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Research Article

On Some Layer-Based Risk Measures with Applications to Exponential Dispersion Models

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Layer-based counterparts of a number of well-known risk measures have been proposed and studied. Namely, some motivations and elementary properties have been discussed, and the analytic tractability has been demonstrated by developing closed-form expressions in the general framework of exponential dispersion models.

1. Introduction

Denote by $\mathcal{X}$ the set of (actuarial) risks, and let $0 \leq X \in \mathcal{X}$ be a random variable (rv) with cumulative distribution function (cdf) $F(x)$, decumulative distribution function (ddf) $\overline{F}(x) = 1 - F(x)$, and probability density function (pdf) $f(x)$. The functional $H : \mathcal{X} \to [0, \infty]$ is then referred to as a risk measure, and it is interpreted as the measure of risk inherent in X. Naturally, a quite significant number of risk measuring functionals have been proposed and studied, starting with the arguably oldest Value-at-Risk or VaR (cf. [1]), and up to the distorted (cf. [2–5]) and weighted (cf. [6, 7]) classes of risk measures.

More specifically, the Value-at-Risk risk measure is formulated, for every $0 < q < 1$, as

$$\mathrm{VaR}_q[X] = \inf\{x : F_X(x) \geq q\}, \tag{1.1}$$

which thus refers to the well-studied notion of the qth quantile. Then the family of distorted risk measures is defined with the help of an increasing and concave function $g : [0,1] \to [0,1]$, such that $g(0) = 0$ and $g(1) = 1$, as the following Choquet integral:

$$H_g[X] = \int_{\mathbf{R}_+} g\left(\overline{F}(x)\right) dx. \tag{1.2}$$

Last but not least, for an increasing nonnegative function $w : [0,\infty) \to [0,\infty)$ and the so-called weighted ddf $\overline{F}_w(x) = \mathbf{E}[\mathbf{1}\{X > x\}w(X)]/\mathbf{E}[w(X)]$ the class of weighted risk measures is given by

$$H_w[X] = \int_{\mathbf{R}_+} \overline{F}_w(x)dx. \tag{1.3}$$

Note that for at least once differentiable distortion function, we have that the weighted class contains the distorted one as a special case, that is, $H_g[X] = \mathbf{E}[Xg'(\overline{F}(X))]$ is a weighted risk measure with a dependent on $\overline{F}$ weight function.

Interestingly, probably in the view of the latter economic developments, the so-called "tail events" have been drawing increasing attention of insurance and general finance experts. Naturally therefore, tail-based risk measures have become quite popular, with the tail conditional expectation (TCE) risk measure being a quite remarkable example. For $0 < q < 1$ and thus $\overline{F}(\mathrm{VaR}_q[X]) \neq 0$, the TCE risk measure is formulated as

$$\mathrm{TCE}_q[X] = \frac{1}{\overline{F}(\mathrm{VaR}_q[X])} \int_{\mathrm{VaR}_q[X]}^{\infty} x\, dF(x). \tag{1.4}$$

Importantly, the TCE belongs in the class of distorted risk measures with the distortion function

$$g(x) = \frac{x}{1-q}\mathbf{1}(x < 1-q) + \mathbf{1}(x \geq 1-q), \tag{1.5}$$

where $\mathbf{1}$ denotes the indicator function (cf., e.g., [8]), as well as in the class of weighted risk measures with the weight function

$$w(x) = \mathbf{1}\{x \geq \mathrm{VaR}_q[X]\} \tag{1.6}$$

(cf., e.g., [6, 7]). The TCE risk measure is often referred to as the expected shortfall (ES) and the conditional Value-at-Risk (CVaR) when the pdf of X is continuous (cf., e.g., [9]).

Functional (1.4) can be considered a tail-based extension of the net premium $H[X] = \mathbf{E}[X]$. Furman and Landsman [10] introduced and studied a tail-based counterpart of the standard deviation premium calculation principle, which, for $0 < q < 1$, the tail variance

$$\mathrm{TV}_q[X] = \mathbf{Var}[X \mid X > \mathrm{VaR}_q[X]], \tag{1.7}$$

and a constant $\alpha \geq 0$, is defined as

$$\mathrm{TSD}_q[X] = \mathrm{TCE}_q[X] + \alpha \cdot \mathrm{TV}_q^{1/2}[X]. \tag{1.8}$$

For a discussion of various properties of the TSD risk measure, we refer to Furman and Landsman [10]. We note in passing that for $q \downarrow 0$, we have that $\mathrm{TSD}_q[X] \to \mathrm{SD}[X] = \mathbf{E}[X] + \alpha \cdot \mathbf{Var}^{1/2}[X]$.

The rest of the paper is organized as follows. In the next section we introduce and motivate layer-based extensions of functionals (1.4) and (1.8). Then in Sections 3 and 4 we analyze the aforementioned layer-based risk measures as well as their limiting cases in the general context of the exponential dispersion models (EDMs), that are to this end briefly reviewed in the appendix. Section 5 concludes the paper.

2. The Limited TCE and TSD Risk Measures

Let $0 < q < p < 1$ and let $X \in \mathcal{X}$ have a continuous and strictly increasing cdf. In many practical situations the degree of riskiness of the layer $(\mathrm{VaR}_q[X], \mathrm{VaR}_p[X])$ of an insurance contract is to be measured (certainly the layer width $\mathrm{VaR}_p[X] - \mathrm{VaR}_q[X] = \Delta_{q,p} > 0$). Indeed, the number of deductibles in a policy is often more than one, and/or there can be several reinsurance companies covering the same insured object. Also, there is the so-called "limited capacity" within the insurance industry to absorb losses resulting from, for example, terrorist attacks and catastrophes. In the context of the aforementioned events, the unpredictable nature of the threat and the size of the losses make it unlikely that the insurance industry can add enough capacity to cover them. In these and other cases neither (1.4) nor (1.8) can be applied since (1) both TCE and TSD are defined for one threshold, only, and (2) the aforementioned pair of risk measures is useless when, say, the expectations of the underlying risks are infinite, which can definitely be the case in the situations mentioned above.

Note 1. As noticed by a referee, the risk measure $H : \mathcal{X} \to [0, \infty]$ is often used to price (insurance) contracts. Naturally therefore, the limited TCE and TSD proposed and studied herein can serve as pricing functionals for policies with coverage modifications, such as, for example, policies with deductibles, retention levels, and so forth (cf., [11, Chapter 8]).

Next, we formally define the risk measures of interest.

