

Exact Bounds for Some Basis Functions of Approximation Operators

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The exact bounds of Bernstein basic functions and Meyer-König and Zeller basis functions have been determined in [*J. Math. Anal. Appl.*, **219** (1998), 364–376]. In this note the exact bounds of some other basis functions of approximation operators and corresponding probability distributions are determined.

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1. INTRODUCTION

In approximation theory the so-called Bernstein basis functions are

$$P_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad (0 \leq k \leq n, x \in [0, 1]), \quad (1)$$

the Meyer-König and Zeller basis functions are

$$M_{nk}(x) = \binom{n+k-1}{k} x^k (1-x)^n \quad (k \in \mathbf{N}, x \in [0, 1]), \quad (2)$$

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and the Szász basis functions are

$$S_{nk}(x) = \frac{(nx)^k}{k!} e^{-nx} \quad (k \in \mathbf{N}, x \in [0, \infty)). \quad (3)$$

Throughout this note, the sign \mathbf{N} denotes the set of nonnegative integer. It is well-known that the basis functions $P_{nk}(x)$, $M_{nk}(x)$ and $S_{nk}(x)$ correspond with the binomial distribution, Pascal distribution and Poisson distribution, respectively in probability theory. If we replace parameter n in Pascal distribution by continuous parameter $\alpha > 0$, we get the so-called negative binomial distribution:

$$\binom{\alpha + k - 1}{k} x^k (1 - x)^\alpha = M_{\alpha,k}(x) = P(X_{\alpha,x} = k) \quad (4)$$

where $X_{\alpha,t}$ denotes a random variable, $x \in (0, 1]$ is a parameter, $k \in \mathbf{N}$, and $\binom{\alpha+k-1}{k} = (\Gamma(\alpha+k)/k! \Gamma(\alpha))$.

In approximation theory it is important to estimate the above-mentioned basis functions and some other basis functions of approximation operators. Specially, these estimations play key roles in studying rates of convergence of approximation operators for functions of bounded variation and bounded functions (*cf.* [1–9]). Recently, the exact bounds of basis functions $P_{nk}(x)$ and $M_{nk}(x)$ (discrete parameter) have been determined in [1]. In this note further research is made for the cases of continuous parameter, for other univariate basis functions of approximation operators and for the corresponding multivariate basis functions of approximation operators.

2. BOUNDS FOR UNIVARIATE BASIS FUNCTIONS

We first consider the case of continuous parameter $\alpha > 0$, *i.e.*, negative binomial distribution (4) and prove the following:

THEOREM 1 *Let j be fixed nonnegative integer and $C_j = ((j+1/2)^{j+1/2}/j!)e^{-(j+1/2)}$. Then for all k, x such that $k \geq j$, $x \in [0, 1]$, there holds*

$$x^{1/2} M_{\alpha,k}(x) < C_j \alpha^{-1/2} \quad (5)$$

Moreover, the coefficients C_j and the asymptotically order $\alpha^{-1/2}$ (for $\alpha \rightarrow +\infty$) are the best possible.

Because the technique of the Lemma 1 of [1] is not valid for the case of continuous parameter $\alpha > 0$, for proving Theorem 1, we need new technique, which mainly is an identical relation concerning Gamma function and its derivative.

LEMMA 1 *Let $\Gamma(t)$ be Gamma function. Then for $\alpha > 0$, and $k = 1, 2, 3, \dots$, we have*

$$\frac{\Gamma'(\alpha + k)}{\Gamma(\alpha + k)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \sum_{i=0}^{k-1} \frac{1}{\alpha + i}. \tag{6}$$

Proof We have

$$\begin{aligned} \Gamma(h) - \frac{\Gamma(h)\Gamma(\alpha)}{\Gamma(h + \alpha)} &= \frac{\Gamma(h + \alpha) - \Gamma(\alpha)}{h} \frac{h\Gamma(h)}{\Gamma(h + \alpha)} \\ &= \frac{\Gamma(h + \alpha) - \Gamma(\alpha)}{h} \frac{\Gamma(h + 1)}{\Gamma(h + \alpha)} \end{aligned}$$

