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Some Remarks on Kato's Inequality

TOSHIO HORIUCHI

Department of Mathematical Sciences, Ibaraki University, Mito, Ibaraki, 310, Japan

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Let $N \geq 1$ and $p > 1$. Let Ω be a domain of $\mathbb{R}^N$. In this article we shall establish Kato's inequalities for p-harmonic operators L_p. Here L_p is defined as $L_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $u \in K_p(\Omega)$, where $K_p(\Omega)$ is an admissible class. If $p=2$ for example, then we have $K_2(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega)\colon \partial_j u, \partial^2_{j,k} u \in L^1_{\mathrm{loc}}(\Omega) \text{ for } j,k = 1,2,\ldots,N\}$. Then we shall prove that $L_p|u| \geq (\operatorname{sgn} u)\, L_p u$ and $L_p u^+ \geq (\operatorname{sgn}^+ u)^{p-1} L_p u$ in $\mathcal{D}'(\Omega)$ with $u \in K_p(\Omega)$. These inequalities are called Kato's inequalities provided that $p = 2$.

Keywords: Kato's inequality; p-Harmonic operators

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1 INTRODUCTION

Let $N \geq 1$. Let Ω be a domain of $\mathbb{R}^N$. Define

$$M(x, \partial_x) = \partial_{x_j}(a_{jk}(x)\partial_{x_k}), \tag{1.1}$$

where $a_{jk}(x) \in C^1(\Omega)$ is positive definite in the following sense.

$$\sum_{j,k=1}^{N} a_{jk}(x)\xi_j\xi_k \geq C|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^N \backslash \{0\} \text{ and } x \in \Omega. \tag{1.2}$$

Here C is a positive number independent if each x and ξ. First we recall well-known Kato's inequalities. (For the proof, see [1]).

THEOREM 1.1 *For u and $M(x, \partial_x)u \in L^1_{\mathrm{loc}}(\Omega)$, we have*

$$M(x, \partial_x)|u| \geq (M(x, \partial_x)u)\operatorname{sgn} u \quad \text{in } \mathcal{D}'(\Omega), \tag{1.3}$$

$$M(x, \partial_x)u_+ \geq (M(x, \partial_x)u)\operatorname{sgn}^+ u \quad \text{in } \mathcal{D}'(\Omega). \tag{1.4}$$

Here

$$\operatorname{sgn} u(x) = \begin{cases} \dfrac{u(x)}{|u(x)|}, & \textit{for } u \neq 0, \\ 0, & \textit{for } u = 0, \end{cases}$$
$$\operatorname{sgn}^+ u(x) = \begin{cases} 1, & \textit{for } u > 0, \\ 1/2, & \textit{for } u = 0, \\ 0, & \textit{for } u < 0, \end{cases} \tag{1.5}$$

and $u_+ = \max\,[u(x), 0]$. *By* $\mathcal{D}'(\Omega)$ *we denote the set of all distributions on* Ω.

In this paper we shall consider the operators defined by

$$\begin{aligned} L_p u &= \operatorname{div}(|\nabla u|^{p-2}\nabla u), \\ &= |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\sum_{j,k=1}^{N} \partial_j u \partial_k u \partial_{j,k}^2 u, \end{aligned} \tag{1.6}$$

where $p > 1$ and $\partial_j u = \partial u/\partial x_j$, $\partial_{j,k}^2 u = \partial^2 u/(\partial x_j \partial x_k)$ for $j, k = 1, 2, \ldots, N$. Then we shall generalize Theorem 1.1 for the operators L_p in place of linear elliptic operators represented by the Laplacian.

This paper is organized in the following way. In Section 2 we prepare basic inequalities including the p-harmonic operators L_p. In Section 3 we shall state our main result, and the proof is also given there.

2 PRELIMINARY

We shall establish some fundamental inequalities for smooth functions u, which are useful to prove our main result.

LEMMA 2.1 *Assume that* $u \in C^2(\Omega)$. *Then it holds that*

$$\begin{aligned} L_p|u| &\geq (\operatorname{sgn} u)L_p u \quad \textit{in } \mathcal{D}'(\Omega), \\ L_p u_+ &\geq (\operatorname{sgn}^+ u)^{p-1} L_p u \quad \textit{in } \mathcal{D}'(\Omega). \end{aligned} \tag{2.1}$$

Here by $\mathcal{D}'(\Omega)$ *we denote the set of all distributions on* Ω.