Definition 2.1. Let $x_q = \mathrm{VaR}_q[X]$ and $x_p = \mathrm{VaR}_p[X]$, for $0 < q < p < 1$. Then the limited TCE and TSD risk measures are formulated as

$$\mathrm{LTCE}_{q,p}[X] = \mathbf{E}\left[X \mid x_q < X \le x_p\right], \tag{2.1}$$

and

$$\mathrm{LTSD}_{q,p}[X] = \mathbf{E}\left[X \mid x_q < X \le x_p\right] + \alpha \cdot \mathbf{Var}^{1/2}\left[X \mid x_q < X \le x_p\right], \tag{2.2}$$

respectively.

Clearly, the TCE and TSD are particular cases of their limited counterparts. We note in passing that the former pair of risk measures need not be finite for heavy tailed distributions, and they are thus not applicable. In this respect, limited variants (2.1) and (2.2) can provide a partial resolution. Indeed, for $k = 1, 2, \ldots$, we have that

$$\mathbf{E}\left[X^k \mid x_q < X \le x_p\right] = \frac{F(x_p)\mathbf{E}\left[X^k \mid X \le x_p\right] - F(x_q)\mathbf{E}\left[X^k \mid X \le x_q\right]}{F(x_p) - F(x_q)} < \infty, \tag{2.3}$$

regardless of the distribution of X.

We further enumerate some properties of the LTSD risk measure, which is our main object of study.

(1) *Translation Invariance.* For any constant $c \ge 0$, we have that

$$\mathrm{LTSD}_{q,p}[X + c] = \mathrm{LTSD}_{q,p}[X] + c. \tag{2.4}$$

(2) *Positive Homogeneity.* For any constant $d > 0$, we have that

$$\mathrm{LTSD}_{q,p}[d \cdot X] = d \cdot \mathrm{LTSD}_{q,p}[X]. \tag{2.5}$$

(3) *Layer Parity.* We call $X \in \mathcal{X}$ and $Y \in \mathcal{X}$ layer equivalent if for some $0 < q < p < 1$, such that $x_q = y_q$, $x_p = y_p$, and for every pair $\{(t_1, t_2) : q < t_1 < t_2 < p\}$, it holds that $\mathbf{P}[x_{t_1} < X \le x_{t_2}] = \mathbf{P}[y_{t_1} < Y \le y_{t_2}]$. In such a case, we have that

$$\mathrm{LTSD}_{t_1,t_2}[X] = \mathrm{LTSD}_{t_1,t_2}[Y]. \tag{2.6}$$

Literally, this property states that the LTSD risk measure for an arbitrary layer is only dependent on the cdf of that layer. Parity of the ddfs implies equality of LTSDs.

Although looking for original ways to assess the degree of (actuarial) riskiness is a very important task, subsequent applications of various theoretical approaches to a real-world data are not less essential. A significant number of papers have been devoted to deriving explicit formulas for some tail-based risk measures in the context of various loss distributions. The incomplete list of works discussing the TCE risk measure consists of, for example, Hürlimann [12] and Furman and Landsman [13], gamma distributions; Panjer [14], normal family; Landsman and Valdez [15], elliptical distributions; Landsman and Valdez [16], and Furman and Landsman [17], exponential dispersion models; and Chiragiev and Landsman [18], Vernic [19], Asimit et al. [20], Pareto distributions of the second kind.

As we have already noticed, the “unlimited” tail standard deviation risk measure has been studied in the framework of the elliptical distributions by Furman and Landsman [10]. Unfortunately, all members of the elliptical class are symmetric, while insurance risks are generally modeled by nonnegative and positively skewed random variables. These peculiarities can be fairly well addressed employing an alternative class of distribution laws. The exponential dispersion models include many well-known distributions such as normal, gamma, and inverse Gaussian, which, except for the normal, are nonsymmetric, have nonnegative supports, and can serve as adequate models for describing insurance risks’ behavior. In this paper we therefore find it appropriate to apply both TSD and LTSD to EDM distributed risks.

3. The Limited Tail Standard Deviation Risk Measure for Exponential Dispersion Models

An early development of the exponential dispersion models is often attributed to Tweedie [21], however a more substantial and systematic investigation of this class of distributions was documented by Jørgensen [22, 23]. In his *Theory of dispersion models*, Jørgensen [24] writes that the main *raison d'étre* for the dispersion models is to serve as error distributions for generalized linear models, introduced by Nelder and Wedderburn [25]. Nowadays, EDMs play a prominent role in actuarial science and financial mathematics. This can be explained by the high level of generality that they enable in the context of statistical inference for widely popular distribution functions, such as normal, gamma, inverse Gaussian, stable, and many others. The specificity characterizing statistical modeling of actuarial subjects is that the underlying distributions mostly have nonnegative support, and many EDM members possess this important phenomenon, (for a formal definition of the EDMs, as well as for a brief review of some technical facts used in the sequel, cf., the appendix).

We are now in a position to evaluate the limited TSD risk measure in the framework of the EDMs. Recall that, for $0 < q < p < 1$, we denote by (x_q, x_p) an arbitrary layer having "attachment point" x_q and width $\Delta_{q,p}$. Also, let

$$h(x_q, x_p; \theta, \lambda) = \frac{\partial}{\partial \theta} \log\left(F(x_p; \theta, \lambda) - F(x_q; \theta, \lambda)\right) \tag{3.1}$$

denote the generalized layer-based hazard function, such that

$$\begin{aligned} h(x_q, x_1; \theta, \lambda) &= \frac{\partial}{\partial \theta} \log\left(\overline{F}(x_q; \theta, \lambda)\right) = h(x_q; \theta, \lambda), \\ h(x_0, x_p; \theta, \lambda) &= -\frac{\partial}{\partial \theta} \log\left(\overline{F}(x_p; \theta, \lambda)\right) = -h(x_p; \theta, \lambda), \end{aligned} \tag{3.2}$$

and thus

$$\begin{aligned} h(x_q, x_p; \theta, \lambda) = {} & \frac{\overline{F}(x_q; \theta, \lambda)}{\overline{F}(x_q; \theta, \lambda) - \overline{F}(x_p; \theta, \lambda)} h(x_q; \theta, \lambda) \\ & - \frac{\overline{F}(x_p; \theta, \lambda)}{\overline{F}(x_q; \theta, \lambda) - \overline{F}(x_p; \theta, \lambda)} h(x_p; \theta, \lambda). \end{aligned} \tag{3.3}$$

The next theorem derives expressions for the limited TCE risk measure, which is a natural precursor to deriving the limited TSD.

