p+1)/2} \geq \cdots \geq (T^nT^{n^*})^{(p+1)/n} \tag{2.2}$$

hold for all positive integer n.

We remark that Theorem B follows from Theorem 1 by comparing the first and last terms of each of the inequalities, and Theorem C also follows from Theorem 1 by Löwner–Heinz theorem. It is interesting to remark that Theorem D just corresponds to Theorem 1 in case $p = 0$ since log-hyponormal can be considered as 0-hyponormal as mentioned in Section 1.

In order to give a proof of Theorem 1, we use the following Theorem F.

THEOREM F (Furuta inequality [5]) *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $(B^{r/2}A^pB^{r/2})^{1/q} \geq (B^{r/2}B^pB^{r/2})^{1/q}$ *and*
(ii) $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pB^{r/2})^{1/q}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)\,q \geq p+r$.

We remark that Theorem F yields Löwner–Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [3,10] and also an elementary one-page proof in [6]. It is shown in [11] that the domain drawn for p, q and r in Fig. 1 is the best possible for Theorem F.

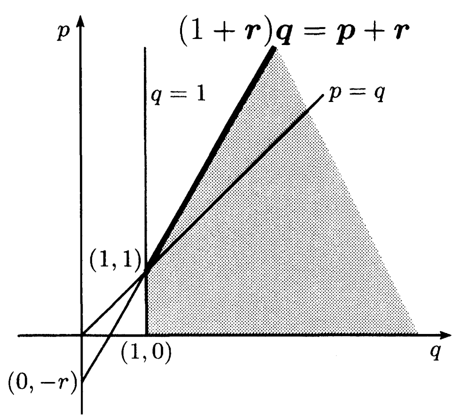

FIGURE 1

We also use the following result which is an application of Theorem F.

THEOREM F′ *If* $A \geq B \geq 0$, *then the following assertions hold:*

(i) *for each* $q \geq 0$ *and* $r \geq 0$, $f(s) = (B^{r/2}A^sB^{r/2})^{(q+r)/(s+r)}$ *is increasing for* $s \geq q$.

(ii) *for each* $q \geq 0$ *and* $r \geq 0$, $g(s) = (A^{r/2}B^sA^{r/2})^{(q+r)/(s+r)}$ *is decreasing for* $s \geq q$.

Proof of Theorem 1 Let $T = U|T|$ be the polar decomposition of T. Then it is well known that the polar decomposition of T^* is $T^* = U^*|T^*|$. Put $A_n = (T^{n^*}T^n)^{p/n} = |T^n|^{2p/n}$ and $B_n = (T^nT^{n^*})^{p/n} = |T^{n^*}|^{2p/n}$ for each positive integer n.

Proof of (2.1) We shall prove that the following (2.3) holds for all positive integer n, which is equivalent to (2.1) obviously:

$$(T^{n+1^*}T^{n+1})^{(p+1)/(n+1)} \geq (T^{n^*}T^n)^{(p+1)/n}. \tag{2.3}$$

(i) Firstly, we prove that (2.3) holds for $n = 1$, that is,

$$(T^{2^*}T^2)^{(p+1)/2} \geq (T^*T)^{p+1}. \tag{2.4}$$

$A_1 = (T^*T)^p \geq (TT^*)^p = B_1$ holds since T is p-hyponormal. By applying (i) of Theorem F to A_1 and B_1 for $1/p \geq 0$, we have

$$\begin{aligned}(T^{2^*}T^2)^{(p+1)/2} &= (U^*|T^*|T^*T|T^*|U)^{(p+1)/2}\\ &= U^*(|T^*|T^*T|T^*|)^{(p+1)/2}U\\ &= U^*(B_1^{1/2p}A_1^{1/p}B_1^{1/2p})^{(1+1/p)/(1/p+1/p)}U\\ &\geq U^*B_1^{1+1/p}U\\ &= U^*|T^*|^{2(p+1)}U\\ &= |T|^{2(p+1)}\\ &= (T^*T)^{p+1},\end{aligned}$$

so that (2.4) is proved.

