J. of Inequal. & Appl., 2002, Vol. 7(5), pp. 623–631

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On Growth of Polynomials

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(Received on 28 December, 2000; Revised on 28 March, 2001)

Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n, $M(p,R) = \max_{|z|=R\geq 1} |p(z)|$ and $\|p\| = \max_{|z|=1} |p(z)|$. If $p(z) \neq 0$ in $|z| < 1$, then according to a well known result of Ankeny and Rivlin, $M(p,R) \leq \{(R^n+1)/2\}\|P\|$ for $R \geq 1$. In this paper, we generalize and sharpen this and some other related inequalities by considering polynomials having no zeros in $|z| < K$, $K \geq 1$.

Keywords: Inequalities in the complex domain; Polynomials; Approximation in the complex domain.

1991 Mathematics Subject Classification. Primary 30A10; Secondary 30C10, 30E10.

1 INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be an algebraic polynomial of degree n, z being a complex variable, $\|p\| = \max_{|z|=1} |p(z)|$ and let $M(p,R) = \max_{|z|=R} |p(z)|$. It is well known that

$$M(p,R) \leq R^n \|p\|, \quad R \geq 1. \tag{1.1}$$

Further, if $p(z)$ has no zeros in $|z| < 1$, then

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right) \|p\|, \quad R \geq 1. \tag{1.2}$$

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ISSN 1025-5834 print; ISSN 1029-242X.
DOI: 10.1080/1025583021000022414

Both the above inequalities are sharp. The inequality (1.1) becomes equality for $p(z) = \lambda z^n, \lambda \in \mathbb{C}$ while in (1.2) the equality holds for $p(z) = \lambda z^n + \mu$, $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}$ and $|\lambda| = |\mu|$.

The inequality (1.1) follows immediately from the maximum modulus principle and the inequality (1.2) is a well-known result due to Ankeny and Rivlin [1]. In the literature there already exists some refinements of these inequalities (for example, see Aziz and Dawood, [2] Frappier, Rahman and Ruscheweyh, [4] Govil, [5,6] Sharma and Singh, [11] Szegö[12]).

For polynomials not vanishing in $|z| < K$, $K \geq 1$, Rahman and Schmeisser [9, p. 87] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < K$, $K \geq 1$, then for $1 \leq R \leq K^2$

$$M(p, R) \leq \left(\frac{R+K}{1+K}\right)^n \| p\|. \tag{1.3}$$

Although the above result (1.3) is sharp, with equality holding for $p(z) = (z+K)^n$, this result unfortunately holds only in the range $1 \leq R \leq K^2$.

As a generalization and refinement of (1.2), Govil [6, Theorem 2] proved that if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \geq 2$, having no zeros in $|z| < K$, $K \geq 1$, and $m = \min_{|z|=K} |p(z)|$, then for $R \geq 1$,

$$\begin{aligned} M(p, R) \leq& \left(\frac{R^n+K}{1+K}\right) \| p\| - \left(\frac{R^n-1}{1+K}\right) m \\ &- |a_1| \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right), \end{aligned} \tag{1.4}$$

if $n > 2$, and

$$M(p, R) \leq \left(\frac{R^n+K}{1+K}\right) \| p\| - \left(\frac{R^n-1}{1+K}\right) m - |a_1| \frac{(R-1)^n}{2}, \tag{1.5}$$

if $n = 2$.

As a generalization of the above inequalities, we prove

THEOREM *If* $p(z) = a_0 + \sum_{v=s}^{n} a_v z^v$ *is a polynomial of degree* $n \geq 2$, *having no zeros in* $|z| < K$, $K \geq 1$, *then for* $R \geq 1$,

$$M(p, R) \leq \left(\frac{R^n + K^s}{1 + K^s}\right) \| p \| - \left(\frac{R^n - 1}{1 + K^s}\right) m - |a_1| \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right), \tag{1.6}$$

if $n > 2$, *and*

$$M(p, R) \leq \left(\frac{R^n + K^s}{1 + K^s}\right) \| p \| - \left(\frac{R^n - 1}{1 + K^s}\right) m - |a_1| \frac{(R-1)^n}{2}, \tag{1.7}$$

if $n = 2$.

Since, as is easy to see, $\{[(R^n + x)/(1 + x)] \| p \| - [(R^n - 1)/(1 + x)]m\}$ is a decreasing function of x, hence for $s > 1$, the bound obtained by inequalities (1.6) and (1.7) is always sharper than the bound obtainable from inequalities (1.4) and (1.5), respectively. For $s = 1$, the inequalities (1.6) and (1.7), respectively reduce to inequalities (1.4) and (1.5), while for $K = 1$, inequalities (1.6) and (1.7) together sharpen both, the inequality (1.2) and the inequality due to Aziz and Dawood [2, Theorem 3].