Proof For any $\varepsilon > 0$ we set

$$u_\varepsilon = (u^2 + \varepsilon^2)^{1/2}. \tag{2.2}$$

Then we see

$$\partial_j u_\varepsilon = \frac{u}{u_\varepsilon}\partial_j u, \tag{2.3}$$

$$\partial_j^2 u_\varepsilon = \frac{u}{u_\varepsilon}\partial_j^2 u + \frac{1}{u_\varepsilon}\left(1 - \left(\frac{u}{u_\varepsilon}\right)^2\right)(\partial_j u)^2 \geq \frac{u}{u_\varepsilon}\partial_j^2 u. \tag{2.4}$$

Here $\partial_j^2 u = \partial^2 u/\partial x_j^2, j = 1, 2, \ldots, N$. Using these we have

$$\begin{aligned} L_p u_\varepsilon &= \left|\frac{u}{u_\varepsilon}\right|^{p-2}\left(\frac{u}{u_\varepsilon}L_p u + (p-1)\frac{1}{u_\varepsilon}\left(1 - \left(\frac{u}{u_\varepsilon}\right)^2 |\nabla u|^p\right)\right) \\ &\geq \left|\frac{u}{u_\varepsilon}\right|^{p-2}\frac{u}{u_\varepsilon}L_p u. \end{aligned} \tag{2.5}$$

In a similar way we have

$$\begin{aligned} L_p\left(\frac{u + u_\varepsilon}{2}\right) &= \left(\frac{1 + (u/u_\varepsilon)}{2}\right)^{p-1}\left(L_p u + (p-1)\frac{2}{u_\varepsilon}\left(1 - \frac{u}{u_\varepsilon}\right)|\nabla u|^p\right) \\ &\geq \left(\frac{1 + (u/u_\varepsilon)}{2}\right)^{p-1} L_p u. \end{aligned} \tag{2.6}$$

Since $2u_+ = u + |u|$ holds, letting $\varepsilon \to 0$ we have the desired inequalities.

In the next we shall consider the operators $L_{p,\eta}$ for $\eta \geq 0$ defined by

$$L_{p,\eta} u = \operatorname{div}\Big((\eta^2 + |\nabla u|^2)^{(p-2)/2}\nabla u\Big). \tag{2.7}$$

Then we see

$$\begin{aligned} L_{p,\eta} u_\varepsilon &= \frac{u}{u_\varepsilon}(\eta^2 + |\nabla u_\varepsilon|^2)^{(p-2)/2}\left(\Delta u + (p-2)\left(\frac{u}{u_\varepsilon}\right)^2 \frac{\partial_j u \partial_k u \partial_{j,k} u}{\eta^2 + |\nabla u_\varepsilon|^2}\right) \\ &\quad + \frac{1}{u_\varepsilon}\left(1 - \left(\frac{u}{u_\varepsilon}\right)^2\right)(\eta^2 + |\nabla u_\varepsilon|^2)^{(p-2)/2}|\nabla u|^2 \\ &\quad \times \left(1 + (p-2)\left(\frac{u}{u_\varepsilon}\right)^2 \frac{|\nabla u|^2}{\eta^2 + |\nabla u_\varepsilon|^2}\right) \\ &\geq \frac{u}{u_\varepsilon}(\eta^2 + |\nabla u_\varepsilon|^2)^{(p-2)/2} \\ &\quad \times \left(\Delta u + (p-2)\left(\frac{u}{u_\varepsilon}\right)^2 \frac{\sum_{j,k=1}^N \partial_j u \partial_k u \partial_{j,k} u}{\eta^2 + |\nabla u_\varepsilon|^2}\right). \end{aligned} \tag{2.8}$$

Similarly we can compute $L_{p,\eta}((u+u_\varepsilon)/2)$ to obtain the following:

$$L_{p,\eta}\left(\frac{u+u_\varepsilon}{2}\right)$$
$$= w_\varepsilon(\eta^2 + w_\varepsilon^2|\nabla u|^2)^{(p-2)/2}\left(\Delta u + (p-2)w_\varepsilon^2 \frac{\sum_{j,k=1}^N \partial_{j,k}u\partial_j u\partial_k u}{\eta^2 + w_\varepsilon^2|\nabla u|^2}\right)$$
$$+ (\nabla w_\varepsilon \cdot \nabla u)(\eta^2 + w_\varepsilon^2|\nabla u|^2)^{(p-2)/2}\left(1 + (p-2)w_\varepsilon^2 \frac{|\nabla u|^2}{\eta^2 + w_\varepsilon^2|\nabla u|^2}\right). \tag{2.9}$$