(ii) Secondly, in order to prove that (2.3) holds for $n \geq 2$, we prove the following (2.5) by induction:

$$(T^{n+1^*}T^{n+1})^{n/(n+1)} \geq T^{n^*}T^n \quad \text{for all positive integer } n. \qquad (2.5)$$

We remark that (2.5) implies that (2.3) holds for $n \geq 2$ by applying Löwner–Heinz theorem to (2.5) for $(p+1)/n \in (0, 1]$.
(2.5) holds for $n = 1$ by (2.4) and Löwner–Heinz theorem. Assume that (2.5) holds for $n = 1, 2, \ldots, k-1$. By applying Löwner–Heinz theorem to (2.5) for $p/n \in (0, 1]$, we have $(T^{n+1^*}T^{n+1})^{p/(n+1)} \geq (T^{n^*}T^n)^{p/n}$, so that

$$A_k = (T^{k^*}T^k)^{p/k} \geq \cdots \geq (T^{2^*}T^2)^{p/2} \geq (T^*T)^p \geq (TT^*)^p = B_1.$$

The last inequality holds since T is p-hyponormal. Put $q_1 = (k-1)/p \geq 0$ and $r_1 = 1/p \geq 0$. Then by (i) of Theorem F′, $f(s) = (B_1^{r_1/2}A_k^sB_1^{r_1/2})^{(q_1+r_1)/(s+r_1)} = (B_1^{1/2p}A_k^sB_1^{1/2p})^{k/(ps+1)}$ is increasing for $s \geq q_1 = (k-1)/p$, so that we have

$$\begin{aligned}
(T^{k+1^*}T^{k+1})^{k/(k+1)} &= (U^*|T^*|T^{k^*}T^k|T^*|U)^{k/(k+1)} \\
&= U^*(|T^*|T^{k^*}T^k|T^*|)^{k/(k+1)}U \\
&= U^*(B_1^{1/2p}A_k^{k/p}B_1^{1/2p})^{k/(k+1)}U \\
&= U^*f\left(\frac{k}{p}\right)U \\
&\geq U^*f\left(\frac{k-1}{p}\right)U \\
&= U^*B_1^{1/2p}A_k^{(k-1)/p}B_1^{1/2p}U \\
&= T^*(T^{k^*}T^k)^{(k-1)/k}T \\
&\geq T^*T^{k-1^*}T^{k-1}T \\
&= T^{k^*}T^k.
\end{aligned}$$

The last inequality holds since we assume that (2.5) holds for $n = k-1$. Hence (2.5) also holds for $n = k$, so that it is proved that (2.5) holds for all positive integer n.

Consequently, the proof of (2.1) is complete by combining (i) and (ii).

Proof of (2.2) We shall prove that the following (2.6) holds for all positive integer n, which is equivalent to (2.2) obviously:

$$(T^nT^{n^*})^{(p+1)/n} \geq (T^{n+1}T^{n+1^*})^{(p+1)/(n+1)}. \tag{2.6}$$

(i) Firstly, we prove that (2.6) holds for $n = 1$, that is,

$$(TT^*)^{p+1} \geq (T^2T^{2^*})^{(p+1)/2}. \tag{2.7}$$

$A_1 = (T^*T)^p \geq (TT^*)^p = B_1$ holds since T is p-hyponormal. By applying (ii) of Theorem F to A_1 and B_1 for $1/p \geq 0$, we have

$$\begin{aligned}
(T^2T^{2^*})^{(p+1)/2} &= (U|T|TT^*|T|U^*)^{(p+1)/2} \\
&= U(|T|TT^*|T|)^{(p+1)/2}U^* \\
&= U(A_1^{1/2p}B_1^{1/p}A_1^{1/2p})^{(1+1/p)/(1/p+1/p)}U^* \\
&\leq UA_1^{1+1/p}U^* \\
&= U|T|^{2(p+1)}U^* \\
&= |T^*|^{2(p+1)} \\
&= (TT^*)^{p+1},
\end{aligned}$$

so that (2.7) is proved.

(ii) Secondly, in order to prove that (2.6) holds for $n \geq 2$, we prove the following (2.8) by induction:

$$T^nT^{n^*} \geq (T^{n+1}T^{n+1^*})^{n/(n+1)} \quad \text{for all positive integer } n. \tag{2.8}$$

We remark that (2.8) implies that (2.6) holds for $n \geq 2$ by applying Löwner–Heinz theorem to (2.8) for $(p+1)/n \in (0, 1]$.

