Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2010, Article ID 730543, 23 pages
doi:10.1155/2010/730543

Research Article

Some Comments on Quasi-Birth-and-Death Processes and Matrix Measures

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Received 12 August 2009; Revised 15 April 2010; Accepted 25 May 2010

Academic Editor: Nikolaos E. Limnios

We explore the relation between matrix measures and quasi-birth-and-death processes. We derive an integral representation of the transition function in terms of a matrix-valued spectral measure and corresponding orthogonal matrix polynomials. We characterize several stochastic properties of quasi-birth-and-death processes by means of this matrixmeasure and illustrate the theoretical results by several examples.

1. Introduction

Let $(\Omega, \mathcal{F}, P, (X_t)_{t\geq 0})$ be a continuous-time two-dimensional homogeneous Markov process with state space

$$E = \{(i,j) \in \mathbb{N}_0 \times \{1,\ldots,d\}\}, \quad d \in \mathbb{N},\ d < \infty \tag{1.1}$$

and infinitesimal generator

$$Q = (Q_{ij})_{i,j=0,1,\ldots} = \begin{pmatrix} B_0 & A_0 & & & \mathbf{0} \\ C_1^T & B_1 & A_1 & & \\ & C_2^T & B_2 & A_2 & \\ & & C_3^T & B_3 & A_3 \\ \mathbf{0} & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.2}$$

where $A_0, A_1, \ldots, B_0, B_1, \ldots, C_1, C_2, \ldots \in \mathbb{R}^{d\times d}$. The transition rate from state (i,j) to state (k,ℓ) is given by the element in the position (j,ℓ) of the matrix Q_{ik}. Markov processes with an infinitesimal generator matrix of the form (1.2) are known as continuous-time quasi-birth-and-death processes. These models have many applications in the evaluation of communicating systems and queueing systems (see, e.g., [1–3]) and have been analyzed by many authors (see, e.g., [4–6]). The case $d = 1$ corresponds to a "classical" birth-and-death process with a tridiagonal infinitesimal generator which has been investigated in great detail using the theory of orthogonal polynomials by Karlin and McGregor [7, 8]. Since this pioneering work several authors have used these techniques to derive interesting properties of birth-and-death processes in terms of orthogonal polynomials and the corresponding measure of orthogonality (see, e.g., [9, 10]).

It is the purpose of the present paper to extend some of these results to quasi-birth-and-death processes with a generator of the form (1.2) using the theory of matrix measures and corresponding orthogonal matrix polynomials.

We associate to a matrix of the form of (1.2) a sequence of matrix polynomials, recursively defined by

$$-xQ_n(x) = A_nQ_{n+1}(x) + B_nQ_n(x) + C_n^TQ_{n-1}(x) \tag{1.3}$$

with initial conditions $Q_{-1}(x) = 0$ and $Q_0(x) = I_d$. A matrix measure $\Sigma = \{\sigma_{ij}\}_{i,j=1,\ldots,d}$ on the real line is a function for which $\Sigma(A) = \{\sigma_{ij}(A)\}_{i,j=1,\ldots,d}$ is a symmetric and nonnegative definite matrix in $\mathbb{R}^{d\times d}$ for each Borel set $A \subset \mathbb{R}$, where the entries σ_{ij} are finite signed measures. In Section 2 we formulate sufficient conditions on the infinitesimal generator (1.2) such that there exists a matrix measure Σ on the real line with

$$\langle Q_i, Q_j\rangle = \int_{\mathbb{R}} Q_i(x)d\Sigma(x)Q_j^T(x) = \delta_{ij}I_d, \tag{1.4}$$

that is, the matrix polynomials are orthonormal with respect to the matrix measure Σ (see [11]). In this case we derive an integral representation for the blocks of the transition function in terms of the orthogonal matrix polynomials Q_i and the matrix measure Σ, which generalize the representation of Karlin and McGregor [7] to the case $d > 1$. We also investigate relations between the Stieltjes transforms of random walk measures corresponding to two quasi-birth-and-death processes, where only a few blocks differ. In Section 3 we discuss several examples to illustrate the theory. Finally, in Section 4 the theoretical results are used to characterize α-recurrence of quasi-birth-and-death processes.