Hence

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \lim_{h \rightarrow 0} \left(\Gamma(h) - \frac{\Gamma(h)\Gamma(\alpha)}{\Gamma(h + \alpha)} \right) \tag{7}$$

Since $\Gamma(h) = \int_0^\infty u^{h-1} e^{-u} du$, $(\Gamma(h)\Gamma(\alpha)/\Gamma(h + \alpha)) = B(\alpha, h) = \int_0^\infty (u^{h-1}/(1 + u)^{h+\alpha}) du$, where $B(\alpha, h)$ is the so-called Beta function, from (7) it follows that

$$\begin{aligned} \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} &= \lim_{h \rightarrow 0} \int_0^\infty u^{h-1} \left(e^{-u} - \frac{1}{(1 + u)^{h+\alpha}} \right) du \\ &= \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{1}{u(1 + u)^\alpha} \right) du \end{aligned}$$

So

$$\frac{\Gamma'(\alpha + k)}{\Gamma(\alpha + k)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \int_0^\infty \left(\frac{1}{u(1 + u)^\alpha} - \frac{1}{u(1 + u)^{\alpha+k}} \right) du.$$

Replacing by $t = (1/(1+u))$ in the above integral we find that

$$\frac{\Gamma'(\alpha + k)}{\Gamma(\alpha + k)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \int_0^1 \left(\frac{t^{\alpha-1} - t^{\alpha+k-1}}{1-t} \right) dt \tag{8}$$

Since $((t^{\alpha-1} - t^{\alpha+k-1})/(1-t)) = \sum_{i=0}^{\infty} (t^{i+\alpha-1} - t^{i+\alpha+k-1})$, integrating term by term on the right hand side of (8), we get identical relation (6).

Proof of Theorem 1 By computing derivative we find for all $x \in [0, 1]$ that

$$\begin{aligned} x^{1/2}M_{\alpha k}(x) &\leq \left(\frac{k + 1/2}{\alpha + k + 1/2} \right)^{1/2} M_{\alpha k} \left(\frac{k + 1/2}{\alpha + k + 1/2} \right) \\ &= \frac{(k + 1/2)^{k+1/2}}{k!} \frac{\Gamma(\alpha + k)\alpha^\alpha}{\Gamma(\alpha)(\alpha + k + 1/2)^{\alpha+k+1/2}} \frac{\sqrt{\alpha}}{\sqrt{\alpha}} \end{aligned} \tag{9}$$

Set

$$G(\alpha, k) = \frac{\Gamma(\alpha + k)\alpha^{\alpha+1/2}}{\Gamma(\alpha)(\alpha + k + 1/2)^{\alpha+k+1/2}} > 0$$

Then, by calculation and using Lemma 1, it follows that

$$\begin{aligned} \frac{d(\log G(\alpha, k))}{d\alpha} &= \frac{\Gamma'(\alpha + k)}{\Gamma(\alpha + k)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \log \alpha + \frac{\alpha + 1/2}{\alpha} \\ &\quad - \log(\alpha + k + 1/2) - \frac{\alpha + k + 1/2}{\alpha + k + 1/2} \\ &= \sum_{i=0}^{k-1} \frac{1}{\alpha + i} + \frac{1}{2\alpha} + \log \frac{\alpha}{\alpha + k + 1/2} \\ &> \int_{\alpha}^{\alpha+k} \frac{1}{x} dx + \int_{2\alpha}^{2\alpha+1} \frac{1}{x} dx + \log \frac{\alpha}{\alpha + k + 1/2} \\ &= \log \frac{(2\alpha + 1)(\alpha + k)}{2\alpha(\alpha + k + 1/2)} > 0 \end{aligned}$$

Therefore $G(\alpha, k)$ is monotone increasing for α by the fact that

$$\frac{d(G(\alpha, k))}{d\alpha} = G(\alpha, k) \frac{d(\log G(\alpha, k))}{d\alpha} > 0$$