Theorem 3.1. *Assume that the natural exponential family (*NEF) which generates EDM is regular or at least steep (cf. [24, page 48]). Then the limited TCE risk measure

(i) for the reproductive EDM $Y \backsim$ ED(¯, œ^2) is given by

$$\mathrm{LTCE}_{q,p}[Y] = \mu + \sigma^2 \cdot h(x_q, x_p; \theta, \lambda) \tag{3.4}$$

and

(ii) for the additive EDM $X \backsim \mathrm{ED}^*(\theta, \lambda)$ is given by

$$\mathrm{LTCE}_{q,p}[X] = \lambda\kappa'(\theta) + h(x_q, x_p; \theta, \lambda). \tag{3.5}$$

Proof. We prove the reproductive case only, since the additive case follows in a similar fashion. By the definition of the limited TCE, we have that

$$\mathrm{LTCE}_{q,p}[Y] = \frac{\overline{F}(y_q)\mathbf{E}[Y \mid Y > y_q] - \overline{F}(y_p)\mathbf{E}[Y \mid Y > y_p]}{F(y_p) - F(y_q)}. \tag{3.6}$$

Further, following Landsman and Valdez [16], it can be shown that for every $0 < q < 1$, we have that

$$\mathbf{E}[Y \mid Y > y_q] = \mu + \sigma^2 \cdot h(y_q; \theta, \lambda), \tag{3.7}$$

which then, employing (3.1) and (3.3), yields

$$\begin{aligned} \mathrm{LTCE}_{q,p}[Y] &= \frac{\overline{F}(y_q; \theta, \lambda)(\mu + \sigma^2 \cdot h(y_q; \theta, \lambda)) - \overline{F}(y_p; \theta, \lambda)(\mu - \sigma^2 \cdot h(y_p; \theta, \lambda))}{\overline{F}(y_q; \theta, \lambda) - \overline{F}(y_p; \theta, \lambda)} \\ &= \mu + \sigma^2 \cdot h(y_q, y_p; \theta, \lambda) \end{aligned} \tag{3.8}$$

and hence completes the proof. □

In the sequel, we sometimes write $\mathrm{LTCE}_{q,p}[Y; \theta, \lambda]$ in order to emphasize the dependence on θ and λ.

Note 2. To obtain the results of Landsman and Valdez [16], we put $p \uparrow 1$, and then, for instance, in the reproductive case, we end up with

$$\lim_{p \uparrow 1} \mathrm{LTCE}_{q,p}[Y] = \mu + \sigma^2 \cdot h(y_q; \theta, \lambda) = \mathrm{TCE}_q[Y], \tag{3.9}$$

subject to the existence of the limit.

Next theorem provides explicit expressions for the limited TSD risk measure for both reproductive and additive EDMs.

Theorem 3.2. *Assume that the* NEF which generates EDM is regular or at least steep. Then the limited TSD risk measure

(i) for the reproductive EDM $Y \backsim$ ED(¯, œ^2) is given by

$$\mathrm{LTSD}_{q,p}[Y] = \mathrm{LTCE}_{q,p}[Y] + \alpha \cdot \sqrt{\sigma^2 \frac{\partial}{\partial\theta} \mathrm{LTCE}_{q,p}[Y;\theta,\lambda]} \tag{3.10}$$

and

(ii) for the additive EDM $X \backsim \mathrm{ED}^*(\theta,\lambda)$ is given by

$$\mathrm{LTSD}_{q,p}[X] = \mathrm{LTCE}_{q,p}[X] + \alpha \cdot \sqrt{\frac{\partial}{\partial\theta} \mathrm{LTCE}_{q,p}[X;\theta,\lambda]}. \tag{3.11}$$

Proof. We again prove the reproductive case, only. Note that it has been assumed that $\kappa(\theta)$ is a differentiable function, and thus we can differentiate the following probability integral in θ under the integral sign (cf., the appendix):

$$\mathbf{P}(y_q < Y \le y_p) = \int_{y_q}^{y_p} e^{\lambda(\theta y - \kappa(\theta))} d\overline{\nu}_\lambda(y), \tag{3.12}$$

and hence, using Definition 2.1, we have that

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\left(\mathrm{LTCE}_{q,p}[Y;\theta,\lambda]\left(F(y_p;\theta,\lambda) - F(y_q;\theta,\lambda)\right)\right) \\
&\quad = \int_{y_q}^{y_p} \frac{\partial}{\partial\theta} y e^{\lambda(\theta y - \kappa(\theta))} d\overline{\nu}_\lambda(y) \\
&\quad = \lambda \int_{y_q}^{y_p} \left(y^2 e^{\lambda(\theta y - \kappa(\theta))} - y\kappa'(\theta) e^{\lambda(\theta y - \kappa(\theta))}\right) d\overline{\nu}_\lambda(y) \\
&\quad = \sigma^{-2}\left(\mathbf{E}\left[Y^2 \mid \mathbf{1}\{y_q < Y \le y_p\}\right] - \mu(\theta) \cdot \mathbf{E}\left[Y \mid \mathbf{1}\{y_q < Y \le y_p\}\right]\right),
\end{aligned} \tag{3.13}$$

with the last line following from the appendix. Further, by simple rearrangement and straightforward calculations, we obtain that

$$\begin{aligned}
\mathbf{E}\left[Y^2 \mid y_q < Y \le y_p\right] &= \frac{\int_{y_q}^{y_p} y^2 e^{\lambda(\theta y - \kappa(\theta))} d\overline{\nu}_\lambda(y)}{F(y_p;\theta,\lambda) - F(y_q;\theta,\lambda)} \\
&= \mu \cdot \mathrm{LTCE}_{q,p}[Y] + \sigma^2 \frac{(\partial/\partial\theta)\mathrm{LTCE}_{q,p}[Y;\theta,\lambda](F(y_p;\theta,\lambda) - F(y_q;\theta,\lambda))}{F(y_p;\theta,\lambda) - F(y_q;\theta,\lambda)} \\
&= \sigma^2 \frac{\partial}{\partial\theta}\mathrm{LTCE}_{q,p}[Y;\theta,\lambda] + \mathrm{LTCE}_{q,p}[Y]\left(\mu + \sigma^2 \frac{\partial}{\partial\theta}\log(F(y_p;\theta,\lambda) - F(y_q;\theta,\lambda))\right) \\
&= \sigma^2 \frac{\partial}{\partial\theta}\mathrm{LTCE}_{q,p}[Y;\theta,\lambda] + \left(\mathrm{LTCE}_{q,p}[Y;\theta,\lambda]\right)^2,
\end{aligned} \tag{3.14}$$

which along with the definition of the limited TSD risk measure completes the proof. □

We further consider two examples to elaborate on Theorem 3.2. We start with the normal distribution, which occupies a central role in statistical theory, and its position in statistical analysis of insurance problems is very difficult to underestimate, for example, due to the law of large numbers.

