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The Upper Bound of a Reserve Hölder's Type Operator Inequality and its Applications

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In our previous paper, we obtained a reverse Hölder's type inequality which gives an upper bound of the difference:

$$(\sum a_k^p)^{1/p}(\sum b_k^q)^{1/q} - \lambda \sum a_k b_k$$

with a parameter $\lambda > 0$, for n-tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots b_n)$ of positive numbers and for $p > 1, q > 1$ satisfying $1/p + 1/q = 1$. In this paper for commutative positive operators A and B on a Hilbert space H and a unit vector $x \in H$, we give an upper bound of the difference

$$\langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} - \lambda\langle ABx, x\rangle.$$

As applications, considering special cases, we induce some difference and ratio operator inequalities. Finally, using the geometric mean in the Kubo-Ando theory we shall give a reverse Hölder's type operator inequality for noncommutative operators.

Keywords: Hölder's inequality; Difference inequality; Ratio inequality; Reverse Hölder's inequality; Geometric mean

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1 INTRODUCTION

This paper is a continuation of our paper [7]. In this paper, we assume that real numbers $p > 1, q > 1$ satisfy $1/p + 1/q = 1$.

The Hölder inequality is one of the most important inequalities in analysis: If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are n-tuples of non-negative numbers, then

$$(\sum a_k^p)^{1/p}(\sum b_k^q)^{1/q} \geq \sum a_k b_k. \tag{1}$$

In [7], we introduced a complementary inequality derived from (1), i.e.,

$$(\sum a_k^p)^{1/p}(\sum b_k^q)^{1/q} - \lambda \sum a_k b_k \leq nM_1M_2F_0(\lambda) \quad \text{for } \lambda > 0$$

under certain conditions (see Theorem A and $F_0(\lambda)$ is defined there). We consider the operator version of this inequality, i.e., we give an estimation of the following difference:

$$\langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} - \lambda\langle ABx, x\rangle \tag{2}$$

for commuting positive operators A and B on a Hilbert space H satisfying

$$0 < m_1 \leq A \leq M_1, \ 0 < m_2 \leq B \leq M_2, \ m_1 < M_1 \text{ and } m_2 < M_2 \tag{3}$$

and for a unit vector $x \in H$. We show $M_1M_2F_0(\lambda)$ as the upper bound of (2), i.e.,

$$\langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} - \lambda\langle ABx, x\rangle \leq M_1M_2F_0(\lambda), \tag{4}$$

which we call a reverse Hölder's type operator inequality. We derive some other inequalities which are given for $p = q = 2$ and $B = I$ (I is the identity operator) in (4). Considering the cases $\lambda = 1$ and λ satisfying $F_0(\lambda) = 0$, we obtain difference and ratio inequalities which are operator versions of known inequalities. Furthermore we obtain a non-commutative version of a reverse Hölder's type operator inequality, using the s-geometric mean $A\natural_s B$ introduced in the Kubo–Ando theory [8], which is defined by

$$A\natural_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2} \quad (0 < s \leq 1)$$

for invertible positive operators A and B.

2 AN OPERATOR VERSION OF A REVERSE HÖLDER'S TYPE INEQUALITY

First we define several notations needed later. Let α and β be real numbers with $0 < \alpha < 1$ and $0 < \beta < 1$, and denote several constants as follows:

$$K_{\gamma,r} = \frac{1-\gamma^r}{1-\gamma}, \quad \tilde{K}_{\gamma,r} = \frac{K_{\gamma,r}}{\gamma^{r-1}}, \quad K = \frac{K_{\alpha,p^{1/p}}K_{\beta,q^{1/q}}}{p^{1/p}q^{1/q}},$$
$$\tilde{K} = \frac{K}{\alpha^{1/q}\beta^{1/p}} \quad (\gamma = \alpha, \beta, \ r = p, q).$$

In our paper [7, Lemma 2.3], we pointed out that for any positive real number λ, the equation

$$(1-\alpha)(\lambda - K\tau^{1/q}) = (1-\beta)(\lambda - K\tau^{-1/p})$$

has a unique positive solution $\tau = \tau_\lambda$. We defined the constant c_λ by

$$c_\lambda = (1-\alpha)(\lambda - K\tau_\lambda^{1/q}) \ (= (1-\beta)(\lambda - K\tau_\lambda^{-1/p})). \tag{5}$$