On the other hand, using Stirling’s formula: $\lim_{\alpha \rightarrow +\infty} ((\Gamma(\alpha + 1))/((\alpha/e)^\alpha \sqrt{2\pi\alpha})) = 1$ (cf. [11, or 12, Chapter 2]), we get by direct calculation

$$\lim_{\alpha \rightarrow +\infty} G(\alpha, k) = \lim_{\alpha \rightarrow +\infty} \frac{\Gamma(\alpha + k)\alpha^{\alpha+1/2}}{\Gamma(\alpha)(\alpha + k + 1/2)^{\alpha+k+1/2}} = e^{-(k+1/2)} \tag{10}$$

Hence from (9)

$$x^{1/2}M_{\alpha,k}(x) \leq \frac{(k + 1/2)^{k+1/2}}{k!} G(\alpha, k) \frac{1}{\sqrt{\alpha}} \leq \frac{(k + 1/2)^{k+1/2}}{k!} e^{-(k+1/2)} \frac{1}{\sqrt{\alpha}}$$

Inequality (5) now follows from the monotonicity of $C_k = ((k + 1/2)^{k+1/2}/k!)e^{-(k+1/2)}$, again, from Theorem 2 of [1], we know that the estimate order $\alpha^{-1/2}$ in (5) is the asymptotically optimal. The proof of Theorem 1 is complete.

Let

$$b_{nk}(x) = \binom{n + k - 1}{k} \left(\frac{x}{1 + x}\right)^k (1 + x)^{-n} (x \in [0, \infty), k \in \mathbf{N})$$

be the so-called Baskatov basis functions. As a key auxiliary result of [2], Wang and Guo gave the upper bounds for the basis functions $b_{nk}(x)$ as follows:

([2, Lemma 3]) For every $x \in (0, \infty)$, $k \in \mathbf{N}$, we have

$$b_{nk}(x) \leq \frac{33}{\sqrt{n}} \left(\frac{1 + x}{x}\right)^{3/2} \tag{11}$$

Now in Theorem 1 by taking $\alpha = n$ and replacing variable x with $(x/(1+x))$ in $M_{\alpha,k}(x)$, we get the exact upper bound for $b_{nk}(x)$ immediately

COROLLARY 1 For every $x \in (0, \infty)$, $k \in \mathbf{N}$, we have

$$b_{nk}(x) \leq \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{n}} \sqrt{\frac{1 + x}{x}}. \tag{12}$$

Corollary 1 can be used to improve the main result of [2], we omit the details.

Below we discuss the Szász basis functions $S_{nk}(x)$. In the Lemma 3 of [7] it is proved that $S_{nk}(x) \leq (1/\sqrt{\pi})(1/\sqrt{nx})$. The following Proposition 1 gives a better estimate.

PROPOSITION 1 *Let $H(j) = ((j+1/2)^{j+1/2}/j!) e^{-(j+1/2)}$. Then for all $k \geq j$ and $x \in [0, \infty)$, there hold*

$$\sqrt{x}S_{nk}(x) \leq H(j) \frac{1}{\sqrt{n}}, \quad (13)$$

where the coefficient $H(j) = ((j+1/2)^{j+1/2}/j!) e^{-(j+1/2)}$ and the estimate order $n^{-1/2}$ are the best possible.

Proof By calculation we find that

$$\begin{aligned} \sqrt{nx}S_{nk}(x) &\leq \sqrt{k+1/2}S_{nk}\left(\frac{k+1/2}{n}\right) \\ &= \frac{(k+1/2)^{k+1/2}}{k!} e^{-(k+1/2)}, \quad \text{for all } x \in [0, \infty), \end{aligned}$$

and $(H(j+1)/H(j)) < 1$, Hence $H(j)$ is monotone decreasing with j . So the inequality (13) holds.