Example 3.3. Let $Y \backsim N(\mu, \sigma^2)$ be a normal random variable with mean μ and variance σ^2, then we can write the pdf of Y as

$$\begin{aligned}
f(y) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}y^2\right)\exp\left(\frac{1}{\sigma^2}\left(\mu y - \frac{1}{2}\mu^2\right)\right), \quad y \in \mathbf{R}.
\end{aligned} \tag{3.15}$$

If we take $\theta = \mu$ and $\lambda = 1/\sigma^2$, we see that the normal distribution is a reproductive EDM with cumulant function $\kappa(\theta) = \theta^2/2$. Denote by $\varphi(\cdot)$ and $\Phi(\cdot)$ the pdf and the cdf, respectively, of the standardized normal random variable. Then using Theorem 3.1, we obtain the following expression for the limited TCE risk measure for the risk Y:

$$\mathrm{LTCE}_{q,p}[Y] = \mu + \sigma\frac{\varphi(\sigma^{-1}(y_q - \mu)) - \varphi(\sigma^{-1}(y_p - \mu))}{\Phi(\sigma^{-1}(y_p - \mu)) - \Phi(\sigma^{-1}(y_q - \mu))}. \tag{3.16}$$

If we put $p \uparrow 1$, then the latter equation reduces to the result of Landsman and Valdez [16]. Namely, we have that

$$\lim_{p \uparrow 1}\mathrm{LTCE}_{q,p}[Y] = \mu + \sigma\frac{\varphi(\sigma^{-1}(y_q - \mu))}{1 - \Phi(\sigma^{-1}(y_q - \mu))} = \mathrm{TCE}_q[Y]. \tag{3.17}$$

Further, let $z_q = (y_q - \mu)/\sigma$ and $z_p = (y_p - \mu)/\sigma$. Then

$$\sigma^2 \frac{\partial}{\partial\theta}\mathrm{LTCE}_{q,p}[Y;\theta,\lambda] = \sigma^2\left(1 + \frac{\varphi(z_q)z_q - \varphi(z_p)z_p}{\Phi(z_p) - \Phi(z_q)} - \left(\frac{\varphi(z_q) - \varphi(z_p)}{\Phi(z_p) - \Phi(z_q)}\right)^2\right). \tag{3.18}$$

Consequently, the limited TSD risk measure is as follows:

$$\begin{aligned}&\mathrm{LTSD}_{q,p}[Y]\\ &\quad = \mu + \sigma\frac{\varphi(z_q) - \varphi(z_p)}{\Phi(z_p) - \Phi(z_q)} + \alpha\sqrt{\sigma^2\left(1 + \frac{\varphi(z_q)z_q - \varphi(z_p)z_p}{\Phi(z_p) - \Phi(z_q)} - \left(\frac{\varphi(z_q) - \varphi(z_p)}{\Phi(z_p) - \Phi(z_q)}\right)^2\right)}.\end{aligned} \tag{3.19}$$

We proceed with the gamma distributions, which have been widely applied in various fields of actuarial science. It should be noted that these distribution functions possess positive support and positive skewness, which is important for modeling insurance losses. In addition, gamma rvs have been well-studied, and they share many tractable mathematical properties which facilitate their use. There are numerous examples of applying gamma distributions for modeling insurance portfolios (cf., e.g., [12, 13, 26, 27]).

Example 3.4. Let $X \backsim \mathrm{Ga}(\gamma,\beta)$ be a gamma rv with shape and rate parameters equal γ and β, correspondingly. The pdf of X is

$$f(x) = \frac{1}{\Gamma(\gamma)}e^{-\beta x}x^{\gamma-1}\beta^\gamma = \frac{1}{\Gamma(\gamma)}x^{\gamma-1}\exp(-\beta x + \gamma\log(\beta)), \quad x > 0. \tag{3.20}$$

Hence the gamma rv can be represented as an additive EDM with the following pdf:

$$f(x) = \frac{1}{\Gamma(\lambda)}x^{\lambda-1}\exp(\theta x + \lambda\log(-\theta)), \tag{3.21}$$

where $x > 0$ and $\theta < 0$. The mean and variance of X are $\mathbf{E}[X] = -\lambda/\theta$ and $\mathbf{Var}[X] = \lambda/\theta^2$. Also, $\theta = -\beta$, $\lambda = \gamma$, and $\kappa(\theta) = -\log(-\theta)$. According to Theorem 3.1, the limited tail conditional expectation is

$$\mathrm{LTCE}_{q,p}[X] = -\frac{\lambda}{\theta}\frac{F(x_p;\theta,\lambda+1) - F(x_q;\theta,\lambda+1)}{F(x_p;\theta,\lambda) - F(x_q;\theta,\lambda)}. \tag{3.22}$$

Putting $p \uparrow 1$ we obtain that

$$\lim_{p\uparrow 1}\left(-\frac{\lambda}{\theta}\right)\frac{F(x_p;\theta,\lambda+1) - F(x_q;\theta,\lambda+1)}{F(x_p;\theta,\lambda) - F(x_q;\theta,\lambda)} = -\frac{\lambda}{\theta}\frac{\overline{F}(x_q;\theta,\lambda+1)}{\overline{F}(x_q;\theta,\lambda)} = \mathrm{TCE}_q[X], \tag{3.23}$$

which coincides with [13, page 643]. To derive an expression for the limited TSD risk measure, we use Theorem 3.2, that is,

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathrm{LTCE}_{q,p}[X;\theta,\lambda] &= \frac{\partial}{\partial\theta}\left(-\frac{\lambda}{\theta}\frac{F(x_p;\theta,\lambda+1)-F(x_q;\theta,\lambda+1)}{F(x_p;\theta,\lambda)-F(x_q;\theta,\lambda)}\right)\\ &= \frac{\lambda}{\theta^2}\frac{F(x_p;\theta,\lambda+1)-F(x_q;\theta,\lambda+1)}{F(x_p;\theta,\lambda)-F(x_q;\theta,\lambda)}\\ &\quad -\frac{\lambda}{\theta}\left(\frac{\partial}{\partial\theta}\frac{F(x_p;\theta,\lambda+1)-F(x_q;\theta,\lambda+1)}{F(x_p;\theta,\lambda)-F(x_q;\theta,\lambda)}\right).\end{aligned} \tag{3.24}$$