2. Quasi-Birth-and-Death Processes and Matrix Polynomials

The moments of the matrix measure Σ are defined by the $d \times d$ matrices

$$S_k = \int x^k d\Sigma(x), \quad k = 0, 1, \ldots, \tag{2.1}$$

and throughout this paper we will only consider matrix measures with existing moments of all order. The "left" inner product with respect to Σ of two matrix polynomials Q and P is defined by

$$\langle Q, P\rangle = \int Q(x)d\Sigma(x)P^T(x). \tag{2.2}$$

If $\{S_n\}_{n\geq 0}$ is a sequence of matrices such that the block Hankel matrices,

$$\underline{H}_{2m} = \begin{pmatrix} S_0 & \cdots & S_m \\ \vdots & & \vdots \\ S_m & \dots & S_{2m} \end{pmatrix}, \quad m \geq 0, \tag{2.3}$$

are positive definite, then there exists a matrix measure Σ with moments S_n, $n \geq 0$, and a sequence of matrix polynomials $\{Q_n(x)\}_{n\geq 0}$ which is orthogonal with respect to Σ (see [12]). The following theorem characterizes the existence of a matrix measure Σ such that there is a sequence of matrix polynomials which is orthogonal with respect to Σ. The proof follows by similar arguments as presented in Theorem 2.1 of [13] and is therefore omitted.

Theorem 2.1. *Let the matrices A_n, $n \geq 0$, and $C_n^T, n \geq 1$, in* (1.2) *be nonsingular and $B_n \geq 0$, and assume that $\{Q_n(x)\}_{n\geq 0}$ is a sequence of matrix polynomials defined by recursion* (1.3)*.*

There exists a matrix measure Σ with positive definite block Hankel matrices $\underline{H}_{2m}$, $m \geq 0$, such that the sequence of matrix polynomials $\{Q_n(x)\}_{n\geq 0}$ is orthogonal with respect to Σ if and only if there is a sequence of nonsingular matrices $\{R_n\}_{n\geq 0}$ with

$$\begin{gathered} R_n B_n R_n^{-1} \text{ symmetric}, \quad \forall n \in \mathbb{N}_0, \\ R_n^T R_n = C_n^{-1} \cdots C_1^{-1}\left(R_0^T R_0\right) A_0 \cdots A_{n-1}, \quad \forall n \in \mathbb{N}. \end{gathered} \tag{2.4}$$

Moreover,

$$R_0^{-1}\left(\left(R_0^T\right)^{-1}\right) = \left(R_0^T R_0\right)^{-1} = S_0, \tag{2.5}$$

and the matrices $\{\widetilde{R}_n\}_{n\geq 0} = \{U_n R_n\}_{n\geq_0}$, where U_n, $n \geq 0$, are orthogonal matrices and also satisfy condition (2.4)*.*

Note that condition (2.4) is crucial for our approach and is always satisfied in the case $d = 1$. If $d > 1$ it has to be checked in concrete examples, but—to our best knowledge—there do not exist any general conditions which imply (2.4). Some examples where (2.4) is satisfied are presented in Section 3. Several other examples can be found in the papers of Grünbaum [14, 15], Grünbaum et al. [16], and Cantero et al. [17]. If condition (2.4) is satisfied, the corresponding measure Σ is called a spectral measure corresponding to $\{Q_n(x)\}_{n\geq 0}$ and the matrix Q in (2.4), respectively. The infinitesimal generator matrix (1.2) is called conservative if

$$(A_0 + B_0)\mathbf{1} = \mathbf{0}, \quad \left(A_n + B_n + C_n^T\right)\mathbf{1} = \mathbf{0}, \quad \forall n \in \mathbb{N}, \tag{2.6}$$

where $\mathbf{1} = (1, 1, \ldots, 1)^T \in \mathbb{R}^d$ and $\mathbf{0} = (0, 0, \ldots, 0)^T \in \mathbb{R}^d$ (see [18]). In this case there exists a transition function

$$P(t) = (P_{ii'}(t))_{i,i'=0,1,\ldots}, \tag{2.7}$$

with $d \times d$ block matrices $P_{ii'}(t) \in \mathbb{R}^{d\times d}$,

$$P(0) = I, \qquad P'(0) = Q, \tag{2.8}$$

which satisfies the Kolmogorov forward differential equation

$$P'(t) = P(t)Q, \quad \forall t \geq 0 \tag{2.9}$$

and the Kolmogorov backward differential equation

$$P'(t) = QP(t), \quad \forall t \geq 0. \tag{2.10}$$

The probability $P(X_t = (i', j') \mid X_0 = (i, j))$ of going from state (i, j) to (i', j') in time t is given by the element in the position (j, j') of the matrix $P_{ii'}(t)$.