Furthermore in [7, Theorem 3.5], we showed the following theorem which gives an upper bound of the difference in Hölder's inequality (1):

THEOREM A *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be n-tuples of positive numbers satisfying $0 < m_1 \leq a_k \leq M_1$, $0 < m_2 \leq b_k \leq M_2$ $(k = 1, 2, \ldots, n)$, $m_1 < M_1$ and $m_2 < M_2$. Put $\alpha = m_1/M_1$ and $\beta = m_2/M_2$. Then for any $\lambda > 0$*

$$\left(\sum a_k^p\right)^{1/p}\left(\sum b_k^q\right)^{1/q} - \lambda \sum a_k b_k \leq nM_1M_2F_0(\lambda), \tag{6}$$

where $F_0(\lambda) = F_0(\lambda; \alpha, \beta, p)$ *is the constant defined by*

$$F_0(\lambda) = \begin{cases} 1-\lambda & \text{if } 0 < \lambda < \min\left\{\dfrac{K_{\alpha,p}}{p}, \dfrac{K_{\beta,q}}{q}\right\} \\ \left\{\dfrac{1}{K_{\alpha,p}}\left(\dfrac{K}{\lambda}\right)^p + \dfrac{1}{K_{\beta,q}} - 1\right\}\lambda & \text{if } \dfrac{K_{\beta,q}}{q}\left(= \min\left\{\dfrac{K_{\alpha,p}}{p}, \dfrac{K_{\beta,q}}{q}\right\}\right) \\ & \qquad \le \lambda < K \\ \left\{\dfrac{1}{K_{\alpha,p}} + \dfrac{1}{K_{\beta,q}}\left(\dfrac{K}{\lambda}\right)^q - 1\right\}\lambda & \text{if } \dfrac{K_{\alpha,p}}{p}\left(= \min\left\{\dfrac{K_{\alpha,p}}{p}, \dfrac{K_{\beta,q}}{q}\right\}\right) \\ & \qquad \le \lambda < K \\ \left(\dfrac{1}{K_{\alpha,p}} + \dfrac{1}{K_{\beta,q}} - 1\right)\lambda & \\ \qquad - \dfrac{1-\alpha^p\beta^q}{(1-\alpha^p)(1-\beta^q)}c_\lambda & \text{if } K \le \lambda \le \tilde{K} \\ \left\{\dfrac{\alpha^p}{K_{\alpha,p}} + \dfrac{1}{K_{\beta,q}}\left(\dfrac{K}{\lambda}\right)^q - \alpha\right\}\beta\lambda & \text{if } \tilde{K} < \lambda \le \dfrac{\tilde{K}_{\alpha,p}}{p} \\ & \quad \left(= \max\left\{\dfrac{\tilde{K}_{\alpha,p}}{p}, \dfrac{\tilde{K}_{\beta,q}}{q}\right\}\right) \\ \left\{\dfrac{1}{K_{\alpha,p}}\left(\dfrac{K}{\lambda}\right)^p + \dfrac{\beta^q}{K_{\beta,q}} - \beta\right\}\alpha\lambda & \text{if } \tilde{K} < \lambda \le \dfrac{\tilde{K}_{\beta,p}}{q} \\ & \quad \left(= \max\left\{\frac{\tilde{K}_{\alpha,p}}{p}, \frac{\tilde{K}_{\beta,q}}{q}\right\}\right) \\ \alpha\beta(1-\lambda) & \text{if } \max\left\{\dfrac{\tilde{K}_{\alpha,p}}{p}, \dfrac{\tilde{K}_{\beta,q}}{q}\right\} < \lambda \end{cases} \tag{7}$$

Since $K \le 1 \le \tilde{K}$, we remark for $\lambda = 1$

$$F_0(1) = \frac{1}{K_{\alpha,p}} + \frac{1}{K_{\beta,q}} - 1 - \frac{1-\alpha^p\beta^q}{(1-\alpha^p)(1-\beta^q)}\, c_1,$$

and the following inequality [6, Theorem 2.2] is obtained

$$\left(\sum a_k^p\right)^{1/p}\left(\sum b_k^q\right)^{1/q} - \sum a_k b_k \le nM_1M_2F_0(1). \tag{8}$$