(2.8) holds for $n = 1$ by (2.7) and Löwner–Heinz theorem. Assume that (2.8) holds for $n = 1, 2, \ldots, k-1$. By applying Löwner–Heinz theorem to (2.8) for $p/n \in (0, 1]$, we have $(T^nT^{n^*})^{p/n} \geq (T^{n+1}T^{n+1^*})^{p/(n+1)}$, so that

$$A_1 = (T^*T)^p \geq (TT^*)^p \geq (T^2T^{2^*})^{p/2} \geq \cdots \geq (T^kT^{k^*})^{p/k} = B_k.$$

The first inequality holds since T is p-hyponormal. Put $q_1=(k-1)/p\geq 0$ and $r_1=1/p\geq 0$. Then by (ii) of Theorem F′, $g(s)=(A_1^{r_1/2}B_k^sA_1^{r_1/2})^{(q_1+r_1)/(s+r_1)}=(A_1^{1/2p}B_k^sA_1^{1/2p})^{k/(ps+1)}$ is decreasing for $s\geq q_1=(k-1)/p$, so that we have

$$\begin{aligned}(T^{k+1}T^{k+1^*})^{k/(k+1)} &= (U|T|T^kT^{k^*}|T|U^*)^{k/(k+1)}\\ &= U(|T|T^kT^{k^*}|T|)^{k/(k+1)}U^*\\ &= U(A_1^{1/2p}B_k^{k/p}A_1^{1/2p})^{k/(k+1)}U^*\\ &= Ug\left(\frac{k}{p}\right)U^*\\ &\leq Ug\left(\frac{k-1}{p}\right)U^*\\ &= UA_1^{1/2p}B_k^{(k-1)/p}A_1^{1/2p}U^*\\ &= T(T^kT^{k^*})^{(k-1)/k}T^*\\ &\leq TT^{k-1}T^{k-1^*}T^*\\ &= T^kT^{k^*}.\end{aligned}$$

The last inequality holds since we assume that (2.8) holds for $n=k-1$. Hence (2.8) also holds for $n=k$, so that it is proved that (2.8) holds for all positive integer n.

Consequently, the proof of (2.2) is complete by combining (i) and (ii).

3 BEST POSSIBILITIES OF THEOREM D AND COROLLARY E

The following Theorem 2 asserts the best possibility of Theorem D.

THEOREM 2 *Let $n\geq 2$ and $\alpha>1$. The the following hold:*

(i) *there exists a log-hyponormal operator T such that $(T^{n^*}T^n)^{\alpha/n}\not\geq(T^*T)^\alpha$.*
(ii) *there exists a log-hyponormal operator T such that $(TT^*)^\alpha\not\geq(T^nT^{n^*})^{\alpha/n}$.*

We remark that $A^\delta\geq B^\delta$ for $\delta>0$ approaches $\log A\geq\log B$ as $\delta\to+0$ for positive invertible operators A and B. In this sense, the following Theorem 3 asserts the best possibilities of all the inequalities of (1.8) in Corollary E.

THEOREM 3 *Let* $n \geq 1$ *and* $\alpha > 0$. *Then the following hold:*

(i) *there exists a log-hyponormal operator* T *such that* $(T^{n^*}T^n)^{\alpha/n} \not\geq (T^nT^{n^*})^{\alpha/n}$.
(ii) *there exists a log-hyponormal operator* T *such that* $(T^{n^*}T^n)^{\alpha/n} \not\geq (TT^*)^{\alpha}$.
(iii) *there exists a log-hyponormal operator* T *such that* $(T^*T)^{\alpha} \not\geq (T^nT^{n^*})^{\alpha/n}$.

To give proofs of Theorem 2 and Theorem 3, we use the following results.