For proving the estimate order $n^{-1/2}$ in (13) is the best. We take $k = [nx]$, then writing $nx = [nx] + \varepsilon$ ($0 \leq \varepsilon < 1$), and using formula $n! \sim (n/e)^n \sqrt{2\pi n}$, we have

$$\begin{aligned} S_{n,[nx]}(x) &= \frac{(nx)^{[nx]}}{[nx]!} e^{-nx} = \frac{([nx] + \varepsilon)^{[nx]}}{[nx]!} e^{-[nx] - \varepsilon} \\ &\sim \frac{([nx] + \varepsilon)^{[nx]}}{[nx]^{[nx]} e^{-[nx]} \sqrt{2\pi [nx]}} e^{-[nx] - \varepsilon} \end{aligned}$$

That is

$$S_{n,[nx]}(x) \sim \left(1 + \frac{\varepsilon}{[nx]}\right)^{[nx]} e^{-\varepsilon} \frac{1}{\sqrt{2\pi [nx]}}. \quad (14)$$

From (14) we deduce that the estimate order $n^{-1/2}$ (for $n \rightarrow +\infty$) in (13) is the best possible.

From [4–8] it is known that in some actual applications on convergence of approximation operators for functions of bounded variation we only need to estimate the values of basis functions $P_{nk}(x)$, $M_{nk}(x)$ and $S_{nk}(x)$ at those points $x = (k/n)$ or $x = (k/(n+k))$. In that cases some better bounds can be obtained. We give a result of this type.

PROPOSITION 2 For $x = k_0/n$ (k_0 is a fixed positive integer, $k_0 < n$) and $k = 0, 1, 2, \dots, n$, there holds

$$P_{nk} \left(\frac{k_0}{n} \right) < \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n} \sqrt{x(1-x)}}. \tag{15}$$

The estimate coefficient $1/\sqrt{2\pi}$ in (15) is the best possible.

Proof If $x = k/n$, then

$$\begin{aligned} \sqrt{n} \sqrt{x(1-x)} P_{nk}(x) &= \frac{\sqrt{n} n!}{k!(n-k)!} \left(\frac{k}{n} \right)^{k+1/2} \left(\frac{n-k}{n} \right)^{n-k+1/2} \\ &= \frac{k^{k+1/2} n!(n-k)^{n-k+1/2}}{k! (n-k)! n^{n+1/2}} \end{aligned}$$

Set $\Delta(n, k) = (n!(n-k)^{n-k+1/2}/(n-k)!n^{n+1/2})$. Then

$$\frac{\Delta(n+1, k)}{\Delta(n, k)} = \left(\frac{n-k+1}{n-k} \right)^{n-k+1/2} \left(\frac{n}{n+1} \right)^{n+1/2} > 1 \tag{16}$$

The right hand inequality of (16) is due to the fact that $(1+(1/n))^{n+1/2}$ is monotone decreasing. Direct calculation gives $\lim_{n \rightarrow +\infty} \Delta(n, k) = e^{-k}$. Hence, it follows for $x = k/n$ that

$$P_{nk} \left(\frac{k}{n} \right) \leq \frac{k^{k+1/2}}{k!} e^{-k} \frac{1}{\sqrt{n} \sqrt{x(1-x)}} < \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n} \sqrt{x(1-x)}} \tag{17}$$

Below we prove that

$$P_{nk} \left(\frac{k_0}{n} \right) \leq P_{nk_0} \left(\frac{k_0}{n} \right). \tag{18}$$

In fact

$$\begin{aligned} \frac{P_{n,k+1}(k_0/n)}{P_{nk}(k_0/n)} &= \frac{(n!/(k+1)!(n-k-1)!(k_0/n)^{k+1}((n-k_0)/n)^{n-k-1}}{(n!/k!(n-k)!(k_0/n)^k((n-k_0)/n)^{n-k}} \\ &= \frac{(n-k)k_0}{(k+1)(n-k_0)} = \frac{nk_0 - kk_0}{nk + n - k_0 - kk_0} \\ &\begin{cases} > 1, & \text{if } k < k_0 - 1 + k_0/n \\ = 1, & \text{if } k = k_0 - 1 + k_0/n. \\ < 1, & \text{if } k > k_0 - 1 + k_0/n \end{cases} \end{aligned}$$

Therefore (18) holds for $k=0, 1, 2, 3, \dots, n$ and fixed k_0 satisfying $0 < k_0 < n$. Inequality (15) now follows from (17) and (18).