Moreover the equation $F_0(\lambda) = 0$ has a unique solution [7, Theorem 3.6 and Lemma 5.1]

$$\lambda = \lambda_0 = \frac{1 - \alpha^p \beta^q}{p^{1/p} q^{1/q} (\beta - \alpha\beta^q)^{1/p} (\alpha - \alpha^p \beta)^{1/q}} \ (\in [K, \tilde{K}]), \tag{9}$$

and the following Gheorghiu inequality [4] (or a reverse Hölder's inequality [10, p.685]) is obtained:

$$(\sum a_k^p)^{1/p} (\sum b_k^q)^{1/q} \leq \lambda_0 \sum a_k b_k. \tag{10}$$

Now we give a reverse Hölder's type operator inequality, that is, an operator version of (6). We also consider the special cases of $\lambda = 1$ and $\lambda = \lambda_0$.

THEOREM 1 *Let A and B be two commuting positive operators on H satisfying* (3). *Put $\alpha = m_1/M_1$, $\beta = m_2/M_2$. Then for any $\lambda > 0$ and any unit vector $x \in H$*

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} - \lambda \langle ABx, x\rangle \leq M_1 M_2 F_0(\lambda), \tag{11}$$

where $F_0(\lambda)$ is the constant defined by (7).

Furthermore the following facts hold:

(i) *If $\lambda = 1$, then*

$$\begin{aligned}&\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} - \langle ABx, x\rangle \\ &\quad \leq M_1 M_2 \left\{ \frac{1}{K_{\alpha,p}} + \frac{1}{K_{\beta,q}} - 1 - \frac{1 - \alpha^p \beta^q}{(1 - \alpha^p)(1 - \beta^q)} c_1 \right\}.\end{aligned} \tag{12}$$

(ii) *The equation $F_0(\lambda) = 0$ has a unique solution $\lambda = \lambda_0 (\in [K, \tilde{K}])$ defined by* (9) *and the following inequality holds*

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \leq \lambda_0 \langle ABx, x\rangle. \tag{13}$$

Proof Let a and b be n-tuples with the same conditions of Theorem A and let $w = (w_1, \ldots, w_n)$ be an n-tuple of nonnegative numbers with

$w = \sum_{k=1}^{n} w_k$. Then by the same method as [6, Theorem 4.1], we have the weighted version of Theorem A, that is, for any $\lambda > 0$

$$(\sum w_k a_k^p)^{1/p}(\sum w_k b_k^q)^{1/q} - \lambda \sum w_k a_k b_k \leq w M_1 M_2 F_0(\lambda). \tag{14}$$

Next let μ be a positive measure on the rectangle $X = [m_1, M_1] \times [m_2, M_2]$ with $\mu(X) = 1$, and let $L^r(X)(r > 1)$ be the set of measurable functions f such that $|f|^r$ are integrable on X. Suppose that $f \in L^p(X)$ and $g \in L^q(X)$ satisfying $0 < m_1 \leq f \leq M_1$ and $0 < m_2 \leq g \leq M_2$. Furthermore let $X_1, X_2, \ldots, X_n$ be a decomposition of X and let $x_k \in X_k (k = 1, 2, \ldots, n)$. Then from (14) we obtain

$$\{\sum f(x_k)^p \mu(X_k)\}^{1/p} \left\{\sum g(x_k)^q \mu(X_k)\right\}^{1/q} - \lambda \sum f(x_k) g(x_k) \mu(X_k) \\ \leq M_1 M_2 F_0(\lambda).$$

Taking the limit of the decomposition, we obtain

$$\left(\int_X f^p d\mu\right)^{1/p} \left(\int_X g^q d\mu\right)^{1/q} - \lambda \int_X fg d\mu \leq M_1 M_2 F_0(\lambda). \tag{15}$$

Now since A and B are commuting, there exist commuting spectral families $E_A(\cdot)$ and $E_B(\cdot)$ corresponding to A and B such that for any polynomial $p(A, B)$ (or a uniform limit of polynomials) in A and B,

$$\langle p(A, B)x, x\rangle = \int_{-\infty}^{\infty} p(s, t) d\langle E^A(s) E^B(t) x, x\rangle \qquad \text{for} \quad x \in H,$$

[13, p.287]. Let $d\mu = d\langle E^A(s)E^B(t)x, x\rangle = d\|E^A(s)E^B(t)x\|^2$. Then from (15) we have

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} - \lambda \langle ABx, x\rangle = \left(\iint_X s^p d\mu\right)^{1/p} \left(\iint_X t^q d\mu\right)^{1/q} \\ - \lambda \iint_X st d\mu \leq M_1 M_2 F_0(\lambda)$$

and hence we have the desired inequality (11).