Here

$$w_\varepsilon = \frac{1}{2}\left(1 + \frac{u}{u_\varepsilon}\right),$$
$$\nabla w_\varepsilon \cdot \nabla u = \frac{w_\varepsilon}{u_\varepsilon}\left(1 - \frac{u}{u_\varepsilon}\right)|\nabla u|^2. \tag{2.10}$$

Therefore we have

LEMMA 2.2 *For* $u \in C^2(\Omega)$ *it holds that*

$$L_{p,\eta}u_\varepsilon \geq \frac{u}{u_\varepsilon}(\eta^2 + |\nabla u_\varepsilon|^2)^{(p-2)/2}$$
$$\times \left(\Delta u + (p-2)\left(\frac{u}{u_\varepsilon}\right)^2 \frac{\sum_{j,k=1}^N \partial_j u\partial_k u\partial_{j,k}u}{\eta^2 + |\nabla u_\varepsilon|^2}\right), \tag{2.11}$$

$$L_{p,\eta}\left(\frac{u+u_\varepsilon}{2}\right) \geq w_\varepsilon(\eta^2 + w_\varepsilon^2|\nabla u|^2)^{(p-2)/2}$$
$$\times \left(\Delta u + (p-2)w_\varepsilon^2 \frac{\sum_{j,k=1}^N \partial_{j,k}u\partial_j u\partial_k u}{\eta^2 + w_\varepsilon^2|\nabla u|^2}\right). \tag{2.12}$$

Letting $\varepsilon \to 0$, we have for $u \in C^2(\Omega)$

LEMMA 2.3 *For* $u \in C^2(\Omega)$ *it holds that in* $\mathcal{D}'(\Omega)$

$$L_{p,\eta}|u| \geq (\operatorname{sgn} u)(\eta^2 + |\nabla u|^2)^{(p-2)/2}\left(\Delta u + (p-2)\frac{\sum_{j,k=1}^N \partial_j u\partial_k u\partial_{j,k}u}{\eta^2 + |\nabla u|^2}\right)$$
$$= (\operatorname{sgn} u)L_{p,\eta}u, \tag{2.13}$$

$$L_{p,\eta}u_+ \geq (\operatorname{sgn}^+ u)(\eta^2 + (\operatorname{sgn}^+ u)^2|\nabla u|^2)^{(p-2)/2}$$
$$\times\left(\Delta u + (p-2)(\operatorname{sgn}^+ u)^2 \frac{\sum_{j,k=1}^N \partial_{j,k}u\partial_j u\partial_k u}{\eta^2 + (\operatorname{sgn}^+ u)^2|\nabla u|^2}\right). \quad (2.14)$$

3 MAIN RESULT

We introduce an admissible class $K_p(\Omega)$ for the operators L_p.

DEFINITION 3.1 *Let $p > 1$ and $p^* = \max(p-1, 1)$. Let us set*

$$K_p(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega)\colon \partial_j u, \partial^2_{jk}u \in L^{p^*}_{\mathrm{loc}}(\Omega),$$
$$|\nabla u|^{p-2}|\partial^2_{j,k}u| \in L^1_{\mathrm{loc}}(\Omega)\ \textit{for}\ j,k = 1,2,\ldots,N\}. \quad (3.1)$$

Now we are in a position to state our main result.