Further, since for $n = 1, 2, \ldots,$

$$\begin{aligned}&\frac{\partial}{\partial\theta}\left(F(x_p;\theta,\lambda+n)-F(x_q;\theta,\lambda+n)\right)\\ &\quad= \int_{x_q}^{x_p}\frac{\partial}{\partial\theta}\left(\frac{1}{\Gamma(\lambda+n)}x^{\lambda+n-1}\exp(\theta x+(\lambda+n)\log(-\theta))\right)dx\\ &\quad= \int_{x_q}^{x_p} f(x;\theta,\lambda+n)\left(x+\frac{\lambda+n}{\theta}\right)dx\\ &\quad= -\frac{\lambda+n}{\theta}\left(\int_{x_q}^{x_p} f(x;\theta,\lambda+n+1)dx-\int_{x_q}^{x_p} f(x;\theta,\lambda+n)dx\right),\end{aligned} \tag{3.25}$$

the limited TSD risk measure for gamma is given by

$$\begin{aligned}&\mathrm{LTSD}_{q,p}[X]\\ &= \left(-\frac{\lambda}{\theta}\right)\frac{F(x_p;\theta,\lambda+1)-F(x_q;\theta,\lambda+1)}{F(x_p;\theta,\lambda)-F(x_q;\theta,\lambda)}\\ &\quad+\alpha\sqrt{\frac{\lambda}{\theta^2}\left((\lambda+1)\frac{F(x_p;\theta,\lambda+2)-F(x_q;\theta,\lambda+2)}{F(x_p;\theta,\lambda)-F(x_q;\theta,\lambda)}-\lambda\left(\frac{F(x_p;\theta,\lambda+1)-F(x_q;\theta,\lambda+1)}{F(x_p;\theta,\lambda)-F(x_q;\theta,\lambda)}\right)^2\right)}.\end{aligned} \tag{3.26}$$

In the sequel, we consider gamma and normal risks with equal means and variances, and we explore them on the interval $(t, 350]$, with $50 < t < 350$. Figure 1 depicts the results. Note that both LTCE and LTSD imply that the normal distribution is riskier than gamma for lower attachment points and vice-versa, that is quite natural bearing in mind the tail behavior of the two.

Although the EDMs are of pivotal importance in actuarial mathematics, they fail to appropriately describe heavy-tailed (insurance) losses. To elucidate on the applicability of the layer-based risk measures in the context of the probability distributions possessing heavy tails, we conclude this section with a simple example.

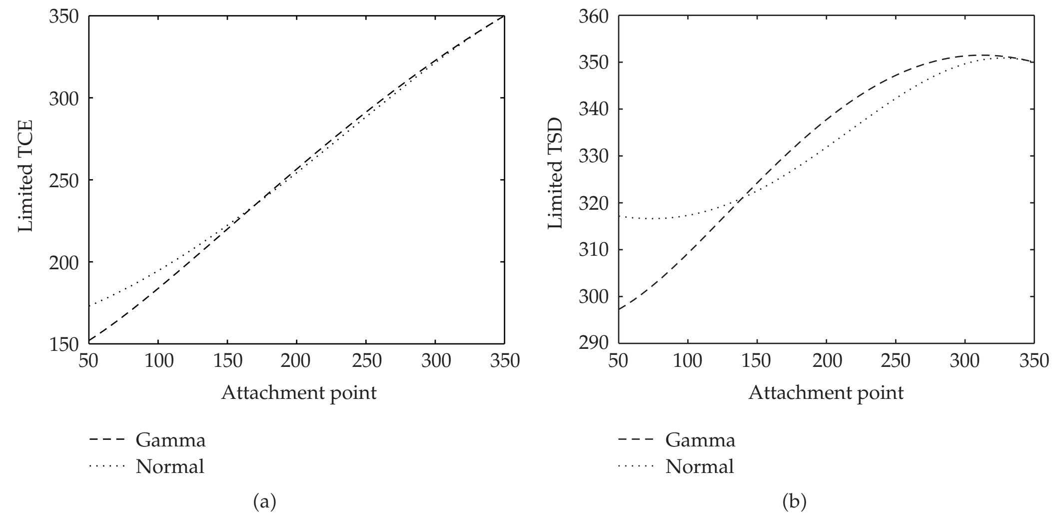

Figure 1: LTCE and LTSD for normal and gamma risks with means 150 and standard deviations 100, alpha = 2.

Example 3.5. Let $\mathrm{X} \backsim \mathrm{Pa}(\gamma, \beta)$ be a Pareto rv with the pdf

$$f(x) = \frac{\gamma\beta^{\gamma}}{x^{\gamma+1}}, \quad x > \beta > 0, \tag{3.27}$$

and $\gamma > 0$. Certainly, the Pareto rv is not a member of the EDMs, though it belongs to the log-exponential family (LEF) of distributions (cf. [7]). The LEF is defined by the differential equation

$$F(dx; \lambda, \nu) = \exp\{\lambda \log(x) - \kappa(\lambda)\}\nu(dx), \tag{3.28}$$

where λ is a parameter, ν is a measure, and $\kappa(\lambda) = \log \int_0^{\infty} x^{\lambda}\nu(dx)$ is a normalizing constant (the parameters should not be confused with the ones used in the context of the EDMs). Then X is easily seen to belong in LEF with the help of the reparameterization $\nu(dx) = x^{-1}dx$, and $\lambda = -\gamma$.

In this context, it is straightforward to see that $\mathbf{E}[\mathrm{X}]$ is infinite for $\gamma \le 1$, which thus implies infiniteness of the TCE risk measure. We can however readily obtain the limited variant as follows:

$$\mathrm{LTCE}_{q,p}[\mathrm{X}] = \frac{1}{\mathbf{P}[x_q < \mathrm{X} \le x_p]} \int_{x_q}^{x_p} \frac{\gamma\beta^{\gamma}}{x^{\gamma}} dx = \frac{\gamma x_p x_q}{\gamma - 1} \left(\frac{x_p^{\gamma-1} - x_q^{\gamma-1}}{x_p^{\gamma} - x_q^{\gamma}} \right), \tag{3.29}$$

that is finite for any $\gamma > 0$. Also, since, for example, for $\gamma < 1$, we have that $x_p^{\gamma-1} - x_q^{\gamma-1} < 0$, the limited TCE risk measure is positive, as expected. The same is true for $\gamma \ge 1$.

We note in passing that, for $\gamma > 1$ and $p \uparrow 1$ and thus $x_p \to \infty$, we have that

$$\mathrm{TCE}_q[X] = \lim_{p\uparrow 1} \frac{\gamma x_p x_q}{\gamma - 1} \left(\frac{x_p^{\gamma-1} - x_q^{\gamma-1}}{x_p^{\gamma} - x_q^{\gamma}} \right) = \frac{\gamma}{\gamma - 1} x_q, \tag{3.30}$$

which confirms the corresponding expression in Furman and Landsman [8].