Furthermore we easily have (12) and (13), putting $\lambda = 1$ and $\lambda = \lambda_0$ in (11), respectively. ■

We remark that the difference inequality (12) was obtained in [6, Theorem 4.3] as an operator version of (8), and that the ratio inequality (13) was obtained in [2, Theorem 4] as an operator version of Gheorghiu's inequality (10).

3 FURTHER APPLICATIONS TO OPERATOR INEQUALITIES

In this section as applications of Theorem 1, we deduce three corollaries which give special inequalities as the cases of $p = q = 2$ or $\beta \to 1$. The inequalities for $\lambda = 1$ correspond with difference inequalities given in [6], and those for the solution λ of $F_0(\lambda) = 0$ correspond with ratio inequalities which are operator versions of known numerical inequalities.

If $\beta \to 1$ in Theorem 1, then we can easily see that $K \to (K_{\alpha,p}/p)^{1/p}$ and $\tilde{K} \to (K_{\alpha,p}/p\alpha^{p-1})^{1/p}$. So we obtain the following corollary:

COROLLARY 2 *Let A be a positive operator on H satisfying* (3). *Put $\alpha = m_1/M_1$. Then for any $\lambda > 0$ and any unit vector $x \in H$*

$$\langle A^p x, x\rangle^{1/p} - \lambda\langle Ax, x\rangle \leq M_1 F_1(\lambda), \tag{16}$$

where $F_1(\lambda)$ is the constant defined by

$$F_1(\lambda) = \begin{cases} 1-\lambda & \text{if } 0 < \lambda < \dfrac{K_{\alpha,p}}{p} \\[2ex] \dfrac{1}{q}\left\{\dfrac{1-\alpha^p}{p(1-\alpha)\lambda}\right\}^{q-1} - \dfrac{\alpha-\alpha^p}{1-\alpha^p}\,\lambda & \text{if } \dfrac{K_{\alpha},p}{p} \leq \lambda \leq \dfrac{K_{\alpha,p}}{p\alpha^{p-1}} \\[2ex] \alpha(1-\lambda) & \text{if } \dfrac{K_{\alpha,p}}{p\alpha^{p-1}} < \lambda. \end{cases}$$

Furthermore the following facts hold:

(i) *If* $\lambda = 1$, *then*

$$\langle A^p x, x\rangle^{1/p} - \langle Ax, x\rangle \leq M_1 \left[\frac{1}{q} \left\{ \frac{1-\alpha^p}{p(1-\alpha)} \right\}^{q-1} - \frac{\alpha - \alpha^p}{1-\alpha^p} \right]. \tag{17}$$

(ii) *The equation* $F_1(\lambda) = 0$ *has a unique solution*

$$\lambda = \lambda_1 = \frac{1-\alpha^p}{p^{1/p} q^{1/q} (1-\alpha)^{1/p} (\alpha - \alpha^p)^{1/q}} \quad \left(\in \left[\frac{K_{\alpha,p}}{p}, \frac{K_{\alpha,p}}{p\alpha^{p-1}} \right] \right),$$

and the following inequality holds

$$\langle A^p x, x\rangle^{1/p} \leq \lambda_1 \langle Ax, x\rangle. \tag{18}$$

Proof Let $M_2 = 1$ and $m_2 = \beta \to 1$ in Theorem 1. Then we obtain (16) by using the same method as in [7, Theorem 4.1]. Moreover (17) is ensured by $\dfrac{K_{\alpha,p}}{p} < 1 < \dfrac{K_{\alpha,p}}{p\alpha^{p-1}}$, and (18) is obtained by an elementary computation, using the fact that a unique solution $\lambda = \lambda_1$ of the equation $F_1(\lambda) = 0$ satisfies $\dfrac{K_{\alpha,p}}{p} \leq \lambda_1 \leq \dfrac{K_{\alpha,p}}{p\alpha^{p-1}}$. ■

The inequality (17) was given in [6] and the inequality (18) was given in [2], [3], [9], [11]. The constant λ_1 coincides with the p-th root of the constant defined by Ky Fan [1], or Furuta [3].