Note that there always exists a transition function $P(t)$ such that the Kolmogorov forward differential equation (2.9) is satisfied. The infinitesimal generator Q is called regular if there exists only one such transition function (see [18]). If additionally a spectral measure Σ corresponding to the generator matrix (1.2) exists, we can derive an integral representation for the block of the transition function $P(t)$ in the position (i, j) in terms of the spectral measure and the corresponding matrix orthogonal polynomials, which generalizes the famous Karlin and McGregor representation.

Theorem 2.2. *Assume that the conditions for the existence of the measure Σ in Theorem 2.1 are satisfied and that there exists a transition function $P(t)$ which satisfies the Kolmogorov forward equation (2.9) for all $t \geq 0$. Then the following representation holds for the block $P_{ij}(t) \in \mathbb{R}^{d\times d}$ in the position (i, j) of the transition function $P(t)$:*

$$P_{ij}(t) = \left(\int e^{-tx} Q_i(x) d\Sigma(x) Q_j^T(x)\right)\left(\int Q_j(x) d\Sigma(x) Q_j^T(x)\right)^{-1}. \tag{2.11}$$

Proof. Let $Q(x) = (Q_0^T(x), Q_1^T(x), \ldots)^T$ denote the vector of orthogonal matrix polynomials $Q_i(x)$ with respect to the spectral measure Σ. Then the recursive relation (1.3) is equivalent to the matrix equation

$$-xQ(x) = QQ(x). \tag{2.12}$$

Defining

$$F(x,t) := P(t)Q(x), \tag{2.13}$$

we obtain the differential equation

$$\frac{d}{dt}F(x,t) = P'(t)Q(x) = P(t)QQ(x) = -xP(t)Q(x) = -x\mathrm{F}(x,t), \tag{2.14}$$

and the condition $P(0) = I$ yields

$$F(x,0) = P(0)Q(x) = Q(x). \tag{2.15}$$

Hence, it follows that

$$F(x,t) = e^{-tx}Q(x) = P(t)Q(x), \tag{2.16}$$

which implies (integrating with respect to $d\Sigma(x)$) that

$$\int e^{-tx}Q(x)d\Sigma(x)Q_j^T(x) = P(t)\int Q(x)d\Sigma(x)Q_j^T(x). \tag{2.17}$$

Because of the orthogonality of the matrix polynomials $Q_n(x)$, $n \geq 0$, we obtain for the blocks $P_{ij}(t)$ of the transition function the representation

$$P_{ij}(t) = \left(\int e^{-tx}Q_i(x)d\Sigma(x)Q_j^T(x)\right)\left(\int Q_j(x)d\Sigma(x)Q_j^T(x)\right)^{-1}, \quad \forall i,j, \tag{2.18}$$

which completes the proof of Theorem 2.2. □

In what follows we present two results, which relate the Stieltjes transforms of the spectral measures of two quasi-birth-and-death processes, which have an infinitesimal generator of similar structure. The first result refers to the case where the entry B_0 has been replaced by the matrix $\overline{B}_0$. The proof is similar to a corresponding result in [13] and is therefore omitted.

Theorem 2.3. *Consider the infinitesimal generator defined by* (1.2) *and the matrix*

$$\overline{Q} = \begin{pmatrix} \overline{B}_0 & A_0 & & & 0 \\ C_1^T & B_1 & A_1 & & \\ & C_2^T & B_2 & A_2 & \\ & & C_3^T & B_3 & A_3 \\ 0 & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.19}$$