Except for the Pareto distribution, the LEF consists of, for example, the log-normal and inverse-gamma distributions, for which expressions similar to (3.29) can be developed in the context of the limited TCE and limited TSD risk measures, thus providing a partial solution to the heavy-tailness phenomenon.

4. The Tail Standard Deviation Risk Measure for Exponential Dispersion Models

The tail standard deviation risk measure was proposed in [10] as a possible quantifier of the so-called tail riskiness of the loss distribution. The above-mentioned authors applied this risk measure to elliptical class of distributions, which consists of such well-known pdfs as normal and student-t. Although the elliptical family is very useful in finance, insurance industry imposes its own restrictions. More specifically, insurance claims are always positive and mostly positively skewed. In this section we apply the TSD risk measure to EDMs.

The following corollary develops formulas for the TSD risk measure both in the reproductive and additive EDMs cases. Recall that we denote the ddf of say X by $\overline{F}(\cdot;\theta,\lambda)$ to emphasize the parameters θ and λ, and we assume that

$$\lim_{p\uparrow 1} \mathrm{LTSD}_{q,p}[X] < \infty. \tag{4.1}$$

The proof of the next corollary is left to the reader.

Corollary 4.1. *Under the conditions in Theorem 3.1, the tail standard deviation risk measure is*

$$\mathrm{TSD}_q[Y] = \mathrm{TCE}_q[Y] + \alpha\sqrt{\sigma^2 \frac{\partial}{\partial\theta} \mathrm{TCE}_q[Y;\theta,\lambda]} \tag{4.2}$$

in the context of the reproductive EDMs, and

$$\mathrm{TSD}_q[X] = \mathrm{TCE}_q[X] + \alpha\sqrt{\frac{\partial}{\partial\theta} \mathrm{TCE}_q[X;\theta,\lambda]} \tag{4.3}$$

in the context of the additive EDMs.

We further explore the TSD risk measure in some particular cases of EDMs, which seem to be of practical importance.

Example 4.2. Let $Y \backsim N(\mu, \sigma^2)$ be again some normal rv with mean μ and variance σ^2. Then we easily evaluate the TSD risk measure using Corollary 4.1 and Example 3.3 as follows:

$$\mathrm{TSD}_q[X] = \mu + \sigma \frac{\varphi(z_q)}{1-\Phi(z_q)} + \alpha \sqrt{\sigma^2 \left(1 + \frac{\varphi(z_q)}{1-\Phi(z_q)} z_q - \left(\frac{\varphi(z_q)}{1-\Phi(z_q)} \right)^2 \right)}, \tag{4.4}$$

which coincides with [10].

Example 4.3. Let $X \backsim \mathrm{Ga}(\gamma, \beta)$ be a gamma rv with shape and scale parameters equal γ and β, correspondingly. Taking into account Example 3.4 and Corollary 4.1 leads to

$$\begin{aligned} \mathrm{TSD}_q[X] & \\ &= -\frac{\lambda}{\theta} \frac{\overline{F}(x_q; \theta, \lambda+1)}{\overline{F}(x_q; \theta, \lambda)} + \alpha \sqrt{\frac{\lambda}{\theta^2} \left((\lambda+1) \frac{\overline{F}(x_q; \theta, \lambda+2)}{\overline{F}(x_q; \theta, \lambda)} - \lambda \left(\frac{\overline{F}(x_q; \theta, \lambda+1)}{\overline{F}(x_q; \theta, \lambda)} \right)^2 \right)} \\ &= \frac{\gamma}{\beta} \frac{\overline{F}(x_q; \gamma+1, \beta)}{\overline{F}(x_q; \gamma, \beta)} + \alpha \sqrt{\frac{\gamma}{\beta^2} \left((\gamma+1) \frac{\overline{F}(x_q; \gamma+2, \beta)}{\overline{F}(x_q; \gamma, \beta)} - \gamma \left(\frac{\overline{F}(x_q; \gamma+1, \beta)}{\overline{F}(x_q; \gamma, \beta)} \right)^2 \right)}, \end{aligned} \tag{4.5}$$

where the latter equation follows because of the reparameterization $\theta = -\beta$ and $\lambda = \gamma$.

We further discuss the inverse Gaussian distribution, which possesses heavier tails than, say, gamma distribution, and therefore it is somewhat more tolerant to large losses.

Example 4.4. Let $Y \backsim \mathrm{IG}(\mu, \lambda)$ be an inverse Gaussian rv. We then can write its pdf as

$$f(y) = \sqrt{\frac{\lambda}{2\pi y^3}} \exp\left(\lambda \left(-\frac{y}{2\mu^2} - \frac{1}{2y} + \frac{1}{\mu} \right) \right), \quad y \in [0, \infty), \tag{4.6}$$

(cf. [24]), which means that Y belongs to the reproductive EDMs, with $\theta = -1/(2\mu^2)$ and $\kappa(\theta) = -1/\mu = -\sqrt{-2\theta}$. Further, due to Corollary 4.1 we need to calculate

$$\frac{\partial}{\partial\theta} \mathrm{TCE}_q[Y; \theta, \lambda] = \frac{\partial}{\partial\theta} \left(\mu(\theta) + \sigma^2 \frac{\partial}{\partial\theta} \log \overline{F}_Y(y_q; \theta, \lambda) \right) = \mu'(\theta) + \sigma^2 \frac{\partial}{\partial\theta} \frac{(\partial/\partial\theta)\overline{F}_Y(y_q; \theta, \lambda)}{\overline{F}_Y(y_q; \theta, \lambda)}. \tag{4.7}$$

To this end, note that the ddf of Y is

$$\overline{F}(y_q; \mu(\theta), \lambda) = \overline{\Phi}\left(\sqrt{\frac{\lambda}{y_q}} \left(\frac{y_q}{\mu(\theta)} - 1 \right) \right) - e^{2\lambda/\mu(\theta)} \Phi\left(-\sqrt{\frac{\lambda}{y_q}} \left(\frac{y_q}{\mu(\theta)} + 1 \right) \right) \tag{4.8}$$

(cf., e.g., [28]), where $\overline{\Phi}(\cdot)$ is the ddf of the standardized normal random variable. Hence, by simple differentiation and noticing that

$$\mu'(\theta) = (-2\theta)^{-3/2} = \mu(\theta)^3, \tag{4.9}$$

we obtain that

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\overline{F}(y_q;\mu(\theta),\lambda) \\
&\quad = \mu(\theta)\left(\sqrt{\lambda y_q}\varphi\left(\sqrt{\frac{\lambda}{y_q}}\left(\frac{y_q}{\mu(\theta)}-1\right)\right) - e^{2\lambda/\mu(\theta)}\sqrt{\lambda y_q}\varphi\left(-\sqrt{\frac{\lambda}{y_q}}\left(\frac{y_q}{\mu(\theta)}+1\right)\right)\right) \\
&\quad + 2\lambda\mu(\theta)e^{2\lambda/\mu(\theta)}\Phi\left(-\sqrt{\frac{\lambda}{y_q}}\left(\frac{y_q}{\mu(\theta)}+1\right)\right).
\end{aligned} \tag{4.10}$$