PROPOSITION 1 [13] *Let* $p > 0$, $q > 0$ *and* $r > 0$. *If* $rq < p + r$, *then the following assertions hold:*

(i) *there exist positive invertible operators* A *and* B *on* $\mathbb{R}^2$ *such that* $\log A \geq \log B$ *and*

$$(B^{r/2}A^pB^{r/2})^{1/q} \not\geq B^{(p+r)/q}.$$

(ii) *there exist positive invertible operators* A *and* B *on* $\mathbb{R}^2$ *such that* $\log A \geq \log B$ *and*

$$A^{(p+r)/q} \not\geq (A^{r/2}B^pA^{r/2})^{1/q}.$$

LEMMA 1 *For positive operators* A *and* B *on* H, *define the operator* T *on* $\oplus_{k=-\infty}^{\infty} H$ *as follows:*

$$T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & B^{1/2} & 0 & & & & \\ & & B^{1/2} & \boxed{0} & & & \\ & & & A^{1/2} & 0 & & \\ & & & & A^{1/2} & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{3.1}$$

where $\square$ *shows the place of the* $(0, 0)$ *matrix element. Then the following assertions hold:*

(i) *T is p-hyponormal for* $p > 0$ *if and only if* $A^p \geq B^p$.

(ii) *T is log-hyponormal if and only if A and B are invertible and* $\log A \geq \log B$.

Furthermore, the following assertions hold for $\beta > 0$ *and integers* $n \geq 2$:

(iii) $(T^{n^*}T^n)^{\beta/n} \geq (T^*T)^{\beta}$ *if and only if*

$$(B^{k/2}A^{n-k}B^{k/2})^{\beta/n} \geq B^{\beta} \quad \textit{holds for } k = 1, 2, \ldots, n-1. \tag{3.2}$$

(iv) $(TT^*)^{\beta} \geq (T^nT^{n^*})^{\beta/n}$ *if and only if*

$$A^{\beta} \geq (A^{k/2}B^{n-k}A^{k/2})^{\beta/n} \quad \textit{holds for } k = 1, 2, \ldots, n-1. \tag{3.3}$$

(v) $(T^{n^*}T^n)^{\beta/n} \geq (T^nT^{n^*})^{\beta/n}$ *if and only if*

$$\begin{cases} A^{\beta} \geq B^{\beta} \quad \textit{holds and} \\ \quad (B^{k/2}A^{n-k}B^{k/2})^{\beta/n} \geq B^{\beta} \quad \textit{and} \\ \qquad A^{\beta} \geq (A^{k/2}B^{n-k}A^{k/2})^{\beta/n} \quad \textit{hold for } k = 1, 2, \ldots, n-1. \end{cases} \tag{3.4}$$

Proof By easy calculation, we have

$$T^*T = \begin{pmatrix} \ddots & & & & & \\ & B & & & & \\ & & B & & & \\ & & & \boxed{A} & & \\ & & & & A & \\ & & & & & A \\ & & & & & & \ddots \end{pmatrix}$$

and

$$TT^* = \begin{pmatrix} \ddots & & & & & \\ & B & & & & \\ & & B & & & \\ & & & \boxed{B} & & \\ & & & & A & \\ & & & & & A \\ & & & & & & \ddots \end{pmatrix},$$

so that (i) and (ii) are obvious by comparing the two $(0,0)$ elements of T^*T and TT^*. Furthermore, the following hold for $n \geq 2$:

$$T^{n^*}T^n = \begin{pmatrix} \ddots &&&&&&&&&& \\ & B^n &&&&&&&&& \\ && B^n &&&&&&&& \\ &&& B^{(n-1)/2}AB^{(n-1)/2} &&&&&&& \\ &&&& \ddots &&&&&& \\ &&&&& B^{k/2}A^{n-k}B^{k/2} &&&&& \\ &&&&&& \ddots &&&& \\ &&&&&&& B^{1/2}A^{n-1}B^{1/2} &&& \\ &&&&&&&& \boxed{A^n} && \\ &&&&&&&&& A^n & \\ &&&&&&&&&& \ddots \end{pmatrix}$$

and

$$T^nT^{n^*} = \begin{pmatrix} \ddots &&&&&&&&&& \\ & B^n &&&&&&&&& \\ && \boxed{B^n} &&&&&&&& \\ &&& A^{1/2}B^{n-1}B^{1/2} &&&&&&& \\ &&&& \ddots &&&&&& \\ &&&&& A^{k/2}B^{n-k}A^{k/2} &&&&& \\ &&&&&& \ddots &&&& \\ &&&&&&& A^{(n-1)/2}BA^{(n-1)/2} &&& \\ &&&&&&&& A^n && \\ &&&&&&&&& A^n & \\ &&&&&&&&&& \ddots \end{pmatrix},$$

so that we have (iii), (iv) and (v) by comparing the corresponding elements of $T^{n^*}T^n$ and $T^nT^{n^*}$.