Let Σ be a spectral measure corresponding to the infinitesimal generator Q with positive definite block Hankel matrices such that the matrix $R_0\overline{B}_0R_0^{-1}$ is symmetric and such that $\{R_n\}_{n\geq 0}$ is a sequence of matrix polynomials which satisfies condition (2.4). *Then there exists a spectral measure $\overline{\Sigma}$ corresponding to $\overline{Q}$. If the spectral measures Σ and $\overline{\Sigma}$ are determined by their moments, then the Stieltjes transforms of the measures satisfy*

$$\Phi(z) = \int \frac{d\Sigma(t)}{z-t} = \left\{ \left(\int \frac{d\overline{\Sigma}(t)}{z-t} \right)^{-1} - S_0^{-1}\left(\overline{B}_0 - B_0\right) \right\}^{-1}. \tag{2.20}$$

Given a sequence $\{Q_n(x)\}_{n\geq 0}$ of matrix polynomials defined by recursion (1.3), the corresponding associated sequence of matrix polynomials $\{Q_n^{(k)}(x)\}_{n\geq 0}$ of order $k, k \geq 1$, is defined by a recursion of the form of (1.3), in which the matrices A_n, B_n, and C_n have been replaced by the matrices A_{n+k}, B_{n+k}, and C_{n+k}, respectively (see [19]). The following result gives a relation between the Stieltjes transform of the spectral measure corresponding to the sequence of matrix polynomials $\{Q_n(x)\}_{n\geq 0}$ and the Stieltjes transform of the spectral measure corresponding to $\{Q_n^{(k)}(x)\}_{n\geq 0}$. The associated quasi-birth-and-death process will be denoted by $(X_t^{(k)})_{t\geq 0}$ with state space E defined by (1.1) (throughout this paper we use the notation $X_t^{(0)} := X_t$).

Theorem 2.4. *Consider the infinitesimal generator Q defined by* (1.2) *and the matrix*

$$Q^{(k)} = \begin{pmatrix} B_k & A_k & & & 0 \\ C_{k+1}^T & B_{k+1} & A_{k+1} & & \\ & C_{k+2}^T & B_{k+2} & A_{k+2} & \\ & & C_{k+3}^T & B_{k+3} & A_{k+3} \\ 0 & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.21}$$

The matrix $Q^{(k)}$ is called the associated matrix of order $k, k \geq 1$, corresponding to Q. Assume that Σ is a spectral measure corresponding to Q with positive definite block Hankel matrices, that is, there exists a sequence $\{R_n\}_{n\geq 0}$ of nonsingular matrices, which satisfies condition (2.4) *of Theorem 2.1. Then there exists a spectral measure $\Sigma^{(k)}$ corresponding to $Q^{(k)}$ with positive definite block Hankel matrices. If the measures are determined by their moments, then the Stieltjes transforms of the measures are related by*

$$\begin{aligned}\int \frac{d\Sigma(x)}{z-x} &= R_0^{-1}\Bigg\{ zI_d - E_0 - D_1 \Bigg\{ zI_d - E_1 - D_2 \Bigg\{ zI_d - E_2 - \cdots \\ &\quad \cdots - D_{k-1}\left\{ zI_d - E_{k-1} - D_k R_k \int \frac{d\Sigma^{(k)}(x)}{z-x} R_k^T D_k^T \right\}^{-1} D_{k-1}^T \Bigg\}^{-1} \cdots \\ &\quad \cdots D_2^T \Bigg\}^{-1} D_1^T \Bigg\}^{-1} \left(R_0^T\right)^{-1},\end{aligned} \tag{2.22}$$

where

$$D_{n+1} = -R_n A_n R_{n+1}^{-1}, \qquad E_n = -R_n B_n R_n^{-1}, \qquad D_n^T = -R_n C_n^T R_{n-1}^{-1}, \tag{2.23}$$

and the Stieltjes transforms of the matrix measures $\Sigma^{(k)}$ *and* $\Sigma^{(k+1)}$ *are related by*

$$\int \frac{d\Sigma^{(k)}(x)}{z-x} = R_k^{-1}\left\{ zI_d + R_k B_k R_k^{-1} - R_k A_k \int \frac{d\Sigma^{(k+1)}(x)}{z-x} R_{k+1}^T R_{k+1} C_{k+1}^T R_k^{-1} \right\}^{-1} \left(R_k^T\right)^{-1}. \tag{2.24}$$