3. BOUNDS FOR MULTIVARIATE BASIS FUNCTIONS

In this section we consider some basis functions of multivariate approximation operators. First we discuss the basis functions of the Bernstein operator over a simplex.

Let $\Delta_k = \{(x_1, \dots, x_k): x_i \geq 0; i = 1, 2, \dots, k; \text{ and } x_1 + \dots + x_k \leq 1\}$ be the standard simplex in \mathbf{R}^k . The basis functions of Bernstein operators over Δ_k are defined as

$$P_{n,j_1,\dots,j_k}(x_1, \dots, x_k) = \frac{n!}{j_1! \dots j_k! (n - j_1 - \dots - j_k)!} x_1^{j_1} \dots x_k^{j_k} (1 - x_1 - \dots - x_k)^{n - j_1 - \dots - j_k}$$

where $j_i \geq 0; i = 1, 2, \dots, k; j_1 + \dots + j_k \leq n$.

We will determine the exact upper bound of the basis functions $P_{n,j_1,\dots,j_k}(x_1, \dots, x_k)$. For convenience we first prove our result for the case $k = 2$.

PROPOSITION 3 *For all nonnegative integers j_1, j_2 satisfying $j_1 + j_2 \leq n$ and $(x_1, x_2) \in \Delta_2$, there holds*

$$P_{n,j_1,j_2}(x_1, x_2) \leq \sqrt{\frac{3}{2}} \frac{1}{8\pi n} \frac{1}{x_1 x_2 (1 - x_1 - x_2)}, \tag{19}$$

where the coefficient $\sqrt{3}/(8\pi\sqrt{2})$ and the estimate order n^{-1} are the best possible.

Proof We can write

$$\begin{aligned}
 P_{n,j_1,j_2}(x_1, x_2) &= \frac{n!}{j_1!j_2!(n-j_1-j_2)!} x_1^{j_1} x_2^{j_2} (1-x_1-x_2)^{n-j_1-j_2} \\
 &= \frac{n!(n-j_1)!}{j_1!j_2!(n-j_1)!(n-j_1-j_2)!} x_1^{j_1} \frac{(1-x_1)^{n-j_1}}{(1-x_1)^{n-j_1}} x_2^{j_2} \\
 &\quad (1-x_1-x_2)^{n-j_1-j_2} \tag{20} \\
 &= \frac{n!}{j_1!(n-j_1)!} x_1^{j_1} (1-x_1)^{n-j_1} \frac{(n-j_1)!}{j_2!(n-j_1-j_2)!} \\
 &\quad \left(\frac{x_2}{1-x_1}\right)^{j_2} \left(1-\frac{x_2}{1-x_1}\right)^{n-j_1-j_2}
 \end{aligned}$$

Using the Proposition 2 of [1] for (20), we get

$$P_{n,j_1,j_2}(x_1, x_2) \leq \frac{1}{8\pi} \frac{1}{\sqrt{n}\sqrt{n-j_1}} \frac{1-x_1}{x_1 x_2 (1-x_1-x_2)}. \tag{21}$$

By symmetry of j_1 and j_2 in $P_{n,j_1,j_2}(x_1, x_2)$, we get as well

$$P_{n,j_1,j_2}(x_1, x_2) \leq \frac{1}{8\pi} \frac{1}{\sqrt{n}\sqrt{n-j_2}} \frac{1-x_2}{x_1 x_2 (1-x_1-x_2)}. \tag{22}$$