Notably,

$$\sqrt{\lambda y_q}\varphi\left(\sqrt{\frac{\lambda}{y_q}}\left(\frac{y_q}{\mu(\theta)}-1\right)\right) = e^{2\lambda/\mu(\theta)}\sqrt{\lambda y_q}\varphi\left(-\sqrt{\frac{\lambda}{y_q}}\left(\frac{y_q}{\mu(\theta)}+1\right)\right), \tag{4.11}$$

and therefore (4.10) results in

$$\frac{\partial}{\partial\theta}\overline{F}(y_q;\mu(\theta),\lambda) = 2\lambda\mu(\theta)e^{2\lambda/\mu(\theta)}\Phi\left(-\sqrt{\frac{\lambda}{y_q}}\left(\frac{y_q}{\mu(\theta)}+1\right)\right). \tag{4.12}$$

Consequently, the expression for the TCE risk measure, obtained by Landsman and Valdez [16], is simplified to

$$\mathrm{TCE}_q[Y;\theta,\lambda] = \mu(\theta) + \frac{2\mu(\theta)}{\overline{F}(y_q;\mu(\theta),\lambda)}e^{2\lambda/\mu(\theta)}\Phi\left(-\sqrt{\frac{\lambda}{y_q}}\left(\frac{y_q}{\mu(\theta)}+1\right)\right). \tag{4.13}$$

In order to derive the TSD risk measure we need to differentiate again, that is,

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{TCE}_q[Y;\theta,\lambda] &= \frac{\partial}{\partial\theta}\left(\mu(\theta) + \frac{2\mu(\theta)}{\overline{F}(y_q;\mu(\theta),\lambda)}e^{2\lambda/\mu(\theta)}\Phi\left(-\sqrt{\frac{\lambda}{y_q}}\left(\frac{y_q}{\mu(\theta)}+1\right)\right)\right) \\
&= \mu(\theta)^3\left(1 + \frac{\partial}{\partial\theta}\frac{2\mu(\theta)e^{2\lambda/\mu(\theta)}\Phi\left(-\sqrt{\lambda/y_q}\left(y_q/\mu(\theta)+1\right)\right)}{\overline{F}(y_q;\mu(\theta),\lambda)}\right),
\end{aligned} \tag{4.14}$$

where we use $\mu'(\theta) = \mu(\theta)^3$. Further, we have that

$$\frac{\partial}{\partial\theta}\frac{2\mu(\theta)e^{2\lambda/\mu(\theta)}\Phi\left(-\sqrt{\lambda/y_q}\left(y_q/\mu(\theta)+1\right)\right)}{\overline{F}(y_q;\mu(\theta),\lambda)}$$
$$= 2\frac{\mu(\theta)^3 e^{2\lambda/\mu(\theta)}\left(\Phi(\tilde{y}_q)\left(1-2\lambda/\mu(\theta)\right)+\left(\sqrt{\lambda y_q}/\mu(\theta)\right)\varphi(\tilde{y}_q)\right)}{\overline{F}(y_q;\mu(\theta),\lambda)}$$
$$-\frac{\lambda\left(2\mu(\theta)e^{2\lambda/\mu(\theta)}\Phi(\tilde{y}_q)\right)^2}{\overline{F}(y_q;\mu(\theta),\lambda)^2}, \tag{4.15}$$

where $\tilde{y}_q = -\sqrt{\lambda/y_q}(y_q/\mu(\theta)+1)$. Therefore

$$\mathrm{TSD}_q[Y] = \mu\left(1+\frac{\Phi(\tilde{y}_q)}{\overline{F}(y_q;\mu,\lambda)}2e^{2\lambda/\mu}\right)$$
$$+\alpha\sqrt{\frac{\mu^3}{\lambda}\left(1+\frac{e^{2\lambda/\mu}\left(\Phi(\tilde{y}_q)\left(1-2\lambda/\mu\right)+\left(\sqrt{\lambda y_q}/\mu\right)\varphi(\tilde{y}_q)\right)}{\overline{F}(y_q;\mu,\lambda)}-\frac{\lambda\left(e^{2\lambda/\mu}\Phi(\tilde{y}_q)\right)^2}{\mu\overline{F}(y_q;\mu,\lambda)^2}\right)} \tag{4.16}$$

subject to $\mathbf{Var}[Y] = \mu^3/\lambda$.

5. Concluding Comments

In this work we have considered certain layer-based risk measuring functionals in the context of the exponential dispersion models. Although we have made an accent on the absolutely continuous EDMs, similar results can be developed for the discrete members of the class. Indeed, distributions with discrete supports often serve as frequency models in actuarial mathematics. Primarily in expository purposes, we further consider a very simple frequency distribution, and we evaluate the TSD risk measure for it. More encompassing formulas can however be developed with some effort for other EDM members of, say, the $(a, b, 0)$ class (cf., [11, Chapter 6]) as well as for limited TCE/TSD risk measures.

Example 5.1. Let $X \backsim \mathrm{Poisson}(\mu)$ be a Poisson rv with the mean parameter μ. Then the probability mass function of X is written as

$$p(x) = \frac{1}{x!}\mu^x e^{-\mu} = \frac{1}{x!}\exp\left(x\log(\mu)-\mu\right), \quad x = 0, 1, \ldots, \tag{5.1}$$

which belongs to the additive EDMs in view of the reparametrization $\theta = \log(\mu)$, $\lambda = 1$, and $\kappa(\theta) = e^{\theta}$.