Proof. Let the sequence of polynomials $\{Q_n(x)\}_{n\geq 0}$ be defined by recursion (1.3) with corresponding spectral measure Σ. Then the polynomials $W_n(x) := R_n Q_n(x)$ are orthonormal with respect to the matrix measure Σ and satisfy the three-term recurrence relation

$$xW_n(x) = D_{n+1}W_{n+1}(x) + E_n W_n(x) + D_n^T W_{n-1}(x) \tag{2.25}$$

with initial conditions $W_{-1}(x) = 0$ and $W_0(x) = R_0$. From Theorem 1.2 and Lemma 1.3 in [20] it follows that

$$\int \frac{d\Sigma(x)}{z-x} = \lim_{n\to\infty} R_0^{-1}\left\{ zI_d - E_0 - D_1\left\{ \cdots zI_d - E_1 - D_2\left\{ zI_d - E_2 - \cdots \cdots - D_n\{zI_d - E_n\}^{-1} D_n^T\right\}^{-1} \cdots \right\}^{-1} D_2^T \right\}^{-1} D_1^T \right\}^{-1} \left(R_0^T\right)^{-1}. \tag{2.26}$$

Assume that the sequence of polynomials $\{Q_n^{(k)}(x)\}_{n\geq 0}$ is defined by recursion (1.3), where the matrices $B_n, A_n,$ and C_n have been replaced by the matrices $B_{n+k}, A_{n+k},$ and C_{n+k}, respectively, that is

$$-xQ_n^{(k)}(x) = A_{n+k}Q_{n+1}^{(k)}(x) + B_{n+k}Q_n^{(k)}(x) + C_{n+k}^T Q_{n-1}^{(k)}(x), \tag{2.27}$$

with $Q_0^{(k)}(x) = I$ and $Q_{-1}^{(k)}(x) = 0$. Define $A_n^{(k)} = A_{n+k}, B_n^{(k)} = B_{n+k}, C_n^{(k)} = C_{n+k}$, and $R_n^{(k)} = R_{n+k}$, $n \geq 0$. From Theorem 2.1 we obtain the symmetry of the matrices

$$-R_n^{(k)} B_n^{(k)} \left(R_n^{(k)}\right)^{-1} = -R_{n+k}B_{n+k}R_{n+k}^{-1}, \quad \forall n \geq 0 \tag{2.28}$$

and the equation

$$\begin{aligned}\left(R_n^{(k)}\right)^T R_n^{(k)} &= R_{n+k}^T R_{n+k}\\ &= C_{n+k}^{-1} C_{n+k-1}^{-1} \cdots C_{k+1}^{-1} C_k^{-1} \cdots C_1^{-1} R_0^T R_0 A_0 A_1 \cdots A_{k-1} A_k \cdots A_{n+k-1}\\ &= C_{n+k}^{-1} C_{n+k-1}^{-1} \cdots C_{k+1}^{-1} R_k^T R_k A_k \cdots A_{n+k-1}\\ &= \left(C_n^{(k)}\right)^{-1}\left(C_{n-1}^{(k)}\right)^{-1} \cdots \left(C_1^{(k)}\right)^{-1}\left(R_0^{(k)}\right)^T R_0^{(k)} A_0^{(k)} \cdots A_{n-1}^{(k)}, \quad \forall n \geq 1.\end{aligned} \tag{2.29}$$

Therefore, from Theorem 2.1 it follows that there exists a spectral measure $\Sigma^{(k)}$ with positive definite block Hankel matrices corresponding to the sequence of polynomials $\{Q_n^{(k)}(x)\}_{n\geq 0}$.

The polynomials $W_n^{(k)}(x) := R_n^{(k)} Q_n^{(k)}(x)$ are orthonormal with respect to the measure $\Sigma^{(k)}$ and satisfy the recursion

$$xW_n^{(k)}(x) = D_{n+1}^{(k)} W_{n+1}^{(k)}(x) + E_n^{(k)} W_n^{(k)}(x) + \left(D_n^{(k)}\right)^T W_{n-1}^{(k)}(x), \qquad W_0^{(k)}(x) = R_0^{(k)} = R_k, \tag{2.30}$$

where

$$D_{n+1}^{(k)} = D_{n+k+1}, \quad E_n^{(k)} = E_{n+k}, \quad \forall n \geq 0. \tag{2.31}$$