Proof of Theorem 2 Put $p_1 = n-1 > 0$, $q_1 = n/\alpha > 0$ and $r_1 = 1 > 0$, then we have $r_1q_1 = n/\alpha > n = p_1 + r_1$.

Proof of (i) By (i) of Proposition 1, there exist positive invertible operators A and B on H such that

$$\log A \geq \log B \tag{3.5}$$

and $(B^{r_1/2}A^{p_1}B^{r_1/2})^{1/q_1} \ngeq B^{(p_1+r_1)/q_1}$, that is,

$$(B^{1/2}A^{n-1}B^{1/2})^{\alpha/n} \ngeq B^{\alpha}. \tag{3.6}$$

Define an operator T on $\oplus_{k=-\infty}^{\infty}H$ as (3.1). Then T is log-hyponormal by (3.5) and (ii) of Lemma 1, and $(T^{n^*}T^n)^{\alpha/n} \ngeq (T^*T)^{\alpha}$ by (iii) of Lemma 1 since the case $k=1$ of (3.2) does not hold for $\beta=\alpha$ by (3.6).

Proof of (ii) By (ii) of Proposition 1, there exist positive invertible operators A and B on H such that

$$\log A \geq \log B \tag{3.7}$$

and $A^{(p_1+r_1)/q_1} \ngeq (A^{r_1/2}B^{p_1}A^{r_1/2})^{1/q_1}$, that is,

$$A^{\alpha} \ngeq (A^{1/2}B^{n-1}A^{1/2})^{\alpha/n}. \tag{3.8}$$

Define an operator T on $\oplus_{k=-\infty}^{\infty}H$ as (3.1). Then T is log-hyponormal by (3.7) and (ii) of Lemma 1, and $(TT^*)^{\alpha} \ngeq (T^nT^{n^*})^{\alpha/n}$ by (iv) of Lemma 1 since the case $k=1$ of (3.3) does not hold for $\beta=\alpha$ by (3.8).

Proof of Theorem 3
Proof of (i) It is well known that there exist positive invertible operators A and B on H such that

$$\log A \geq \log B \tag{3.9}$$

and

$$A^{\alpha} \ngeq B^{\alpha}. \tag{3.10}$$

Define an operator T on $\oplus_{k=-\infty}^{\infty}H$ as (3.1). Then T is log-hyponormal by (3.9) and (ii) of Lemma 1, and $(T^{n^*}T^n)^{\alpha/n} \ngeq (T^nT^{n^*})^{\alpha/n}$ for $n \geq 2$ by (v) of Lemma 1 since the first inequality of (3.4) does not hold for $\beta=\alpha$ by (3.10), and $(T^*T)^{\alpha} \ngeq (TT^*)^{\alpha}$ by (3.10) and (i) of Lemma 1.

Proof of (ii) We have only to prove the case $1 \geq \alpha > 0$ by Löwner–Heinz theorem. Assume

$$(T^{n^*}T^n)^{\alpha/n} \geq (TT^*)^{\alpha}. \tag{3.11}$$

Then we have

$$(T^{n^*}T^n)^{\alpha/n} \geq (TT^*)^{\alpha} \geq (T^nT^{n^*})^{\alpha/n}.$$

The first inequality is (3.11) itself, and the second inequality holds by (1.7) in Theorem D and Löwner–Heinz theorem. This is a contradiction to (i) of Theorem 3.

Proof of (iii) We have only to prove the case $1 \geq \alpha > 0$ by Löwner–Heinz theorem. Assume

$$(T^*T)^{\alpha} \geq (T^nT^{n^*})^{\alpha/n}. \tag{3.12}$$

Then we have

$$(T^{n^*}T^n)^{\alpha/n} \geq (T^*T)^{\alpha} \geq (T^nT^{n^*})^{\alpha/n}.$$

The first inequality holds by (1.6) in Theorem D and Löwner–Heinz theorem, and the second inequality is (3.12) itself. This is a contradiction to (i) of Theorem 3.

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