THEOREM 3.1 *Let $p > 1$. Assume that $u \in K_p(\Omega)$, then it holds that in $\mathcal{D}'(\Omega)$*

$$\begin{aligned} L_p|u| &\geq (\operatorname{sgn} u)L_p u, \\ L_p u_+ &\geq (\operatorname{sgn}^+ u)^{p-1}L_p u. \end{aligned} \quad (3.2)$$

Remark 1 (1) If $p = 2$, then $K_2(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega)\colon \partial_j u, \partial^2_{jk}u \in L^1_{\mathrm{loc}}(\Omega)$, for $j,k = 1,2,\ldots,N\}$. Since $L_2 = \Delta$ in this case, it is known that Kato's inequalities hold under the assumptions that $u, \Delta u \in L^1_{\mathrm{loc}}(\Omega)$. But if $p \neq 2$, the operator L_p is nonlinear. Hence it was needed to introduce the class K_p. If $p > 2$, we see $|\nabla u|^{p-2}|\partial^2_{jk}u| \in L^1_{\mathrm{loc}}(\Omega)$ by a Young's inequality.

(2) We can also establish the same type results for the operators with variable coefficients.

Proof Without loss of generality, we assume that $\Omega = \mathbb{R}^N$. If $u \in C^2(\mathbb{R}^N)$, then the assertions follow from Lemma 2.1. Hence we approximate a locally integrable function u by smooth functions u_ρ ($\rho > 0$) as follows: Let us set $B_r = \{x \in \mathbb{R}^N\colon |x| < r\}$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfy $\varphi \geq 0$, $\int_{\mathbb{R}^N}\varphi(x)\,\mathrm{d}x = 1$ and $\varphi = 0$ in B_2^c. Now we set

$\varphi_\rho(x) = \rho^{-N}\varphi(x/\rho)$ for $\rho > 0$ and define

$$u_\rho(x) = u * \varphi_\rho(x) \equiv \int_{\mathbb{R}^N} u(x-y)\varphi_\rho(y)\,\mathrm{d}y. \tag{3.3}$$

Then it is clear from the assumptions on u that as $\rho \to 0$

$$\begin{gathered} u_\rho \to u \quad \text{almost everywhere} \\ u_\rho, \partial_j u_\rho, \partial^2_{j,k} u_\rho \to u, \partial_j u, \partial^2_{j,k} u \quad \text{in } L^{p^*}_{\mathrm{loc}}(\mathbb{R}^N) \text{ respectively.} \end{gathered} \tag{3.4}$$

Moreover we shall show that as $\rho \to 0$

$$L_p u_\rho \to L_p u \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^N). \tag{3.5}$$

First it follows from the definition of the operator L_p that for a smooth function v

$$|L_p v| \le (p-1)|\nabla v|^{p-2} \sum_{j=1}^{N} |\partial^2_{j,k} v|. \tag{3.6}$$

Therefore we see $L_p u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

Now we assume that $p \ge 2$. Then from Hölder's inequality it holds that for any $\rho > 0$ and any compact set K

$$\begin{aligned} \int_K |L_p u_\rho|\,\mathrm{d}x &\le (p-1) \int_K |\nabla u_\rho|^{p-2} \sum_{j=1}^{N} |\partial^2_{j,k} u_\rho|\,\mathrm{d}x \\ &\le (p-1) \sum_{j,k=1}^{N} \left(\int_K |\nabla u_\rho|^{p-1} \mathrm{d}x \right)^{(p-2)/(p-1)} \\ &\quad \times \left(\int_K |\partial^2_{j,k} u_\rho|^{p-1} \mathrm{d}x \right)^{1/(p-1)} \\ &\le C(K) < +\infty. \end{aligned} \tag{3.7}$$

Here $C(K)$ is a positive number independent of each $\rho > 0$. Hence by (3.4) and the dominated convergence theorem we have $L_p u_\rho \to L_p u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$

as $\rho \to 0$. From Lemma 1.1 and the dominated convergence theorem we see

$$L_p(u_\rho)_\varepsilon \geq \left|\frac{u_\rho}{(u_\rho)_\varepsilon}\right|^{p-2} \frac{u_\rho}{(u_\rho)_\varepsilon} L_p u_\rho \tag{3.8}$$

$$= \left|\frac{u_\rho}{(u_\rho)_\varepsilon}\right|^{p-2} \frac{u_\rho}{(u_\rho)_\varepsilon} (L_p u_\rho - L_p u)$$

$$+ L_p u \left(\left|\frac{u_\rho}{(u_\rho)_\varepsilon}\right|^{p-2} \frac{u_\rho}{(u_\rho)_\varepsilon} - \left|\frac{u}{(u)_\varepsilon}\right|^{p-2} \frac{u}{(u)_\varepsilon} \right)$$

$$+ \left|\frac{u}{(u)_\varepsilon}\right|^{p-2} \frac{u}{(u)_\varepsilon} L_p u$$

$$\to \left|\frac{u}{(u)_\varepsilon}\right|^{p-2} \frac{u}{(u)_\varepsilon} L_p u, \quad \text{as } \rho \to 0. \tag{3.9}$$