Motivated by Corollary 4.1, we differentiate (cf. [16], for the formula for the TCE risk measure)

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{TCE}_q(X;\theta,\lambda) &= \frac{\partial}{\partial\theta}\left(e^{\theta}\left(1+\frac{p(x_q;\theta,1)}{\overline{F}(x_q;\theta,1)}\right)\right)\\
&= e^{\theta}\left(1+\frac{p(x_q;\theta,1)}{\overline{F}(x_q;\theta,1)}+\frac{p(x_q;\theta,1)}{\overline{F}(x_q;\theta,1)}\left(x_q-e^{\theta}\right)-e^{\theta}\left(\frac{p(x_q;\theta,1)}{\overline{F}(x_q;\theta,1)}\right)^2\right)\\
&= e^{\theta}\left(\frac{\overline{F}(x_q-1;\theta,1)}{\overline{F}(x_q;\theta,1)}+\frac{p(x_q;\theta,1)}{\overline{F}(x_q;\theta,1)}\left(x_q-e^{\theta}\right)-e^{\theta}\left(\frac{p(x_q;\theta,1)}{\overline{F}(x_q;\theta,1)}\right)^2\right),
\end{aligned} \tag{5.2}$$

where the latter equation follows because

$$\overline{F}(x_q;\theta,1)+p(x_q;\theta,1)=\overline{F}(x_q-1;\theta,1). \tag{5.3}$$

The formula for the TSD risk measure is then

$$\begin{aligned}
&\mathrm{TSD}_q(X)\\
&= e^{\theta}\left(1+\frac{p(x_q;\theta,1)}{\overline{F}(x_q;\theta,1)}\right)+\alpha\sqrt{e^{\theta}\left(\frac{\overline{F}(x_q-1;\theta,1)}{\overline{F}(x_q;\theta,1)}+\frac{p(x_q;\theta,1)}{\overline{F}(x_q;\theta,1)}z_q-e^{\theta}\left(\frac{p(x_q;\theta,1)}{\overline{F}(x_q;\theta,1)}\right)^2\right)},
\end{aligned} \tag{5.4}$$

where $\mathbf{E}[X]=\mathbf{Var}[X]=e^{\theta}$ and $z_q=x_q-e^{\theta}$.

Appendix

A. Exponential Dispersion Models

Consider a σ-finite measure ν on $\mathbf{R}$ and assume that ν is nondegenerate. Next definition is based on [24].

Definition A.1. The family of distributions of $X \backsim \mathrm{ED}^*(\theta,\lambda)$ for $(\theta,\lambda)\in\Theta\times\Lambda$ is called the additive exponential dispersion model generated by ν. The corresponding family of distributions of $Y=X/\lambda \backsim \mathrm{ED}(\mu,\sigma^2)$, where $\mu=\tau(\theta)$ and $\sigma^2=1/\lambda$ are the mean value and the dispersion parameters, respectively, is called the reproductive exponential dispersion model generated by ν. Moreover, given some measure ν_λ the representation of $X \backsim \mathrm{ED}^*(\theta,\lambda)$ is as follows:

$$\exp(\theta x-\lambda\kappa(\theta))\nu_\lambda(dx). \tag{A.1}$$

If in addition the measure ν_λ has density $c^*(x;\lambda)$ with respect to some fixed measure (typically Lebesgue measure or counting measure), the density for the additive model is

$$f^*(x;\theta,\lambda) = c^*(x;\lambda)\exp(\theta x - \lambda\kappa(\theta)), \quad x \in \mathbf{R}. \tag{A.2}$$

Similarly, we obtain the following representation of $Y \backsim \mathrm{ED}(\mu,\sigma^2)$ as

$$\exp(\lambda(y\theta - \kappa(\theta)))\overline{\nu}_\lambda(dy), \tag{A.3}$$

where $\overline{\nu}_\lambda$ denotes ν_λ transformed by the duality transformation $X = Y/\sigma^2$. Again if the measure $\overline{\nu}_\lambda$ has density $c(y;\lambda)$ with respect to a fixed measure, the reproductive model has the following pdf:

$$f(y;\theta,\lambda) = c(y;\lambda)\exp(\lambda(\theta y - \kappa(\theta))), \quad y \in \mathbf{R}. \tag{A.4}$$

Note that θ and λ are called canonical and index parameters, $\Theta = \{\theta \in \mathbf{R} : \kappa(\theta) < \infty\}$ for some function $\kappa(\theta)$ called the cumulant, and Λ is the index set. Throughout the paper, we use $X \backsim \mathrm{ED}^*(\mu,\sigma^2)$ and $X \backsim \mathrm{ED}(\theta,\lambda)$ for the additive form with parameters μ and σ^2 and the reproductive form with parameters θ and λ, correspondingly, depending on which notation is more convenient.

We further briefly review some properties of the EDMs related to this work. Consider the reproductive form first, that is, $Y \backsim \mathrm{ED}(\mu,\sigma^2)$, then

(i) the cumulant generating function (cgf) of Y is, for $\theta' = \theta + t/\lambda$,

$$\begin{aligned} K(t) = \log \mathbf{E}\left[e^{tY}\right] &= \log\left(\int_{\mathbf{R}} \exp\left(\lambda\left(y\left(\theta + \frac{t}{\lambda}\right) - \kappa(\theta)\right)\right) d\overline{\nu}_\lambda(y)\right) \\ &= \log\left(\exp\left(\lambda\left(\kappa\left(\theta + \frac{t}{\lambda}\right) - \kappa(\theta)\right)\right)\int_{\mathbf{R}} \exp(\lambda[\theta' y - \kappa(\theta')]) d\overline{\nu}_\lambda(y)\right) \\ &= \lambda\left(\kappa\left(\theta + \frac{t}{\lambda}\right) - \kappa(\theta)\right), \end{aligned} \tag{A.5}$$

(ii) the moment generating function (mgf) of Y is given by

$$M(t) = \exp\left(\lambda\left(\kappa\left(\frac{\theta + t}{\lambda}\right) - \kappa(\theta)\right)\right), \tag{A.6}$$

(iii) the expectation of Y is

$$\mathbf{E}[Y] = \left.\frac{\partial K(t)}{\partial t}\right|_{t=0} = \kappa'(\theta) = \mu, \tag{A.7}$$

(iv) the variance of Y is

$$\mathbf{Var}[Y] = \left.\frac{\partial^2 K(t)}{\partial t^2}\right|_{t=0} = \sigma^2 \kappa^{(2)}(\theta). \tag{A.8}$$

Consider next an rv X following an additive EDM, that is, $X \backsim \mathrm{ED}^*(\theta, \lambda)$. Then, in a similar fashion,

(i) the cgf of X is

$$K(t) = \lambda(\kappa(\theta + t) - \kappa(\theta)), \tag{A.9}$$

(ii) the mgf of X is

$$M(t) = \exp(\lambda(\kappa(\theta + t) - \kappa(\theta))), \tag{A.10}$$

(iii) the expectation of X is

$$\mathbf{E}[X] = \lambda\kappa'(\theta), \tag{A.11}$$

(iv) the variance of X is

$$\mathbf{Var}[X] = \lambda\kappa^{(2)}(\theta). \tag{A.12}$$

For valuable examples of various distributions belonging in the EDMs we refer to Jørgensen [24].

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