On the other hand, note that

$$\begin{aligned}
 P_{n,j_1,j_2}(x_1, x_2) &= \frac{n!}{j_1!j_2!(n-j_1-j_2)!} x_1^{j_1} x_2^{j_2} (1-x_1-x_2)^{n-j_1-j_2} \\
 &= \frac{n!(j_1+j_2)!}{j_1!j_2!(j_1+j_2)!(n-j_1-j_2)!} x_1^{j_1} x_2^{j_2} \frac{(x_1+x_2)^{j_1+j_2}}{(x_1+x_2)^{j_1+j_2}} \\
 &\quad (1-x_1-x_2)^{n-j_1-j_2} \\
 &= \frac{n!}{(j_1+j_2)!(n-j_1-j_2)!} (x_1+x_2)^{j_1+j_2} \\
 &\quad (1-x_1-x_2)^{n-j_1-j_2} \frac{(j_1+j_2)!}{j_1!j_2!} \\
 &\quad \left(\frac{x_1}{x_1+x_2}\right)^{j_1} \left(1-\frac{x_1}{x_1+x_2}\right)^{j_2} \tag{23}
 \end{aligned}$$

Using the Proposition 2 of [1] for (23), we get

$$\begin{aligned}
 P_{n,j_1,j_2}(x_1, x_2) &\leq \frac{1}{\sqrt{8\pi}} \frac{1}{\sqrt{n}(x_1 + x_2)(1 - x_1 - x_2)} \frac{1}{\sqrt{8\pi}} \frac{(x_1 + x_2)^2}{\sqrt{j_1 + j_2}x_1x_2} \\
 &= \frac{1}{8\pi} \frac{1}{\sqrt{n}\sqrt{j_1 + j_2}} \frac{x_1 + x_2}{x_1x_2(1 - x_1 - x_2)} \tag{24}
 \end{aligned}$$

Again, we find that for any nonnegative integers, j_1, j_2 satisfying $j_1 + j_2 \leq n$, there holds

$$\min \left\{ \frac{1}{\sqrt{n - j_1}}, \frac{1}{\sqrt{n - j_2}}, \frac{1}{\sqrt{j_1 + j_2}} \right\} \leq \sqrt{\frac{3}{2}} \frac{1}{\sqrt{n}}. \tag{25}$$

The sign of equality holds in (25) if and only if $j_1 = j_2 = n/3$.

Note that $0 \leq x_1 + x_2 \leq 1$, from (21), (22), (24) and (25), we get (19). By the Proposition 2 of [1] and (25), we deduce that the coefficient $\sqrt{3}/(8\pi\sqrt{2})$ and the estimate order n^{-1} in (19) are the best possible. The proof of Proposition 3 is complete.

From the Proposition 2 of [1] and Proposition 3, we can get the following Theorem with the recurrence method.

THEOREM 2 For $k \geq 1$ and all j_i satisfying $j_i \geq 0; i = 1, 2, \dots, k; j_1 + \dots + j_k \leq n$ and $(x_1, \dots, x_k) \in \Delta_k$, there holds

$$\begin{aligned}
 P_{n,j_1,\dots,j_k}(x_1, \dots, x_k) &\leq \sqrt{\frac{(k + 1)^k}{(k + 1)!}} \frac{1}{(8\pi)^{k/2}} \frac{1}{n^{k/2}} \\
 &\frac{1}{x_1 \dots x_k (1 - x_1 - \dots - x_k)} \tag{26}
 \end{aligned}$$

Moreover, the coefficient $\left(\sqrt{(k + 1)^k / (k + 1)!} \right) (1 / (8\pi)^{k/2})$, and the estimate order $n^{-k/2}$ in (26) are the best possible.

It is known that $P_{n,j_1,\dots,j_k}(x_1, \dots, x_k)$ corresponds with the multinomial distribution $(D_{n,x_1}, \dots, D_{n,x_k})$ with parameters $(n, x_1, \dots, x_k, 1 - x_1 - \dots - x_k)$ in probability theory, i.e.,

$$P((D_{n,x_1}, \dots, D_{n,x_k}) = (j_1, \dots, j_k)) = P_{n,j_1,\dots,j_k}(x_1, \dots, x_k).$$

Hence formula (26) also is an exact upper bound for the multinomial distribution in probability theory.