Next we take $p = q = 2$ in Theorem 1:

COROLLARY 3 *Let A and B be two commuting positive operators on H satisfying (3). Put* $\alpha = \min\{m_1/M_1, m_2/M_2\}$, $\beta = \max\{m_1/M_1, m_2/M_2\}$, $\gamma = (1+\alpha)^{1/2}(1+\beta)^{1/2}/2$ *and* $\tilde{\gamma} = \gamma/\alpha^{1/2}\beta^{1/2}$. *Write* c'_λ *the constant of (5) with respect to* $p = q = 2$. *Then for any* $\lambda > 0$ *and any unit vector* $x \in H$

$$\langle A^2 x, x\rangle^{1/2} \langle B^2 x, x\rangle^{1/2} - \lambda \langle ABx, x\rangle \leq M_1 M_2 F_2(\lambda),$$

where $F_2(\lambda)$ is the constant defined by

$$F_2(\lambda) = \begin{cases} 1-\lambda & \textit{if } 0 < \lambda < \dfrac{1+\alpha}{2} \\ \left(\dfrac{1+\alpha}{4\lambda^2} - \dfrac{\alpha}{1+\alpha}\right)\lambda & \textit{if } \dfrac{1+\alpha}{2} \leq \lambda < \gamma \\ \dfrac{(1-\alpha\beta)\lambda}{(1+\alpha)(1+\beta)} - c_{/\lambda}\left\{\dfrac{1-\alpha^2\beta^2}{(1-\alpha^2)(1-\beta^2)}\right\} & \textit{if } \gamma \leq \lambda \leq \tilde{\gamma} \\ \left(\dfrac{1+\alpha}{4\lambda^2} - \dfrac{\alpha}{1+\alpha}\right)\beta\lambda & \textit{if } \tilde{\gamma} < \lambda \leq \dfrac{1+\alpha}{2\alpha} \\ \alpha\beta(1-\lambda) & \textit{if } \dfrac{1+\alpha}{2\alpha} < \lambda. \end{cases}$$

Furthermore the following facts hold:

(i) *If* $\lambda = 1$, *then*

$$\langle A^2x, x\rangle^{1/2}\langle B^2x, x\rangle^{1/2} - \langle ABx, x\rangle \leq M_1M_2\,\frac{(1-\alpha\beta)^2}{2(1+\alpha)(1+\beta)}. \tag{19}$$

(ii) *The equation* $F_2(\lambda) = 0$ *has a unique solution*

$$\lambda = \lambda_2 = \frac{1+\alpha\beta}{2\alpha^{1/2}\beta^{1/2}} \ (\in [\gamma, \tilde{\gamma}]),$$

and the following inequality holds

$$\langle A^2x, x\rangle^{1/2}\langle B^2x, x\rangle^{1/2} \leq \lambda_2\langle ABx, x\rangle. \tag{20}$$

Proof Let $p = q = 2$ in Theorem 1. Then we obtain the desired inequalities by using the same method as [7, Theorem 4.3]. ■

The inequality (19) was given in [6]. The inequality (20) is a commutative operator version of the Pólya-Szegö inequality [12], [10, p.684]:

$$\sum a_k^2 \sum b_k^2 \leq \frac{(M_1M_2 + m_1m_2)^2}{4M_1M_2m_1m_2}\left(\sum a_kb_k\right)^2,$$

or Greub-W. Rheinboldt inequality [5]:

$$\sum p_k a_k^2 \sum p_k b_k^2 \leq \frac{(M_1M_2 + m_1m_2)^2}{4M_1M_2m_1m_2} \left(\sum p_k a_k b_k\right)^2.$$

with a weight $p_k \geq 0$ $(k = 1, 2, \ldots, n)$ with $\sum p_k = 1$.