If in (1.6) and (1.7) we divide both sides by R^n and make $R \to \infty$, we get

COROLLARY *If* $p(z) = a_0 + \sum_{v=s}^{n} a_v z^v$ *is a polynomial of degree* $n \geq 2$, *having no zeros in* $|z| < K$, $K \geq 1$, *then*

$$|a_n| + \frac{|a_1|}{n} \leq \left(\frac{\| p \| - m}{1 + K^s}\right). \tag{1.8}$$

Inequality (1.8) for $K = 1$ clearly sharpens the well known inequality

$$|a_n| + \frac{|a_1|}{n} \leq \frac{\| p \|}{2}, \tag{1.9}$$

which can be obtained by applying Visser's inequality [13] to the polynomial $p'(z)$ and then combining it with the well known inequality of Lax [8] conjectured by Erdös.

2 LEMMAS

For the proof of the Theorem, we shall need the following lemmas.

LEMMA 1 *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n, $n \leq 2$, *then for all* $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq R^n \|p\| - (R^n - R^{n-2})|p(0)|, \tag{2.1}$$

if $n > 2$, *and*

$$\max_{|z|=R} |p(z)| \leq R^n \|p\| - |a_1| \frac{(R-1)^n}{2}, \tag{2.2}$$

if $n = 2$.

The above result is due to Frappier, Rahman and Ruscheweyh [4].

LEMMA 2 *If* $p(z) = a_0 + \sum_{\nu=s}^{n} a_\nu z^\nu$ *is a polynomial of degree* n, *having no zeros in* $|z| < K$, $K \geq 1$ *then on* $|z| = 1$,

$$K^s |p'(z)| \leq |q'(z)|, \tag{2.3}$$

where $q(z) = z^n \overline{p(1/\bar{z})}$.

This lemma is due to Chan and Malik [3], however for the sake of completeness, we present here the brief outlines of its proof.

Since $p(z) \neq 0$ for $|z| < K$, $K \geq 1$ it follows from a well known theorem of Laguerre (see for example [10, p. 463]) that

$$np(z) - zp'(z) \neq -\zeta p'(z) \quad \text{for} \quad |\zeta| < K, |z| < K,$$

which implies

$$\left|\frac{p'(z)}{np(z) - zp'(z)}\right| \le \frac{1}{K} \quad \text{for} \quad |z| \le K. \tag{2.4}$$

The function

$$f(z) = \frac{Kp'(Kz)}{np(Kz) - Kzp'(Kz)}$$

is therefore holomorphic in $|z| \le 1$, where it satisfies $|f(z)| \le 1$. Further $f(0) = f'(0) = \cdots = f^{(s-2)}(0) = 0$, hence by Schwarz's lemma

$$|f(z)| \le |z|^{s-1},$$

for $|z| < 1$. In particular, this holds also for $|z| = 1/K$, and so

$$K\left|\frac{p'(z)}{np(z) - zp'(z)}\right| \le \frac{1}{K^{s-1}} \quad \text{for} \quad |z| = 1. \tag{2.5}$$

On combining (2.5) with the fact that on $|z| = 1$, we have $|np(z) - zp'(z)| = |q'(z)|$, Lemma 2 follows.

LEMMA 3 *If $p(z) = a_0 + \sum_{v=s}^{n} a_v z^v$ is a polynomial of degree n, having no zeros in $|z| < K$, $K \ge 1$ then on $|z| = 1$*

$$|q'(z)| \ge mn, \tag{2.6}$$

where $q(z) = z^n \overline{p(1/\bar{z})}$ is as in Lemma 2 and $m = \min_{|z|=K} |p(z)|$.

Proof We can assume without loss of generality that $p(z)$ has no zeros in $|z| \le K$, for if $p(z)$ has a zero on $|z| = K$, the result holds trivially.

Since $p(z)$ has no zeros in $|z| \leq K, K \geq 1$ the polynomial $q(z)$ has no zeros in $|z| \geq 1/K$, and therefore by the maximum modulus principle for unbounded domains

$$\frac{|z|^n}{|q(z)|} \leq \frac{1/K^n}{\min_{|z|=1/K}|q(z)|} \quad \text{for} \quad |z| \geq \frac{1}{K},$$

and because $\min_{|z|=1/K}|q(z)| = m/K^n$, the above inequality is equivalent to

$$|q(z)| \geq m|z|^n \quad \text{for} \quad |z| \geq \frac{1}{K}. \tag{2.7}$$

It clearly follows from (2.7) that for every λ, $|\lambda| < 1$, the polynomial $q(z) - \lambda m z^n$ has all its zeros in $|z| < 1/K$ and therefore by Gauss–Lucas Thereom, the polynomial $q'(z) - \lambda mnz^{n-1}$ also has all its zeros in $|z| < 1/K$, which implies

$$|q'(z)| \geq |\lambda| mn|z|^{n-1} \quad \text{for} \quad |z| \geq \frac{1}{K}. \tag{2.8}$$

As $1/K \leq 1$, the inequality (2.8) in particular implies that

$$|q'(z)| \geq |\lambda| mn \quad \text{for} \quad |z| = 1. \tag{2.9}$$

Since (2.9) holds for every λ, $|\lambda| < 1$, making $|\lambda| \to 1$ in (2.9), we get (2.6).