Next we discuss the so-called negative multinomial distribution:

$$M_{n,j_1,j_2}(x_1, x_2) = \frac{(n + j_1 + j_2 - 1)!}{j_1!j_2!(n - 1)!} x_1^{j_1} x_2^{j_2} (1 - x_1 - x_2)^n, \\ j_1, j_2 \in \mathbb{N} \text{ and } (x_1, x_2) \in \Delta_2,$$

which corresponds with two-dimension Meyer-könig and Zeller basis functions over a simplex (cf. [13, 14]). The higher dimensional case $M_{n,j_1,\dots,j_k}(x_1, \dots, x_k)$ can be defined similarly.

It is somewhat difficult to decompose $M_{n,j_1,j_2}(x_1, x_2)$ directly like we have done for $P_{n,j_1,j_2}(x_1, x_2)$. Hence we first need a replacement in $M_{n,j_1,j_2}(x_1, x_2)$ with

$$\begin{cases} x_1 = t_1/(1 + t_1 + t_2) \\ x_2 = t_2/(1 + t_1 + t_2) \end{cases} \tag{27}$$

Then

$$M_{n,j_1,j_2}(x_1, x_2) = \frac{(n + j_1 + j_2 - 1)!}{j_1!j_2!(n - 1)!} t_1^{j_1} t_2^{j_2} (1 + t_1 + t_2)^{-n-j_1-j_2} \\ = \frac{(n + j_1 - 1)!}{j_1!(n - 1)!} t_1^{j_1} (1 + t_1)^{-n-j_1} \frac{(n + j_1 + j_2 - 1)!}{j_2!(n + j_1 - 1)!} \\ \left(\frac{t_2}{1 + t_2}\right)^{j_2} \left(1 + \frac{t_2}{1 + t_2}\right)^{-n-j_1-j_2}$$

Hence from Corollary 1 and (27) it follows for all $j_2 \in \mathbb{N}$ that

$$M_{n,j_1,j_2}(x_1, x_2) \leq \frac{1}{2e} \frac{1}{\sqrt{n(n + j_1)}} \frac{\sqrt{1 + t_1} \sqrt{1 + t_1 + t_2}}{\sqrt{t_1 t_2}} \\ = \frac{1}{2e} \frac{1}{\sqrt{n(n + j_1)}} \frac{\sqrt{1 - x_1}}{\sqrt{x_1 x_2}} \tag{28}$$

By symmetry, it follows for all $j_1 \in \mathbb{N}$ that

$$M_{n,j_1,j_2}(x_1, x_2) \leq \frac{1}{2e} \frac{1}{\sqrt{n(n + j_2)}} \frac{\sqrt{1 - x_2}}{\sqrt{x_1 x_2}}. \tag{29}$$

Inequalities (28) and (29) derive

PROPOSITION 4 *There holds uniformly for all $j_1, j_2 \in N$*

$$M_{n,j_1,j_2}(x_1, x_2) \leq \frac{1}{2en\sqrt{x_1x_2}}, \quad (30)$$

where the coefficient $1/(2e)$ and the estimate order n^{-1} are the best possible.

Similar discussion, for the case of higher dimension we get

THEOREM 3 *There holds uniformly for all $j_1, \dots, j_k \in N$*

$$M_{n,j_1,\dots,j_k}(x_1, \dots, x_k) \leq \frac{1}{(2en)^{k/2} \sqrt{x_1, \dots, x_k}}, \quad (31)$$

where the coefficient $1/(2e)^{k/2}$ and the estimate order $n^{-k/2}$ are the best possible.

For the bounds of basis functions of the tensor product operators formed by the Bernstein, Szász, Baskakov, Meyer-könig and Zeller, the results can be get easily from the results of correspondent univariate operators. We omit the discussion. We conclude this note with an interesting result by combining Theorems 2, 3, and the Theorem 2, the Proposition 2 of [1].

THEOREM 4 *For $k > 1$, Meyer-könig and Zeller basis functions over simplex Δ_k and Meyer-könig and Zeller basis functions of k -dimension tensor product have the same optimal upper bound. However for Bernstein basis functions over simplex Δ_k and Bernstein basis functions of k -dimension tensor product, the conclusion is quite the contrary.*

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