In particular, we obtain the following corollary, putting $p = q = 2$ in Corollary 2 or $\beta \to 1$ in Corollary 3:

COROLLARY 4 *Let A be a positive operator on H satisfying (3). Put $\alpha = m_1/M_1$. Then for any $\lambda > 0$ and any unit vector $x \in H$*

$$\langle A^2x, x\rangle^{1/2} - \lambda\langle Ax, x\rangle \leq M_1F_3(\lambda),$$

where $F_3(\lambda)$ is the constant defined by

$$F_3(\lambda) = \begin{cases} 1 - \lambda & \text{if } 0 < \lambda < \dfrac{1+\alpha}{2} \\ \dfrac{1+\alpha}{4\lambda} - \dfrac{\alpha}{1+\alpha}\lambda & \text{if } \dfrac{1+\alpha}{2} \leq \lambda \leq \dfrac{1+\alpha}{2\alpha} \\ \alpha(1-\lambda) & \text{if } \dfrac{1+\alpha}{2\alpha} < \lambda. \end{cases}$$

Furthermore the following facts hold:

(i) *If $\lambda = 1$, then*

$$\langle A^2x, x\rangle^{1/2} - \langle Ax, x\rangle \leq \frac{(M_1 - m_1)^2}{4(M_1 + m_1)}. \tag{21}$$

(ii) *The equation $F_3(\lambda) = 0$ has a unique solution*

$$\lambda = \lambda_3 = \frac{1+\alpha}{2\alpha^{1/2}} \left(\in \left[\frac{1+\alpha}{2}, \frac{1+\alpha}{2\alpha}\right]\right),$$

and the following inequality holds

$$\langle A^2x, x\rangle^{1/2} \leq \lambda_3\langle Ax, x\rangle. \tag{22}$$

The inequalities (21) and (22) are well-known inequalities (cf. [6]) related to the following celebrated Kantorovich inequality:

$$\langle Ax, x\rangle \langle A^{-1}x, x\rangle \le \frac{(M+m)^2}{4mM}.$$

As an application of Theorem 1 (or Corollary 2), we shall show some operator inequalities without commutativity assumption. In [8], F. Kubo and T. Ando introduced the s-geometric mean $A\sharp_s B$ defined by

$$A\sharp_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2} \quad (0 < s \le 1)$$

for positive invertible operators A and B. We note that $B^q\sharp_{1/p}A^p = AB$ if A and B commute.

Using the s-geometric mean, we have a noncommutative version of Theorem 1:

THEOREM 5 *Let A and B be two positive invertible operators on H satisfying (3). Put $\alpha = m_1/M_1, \beta = m_2/M_2, \gamma = \alpha\beta^{q-1} = \dfrac{m_1 m_2^{q-1}}{M_1 M_2^{q-1}}$ and $K_\gamma (= K_{\gamma,p}) = \dfrac{1-\alpha^p\beta^q}{1-\alpha\beta^{q-1}}$. Then for any $\lambda > 0$ and any unit vector $x \in H$*

$$\langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} - \lambda\langle B^q\sharp_{1/p}A^p x, x\rangle \le \frac{M_1 M_2}{\beta^{q-1}} F_\sharp(\lambda), \tag{23}$$

where $F_\sharp(\lambda)$ is the constant defined by

$$F_\sharp(\lambda) = \begin{cases} 1-\lambda & \text{if } 0 < \lambda < \dfrac{K_\gamma}{p} \\ \dfrac{1}{q}\left\{\dfrac{1-\gamma^p}{p(1-\gamma)\lambda}\right\}^{q-1} - \dfrac{\gamma-\gamma^p}{1-\gamma^p}\lambda & \text{if } \dfrac{K_\gamma}{p} \le \lambda \le \dfrac{K_\gamma}{p\gamma^{p-1}} \\ \gamma(1-\lambda) & \text{if } \dfrac{K_\gamma}{p\gamma^{p-1}} < \lambda. \end{cases}$$

Proof In Corollary 2, $F_1(\lambda)$ is determined by λ, α (and p), and hence we may write $F_1(\lambda) = F_1(\lambda, \alpha)$. If C is a positive operator such that