Therefore, it follows from Theorem 1.2 and Lemma 1.3 in [20] that

$$\begin{aligned}&\int \frac{d\Sigma^{(k)}(x)}{z-x}\\ &= \lim_{n\to\infty}\left(R_0^{(k)}\right)^{-1}\Bigg\{zI_d - E_0^{(k)} - D_1^{(k)}\Bigg\{zI_d - E_1^{(k)} - D_2^{(k)}\Bigg\{zI_d - E_2^{(k)} - \cdots\\ &\qquad \cdots - D_n^{(k)}\left\{zI_d - E_n^{(k)}\right\}^{-1}\left(D_n^{(k)}\right)^T\Bigg\}^{-1}\\ &\qquad \cdots\Bigg\}^{-1}\left(D_2^{(k)}\right)^T\Bigg\}^{-1}\left(D_1^{(k)}\right)^T\Bigg\}^{-1}\left(\left(R_0^{(k)}\right)^T\right)^{-1}\\ &= \lim_{n\to\infty} R_k^{-1}\Bigg\{zI_d - E_k - D_{k+1}\Bigg\{\ldots zI_d - E_{k+1} - D_{k+2}\Bigg\{zI_d - E_{k+2} - \cdots\\ &\qquad \cdots - D_{n+k}\{zI_d - E_{n+k}\}^{-1} D_{n+k}^T\Bigg\}^{-1}\ldots\Bigg\}^{-1} D_{k+2}^T\Bigg\}^{-1} D_{k+1}^T\Bigg\}^{-1}\left(R_k^T\right)^{-1}.\end{aligned} \tag{2.32}$$

A combination of (2.26) and (2.32) yields

$$\int \frac{d\Sigma(x)}{z-x} = R_0^{-1}\left\{ zI_d - E_0 - D_1 \left\{ zI_d - E_1 - D_2 \left\{ zI_d - E_2 - \cdots \cdots - D_{k-1}\left\{ zI_d - E_{k-1} - D_k R_k \int \frac{d\Sigma^{(k)}(x)}{z-x} R_k^T D_k^T \right\}^{-1} D_{k-1}^T \right\}^{-1} \cdots \cdots D_2^T \right\}^{-1} D_1^T \right\}^{-1} \left(R_0^T\right)^{-1}, \tag{2.33}$$

and from (2.32) and (2.23) we obtain

$$\begin{aligned} \int \frac{d\Sigma^{(k)}(x)}{z-x} &= R_k^{-1}\left\{ zI_d - E_k - D_{k+1}R_{k+1} \int \frac{d\Sigma^{(k+1)}(x)}{z-x} R_{k+1}^T D_{k+1}^T \right\}^{-1} \left(R_k^T\right)^{-1} \\ &= R_k^{-1}\left\{ zI_d + R_k B_k R_k^{-1} - R_k A_k \int \frac{d\Sigma^{(k+1)}(x)}{z-x} R_{k+1}^T R_{k+1} C_{k+1}^T R_k^{-1} \right\}^{-1} \left(R_k^T\right)^{-1}, \end{aligned} \tag{2.34}$$

which completes the proof of the theorem. □

Remark 2.5. Note that in the literature, many queueing models are considered, where the matrices C_n do not have full rank (see [21]). Following the arguments used in Remark 2.7 in [13] the conditions

$$\begin{aligned} R_n B_n &= E_n R_n, \quad n \geq 0, \\ C_{n+1} R_{n+1}^T R_{n+1} &= R_n^T R_n A_n, \quad n \geq 1 \end{aligned} \tag{2.35}$$

are sufficient for the existence of a spectral measure Σ corresponding to Q, where $\{E_n\}_{n\geq 0}$ is a sequence of symmetric matrices and

$$\int Q_i(x) d\Sigma(x) Q_j^T(x) = \delta_{ij} R_j^T R_j. \tag{2.36}$$

In other words, the assumption of nonsingularity of the matrices C_n can be relaxed. The same arguments as those used in Theorem 2.2 then imply that