Since $L_p(u_\rho)_\varepsilon \to L_p u_\varepsilon$ in the sense of the distribution, we get

$$L_p u_\varepsilon \geq \left|\frac{u}{u_\varepsilon}\right|^{p-2} \frac{u}{u_\varepsilon} L_p u \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{3.10}$$

Then by letting $\varepsilon \to 0$, we see $L_p u_\varepsilon \to L_{\mathrm{p}}|u|$ in the sense of the distribution, and the right-hand side tends to $(\operatorname{sgn} u)L_p u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore we get the desired inequality.

We proceed to the case that $1 < p < 2$. In this case we make use of $L_{p,\eta}$ instead. First we see for any compact set K of $\mathbb{R}^N$ and any $\eta > 0$,

$$\int_K |L_{p,\eta} u| \, \mathrm{d}x \leq (p-1) \int_K (\eta^2 + |\nabla u|^2)^{(p-2)/2} \sum_{j,k=1}^{N} |\partial^2_{j,k} u| \, \mathrm{d}x < \infty. \tag{3.11}$$

Here we note that $1 < p < 2$ and $\partial^2_{j,k} u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for $j, k = 1, 2, \ldots, N$. Let u_ρ be defined by (3.3). Then it follows from Lemma 2.2 that $(u_\rho)_\varepsilon$ satisfies

$$L_{p,\eta}(u_\rho)_\varepsilon \geq \frac{u_\rho}{(u_\rho)_\varepsilon} (\eta^2 + |\nabla (u_\rho)_\varepsilon|^2)^{(p-2)/2}$$

$$\times \left(\Delta u_\rho + (p-2) \left(\frac{u_\rho}{(u_\rho)_\varepsilon} \right)^2 \frac{\sum_{j,k=1}^N \partial_j u_\rho \partial_k u_\rho \partial_{j,k} u_\rho}{\eta^2 + |\nabla (u_\rho)_\epsilon|^2} \right). \tag{3.12}$$

As $\rho \to 0$, we see $L_{p,\eta}(u_\rho)_\varepsilon \to L_{p,\eta}(u)_\varepsilon$ in the sense of distribution, and the terms in the right-hand side also converges in $L^1_{\text{loc}}(\mathbb{R}^n)$. Therefore we get in $\mathcal{D}'(\mathbb{R}^N)$

$$L_{p,\eta}u_\varepsilon \geq \frac{u}{u_\varepsilon}(\eta^2 + |\nabla u_\varepsilon|^2)^{(p-2)/2} \times \left(\Delta u + (p-2)\left(\frac{u}{u_\varepsilon}\right)^2 \frac{\sum_{j,k=1}^N \partial_j u \partial_k u \partial_{j,k} u}{\eta^2 + |\nabla u_\varepsilon|^2}\right). \tag{3.13}$$

Letting $\varepsilon \to 0$ we have in a similar way in $\mathcal{D}'(\mathbb{R}^N)$

$$L_{p,\eta}|u| \geq (\operatorname{sgn} u)(\eta^2 + |\nabla u|^2)^{(p-2)/2} \times \left(\Delta u + (p-2)\frac{\sum_{j,k=1}^N \partial_j u \partial_k u \partial_{j,k} u}{\eta^2 + |\nabla u|^2}\right). \tag{3.14}$$

Finally by letting $\eta \to 0$, we have in the sense of distribution $L_{p,\eta}|u| \to L_p|u|$, and the right-hand side also converges in $L^1_{\text{loc}}(\mathbb{R}^N)$. After all we get

$$L_p|u| \geq (\operatorname{sgn} u)L_p u \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{3.15}$$

In a similar way we can show

$$L_p u_+ \geq (\operatorname{sgn}^+ u)^{p-1} L_p u \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{3.16}$$

by making use of Lemma 2.2. Therefore the assertions are proved.

Reference

[1] T. Kato, Schrödinger operators with singular potentials, *Israel J. Math.*, **13** (1972), 135–148.