LEMMA 4 *If $p(z) = a_0 + \sum_{v=s}^{n} a_v z^v$ is a polynomial of degree n, having no zeros in $|z| < K$, $K \geq 1$, then on $|z| = 1$,*

$$|q'(z)| \geq K^s |p'(z)| + mn, \tag{2.10}$$

where $q(z) = z^n\overline{p(1/\bar{z})}$ and $m = \min_{|z|=K} |p(z)|$ are as in Lemma 3.

Proof For every λ, $|\lambda| < 1$, the polynomial $p(z) - \lambda m$ has no zeros in $|z| < K$, $K \geq 1$, hence applying Lemma 2 to the polynomial $p(z) - \lambda m$, $|\lambda| < 1$, we get

$$K^s |p'(z)| \leq |q'(z) - \bar{\lambda} mnz^{n-1}| \quad \text{on} \quad |z| = 1. \tag{2.11}$$

Since by Lemma 3, $|q'(z)| \geq mn$ on $|z| = 1$, we can choose $\arg \lambda$ suitably so that

$$|q'(z) - \bar{\lambda} mnz^{n-1}| = |q'(z)| - |\lambda| mn \quad \text{on} \quad |z| = 1, \tag{2.12}$$

and (2.12) when combined with (2.11) gives

$$K^s |p'(z)| \leq |q'(z)| - |\lambda| mn, \quad \text{on} \quad |z| = 1. \tag{2.13}$$

Now, if we make $|\lambda| \to 1$ in (2.13), we get (2.10) and Lemma 4 is thus established. ■

LEMMA 5 *Let $p(z)$ be a polynomial of degree n. Then on $|z| = 1$,*

$$|p'(z)| + |q'(z)| \leq n\|p\|, \tag{2.14}$$

where $q(z) = z^n\overline{p(1/\bar{z})}$ is as defined in Lemma 2.

The above Lemma is a special case of a result due to Govil and Rahman [7, Lemma 10].

LEMMA 6 *If $p(z) = a_0 + \sum_{v=s}^{n} a_v z^v$ is a polynomial of degree n, having no zeros in $|z| < K$, $K \geq 1$ then on $|z| = 1$,*

$$|p'(z)| \leq \left(\frac{n}{1 + K^s}\right)(\|p\| - m). \tag{2.15}$$

The result is best possible and the equality holds for the polynomial $p(z) = (z^s + K^s)^{n/s}$, n being a multiple of s.

Proof Since (2.14) holds for any polynomial $p(z)$, inequality (2.15) follows immediately from Lemma 4 and Lemma 5.

It may be remarked here that the above Lemma 6 generalizes and sharpens Erdös–Lax Theorem (see [8]) and so is of interest in itself. In general, it sharpens a result of Chan and Malik [3, Theorem 1], while for $K = 1$, it reduces to a result of Aziz and Dawood [2, Theorem 2].

3 PROOF OF THE THEOREM

We first consider the case when $p(z)$ is of degree $n > 2$. Note that for every $\theta, 0 \leq \theta < 2\pi$

$$
\begin{aligned}
|p(Re^{i\theta}) - p(e^{i\theta})| &= \left| \int_1^R p'(re^{i\theta}) e^{i\theta} dr \right| \\
&\leq \int_1^R |p'(re^{i\theta})| dr. \qquad (3.1)
\end{aligned}
$$

Since $p(z)$ is of degree $n > 2$, the polynomial $p'(z)$ is of degree $(n-1) \geq 2$, hence applying (2.1) to $p'(z)$ in (3.1) we get

$$
\begin{aligned}
|p(Re^{i\theta}) - p(e^{i\theta})| &\leq \int_1^R (r^{n-1} \|p'\| - (r^{n-1} - r^{n-3})|p'(0)|) dr \\
&= \left(\frac{R^n - 1}{n}\right) \|p'\| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) |p'(0)| \qquad (3.2)
\end{aligned}
$$

Combining (3.2) with Lemma 6, we get for $n > 2, R \geq 1$ and $0 \leq \theta < 2\pi$

$$
\begin{aligned}
|p(Re^{i\theta}) - p(e^{i\theta})| \leq &\left(\frac{R^n - 1}{1 + K^s}\right) (\|p\| - m) \\
&- \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) |p'(0)|, \qquad (3.3)
\end{aligned}
$$

implying that for $n > 2, R \geq 1$ and $0 \leq \theta < 2\pi$

$$
\begin{aligned}
|p(Re^{i\theta})| \leq &\left(\frac{R^n + K^s}{1 + K^s}\right) \|p\| - \left(\frac{R^n - 1}{1 + K^s}\right) m \\
&- |a_1| \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right), \qquad (3.4)
\end{aligned}
$$

which gives (1.6). The proof of (1.7) follows on the same lines as the proof of (1.6) except that instead of using (2.1), here we use (2.2). We omit the details.

Acknowledgement

The author is extremely grateful to Professor Q. I. Rahman for his suggestions.

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