$0 < m \le C \le M$, then from (16) we have for any $\lambda > 0$ and any vector $x \in H$

$$\langle C^p x, x\rangle^{1/p}\langle x, x\rangle^{1/q} - \lambda\langle Cx, x\rangle \le MF_1(\lambda, \gamma_0)\langle x, x\rangle \tag{24}$$

holds for $\gamma_0 = m/M$. (Correspondingly we replace the constant $K_{\alpha,p}$ in Corollary 2 by $K_{\gamma_0} = \frac{1-\gamma_0^p}{1-\gamma_0}$.) Now, we replace C and x by $(B^{-q/2}A^pB^{-q/2})^{1/p}$ and $B^{q/2}x$ with x having unit norm in (24), respectively. Then since

$$0 < \frac{m_1}{M_2^{q-1}} \le (B^{-q/2}A^pB^{-q/2})^{1/p} \le \frac{M_1}{m_2^{q-1}},$$

we have

$$\begin{aligned}\langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} &- \lambda\langle B^{q/2}(B^{-q/2}A^pB^{-q/2})^{1/p}B^{q/2}x, x\rangle\\ &\le \frac{M_1}{m_2^{q-1}}\, F_1(\lambda, \gamma)\langle B^q x, x\rangle\\ &\le \frac{M_1M_2}{\beta^{q-1}}\, F_1(\lambda, \gamma)\end{aligned}$$

for $\gamma = \frac{\frac{m_1}{M_2^{q-1}}}{\frac{M_1}{m_2^{q-1}}} = \alpha\beta^{q-1}$. Putting $F_\sharp(\lambda) = F_1(\lambda, \gamma)$, we obtain the desired inequality (23). ■

If we put $\lambda = 1$ in (23), then we have the following inequality [6, Theorem 4.5] which is the noncommutative version of (12):

$$\begin{aligned}&\langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} - \langle B^q \sharp_{1/p} A^p x, x\rangle\\ &\quad\le \frac{M_1M_2}{\beta^{q-1}}\left\{\frac{1}{q}\left(\frac{1-\gamma^p}{p(1-\gamma)}\right)^{q-1} - \frac{\gamma-\gamma^p}{1-\gamma^p}\right\}.\end{aligned} \tag{25}$$

By an elementary computation we can see that $F_\sharp(\lambda) = 0$ has a unique solution $\lambda = \lambda_0\left(\in\left[\frac{K_\gamma}{p}, \frac{K_\gamma}{p\gamma^{p-1}}\right]\right)$ defined by (9). So we have the following result [2, Theorem 4]:

COROLLARY 6 *Let A and B be two positive invertible operators on H satisfying (3). Put $\alpha = m_1/M_1$ and $\beta = m_2/M_2$. Then for any unit vector $x \in H$*

$$\langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} \le \lambda_0 \langle B^q \sharp_{1/p} A^p x, x\rangle, \tag{26}$$

where $\lambda_0 \left(\in \left[\frac{K_\gamma}{p}, \frac{K_\gamma}{p\gamma^{p-1}}\right]\right)$ is the constant defined by (9).

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References

[1] Ky. Fan, *Some matrix inequalities*, Abh. Math. Sem. Univ. Hamburg, **29** (1966), 185–196.
[2] M. Fujii, S. Izumino, R. Nakamoto and Y. Seo, *Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities*, Nihonkai Math. J., **8**(1997), 117–122.
[3] T. Furuta, *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities*, J. Inequal. Appl., **2**(1998), 137–148.
[4] S. A. Gheorghiu, *Note sur une inégalité de Cauchy*, Bull. Math. Soc. Roumainie Sci., **35**(1933), 117–119.
[5] W. Greub and W. Rheinboldt, *On a generalization of an inequality of L. V. Kantorovich*, Proc. Amer. Math. Soc., **10**(1959), 407–415.
[6] S. Izumino, *Ozeki's method on Hölder's inequality*, Math. Japon., **50**(1999), 41–55.
[7] S. Izumino and M. Tominaga, *Extensions in Hölder's type inequalities*, Math. Ineq. Appl., **4**(2001), 163–187.
[8] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246**(1980), 205–224.
[9] C. A. McCarthy, C_p, Israel J. Math., **5**(1967), 249–271.
[10] D. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Boston, London 1993.
[11] B. Mond and O. Shisha, *Difference and ratio inequalities in Hilbert spaces*, "Inequalities,II", Academic Press, New York, 1967.
[12] G. Pólya, and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Berlin, 1925, pp. 57 and 213–214.
[13] F. Riesz and Bz. -Nagy, *Functional Analysis*, Ungar, New York (1952).