$$P_{ij}(t) R_j^T R_j = \int e^{-tx} Q_i(x) d\Sigma(x) Q_j^T(x). \tag{2.37}$$

3. Examples

Example 3.1. Dayar and Quessette [3] considered a queuing system consisting of an $M/M/1$-system and an $M/M/1/d-1$-system. Both queues have Poisson arrival processes with rate λ_i, $i = 1, 2$, and exponential service distributions with rate μ_i, $i = 1, 2$, and it was assumed that $\gamma = \lambda_1 + \lambda_2 + \mu_1 + \mu_2$. The level represents the length of queue 1, which is unbounded, and the phase represents the length of queue 1, which can range from 0 to $d-1$. The process is of interest because of its level geometric stationary distribution. This system can be described by a homogeneous Markov process $X(t) = (L_1(t), L_2(t))_{t\in\mathbb{R}^+}$ with state space $E = \mathbb{N} \times \{0, \dots, d-1\}$, where $L_1(t)$ and $L_2(t)$ denote the length of the first queue at time t and the length of the second queue at time t, respectively. The entries of the corresponding infinitesimal generator (1.2) have the form

$$B_0 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_2 & & & \\ \mu_2 & -(\gamma-\mu_1) & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_2 & -(\gamma-\mu_1) & \lambda_2 \\ & & & \mu_2 & -(\lambda_1+\mu_2) \end{pmatrix},$$
$$B_i = \begin{pmatrix} -(\gamma-\mu_2) & \lambda_2 & & & \\ \mu_2 & -\gamma & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_2 & -\gamma & \lambda_2 \\ & & & \mu_2 & -(\gamma-\lambda_2) \end{pmatrix}, \quad i \geq 1, \tag{3.1}$$

$A_i = \lambda_1 I_d$, $i \geq 0$, and $C_i^T = \mu_1 I_d$, $i \geq 1$. It is easy to see that Q is conservative. A straightforward calculation shows that the conditions of Theorem 2.1 are satisfied with the matrices

$$R_0 = \operatorname{diag}\left(1, \sqrt{\frac{\lambda_2}{\mu_2}}, \left(\sqrt{\frac{\lambda_2}{\mu_2}}\right)^2, \dots, \left(\sqrt{\frac{\lambda_2}{\mu_2}}\right)^{d-1}\right),$$
$$R_i = \left(\sqrt{\frac{\lambda_1}{\mu_1}}\right)^i R_0, \quad i \in \mathbb{N}. \tag{3.2}$$

This implies the existence of a spectral measure.

Example 3.2. In general, the spectral distribution can only be identified in special cases. Even if the Stieltjes transform can be determined, its inversion is usually difficult (see, e.g., [22, Chapter 3]). We now present an example where the spectral measure can be found explicitly.

To be precise consider a homogeneous Markov process $(X_t)_{t\geq 0}$ with infinitesimal generator (1.2), where

$$B_0 = \begin{pmatrix} -\gamma & \beta_1 & & & \\ \beta_2 & -\gamma & \beta_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_2 & -\gamma & \beta_1 \\ & & & \beta_2 & -\gamma \end{pmatrix}, \quad \gamma \neq 0, \qquad B_i = \begin{pmatrix} -\delta & \beta_1 & & & \\ \beta_2 & -\delta & \beta_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_2 & -\delta & \beta_1 \\ & & & \beta_2 & -\delta \end{pmatrix}, \quad i \geq 1, \ \delta \neq 0, \tag{3.3}$$

$A_i = \alpha_1 I_d$, $i \geq 0$, and $C_i^T = \alpha_2 I_d$, $i \geq 1$. A generator matrix of this form can be associated to a queueing model which consists of d different $M/M/1$-systems. Each $M/M/1$-system has a Poisson arrival process with rate α_1 and an exponential service time distribution with rate α_2. If the customer is situated in system i, then it changes to the system $i-1$ and $i+1$ with the rate β_2 and β_1, respectively. This model can be described by the two-dimensional homogeneous Markov process $(N_t, S_t)_{t\geq 0}$ with state space $E = \mathbb{N}_0 \times \{0, \ldots, d-1\}$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, N_t denotes the number of customers in the whole model at time t, and S_t denotes the number of the system at time t.

If $\beta_1 \neq 0$ and $\beta_2 \neq 0$ the conditions of Theorem 2.1 are satisfied with

$$\begin{aligned} R_0 &= \operatorname{diag}\left(\left(\frac{\beta_2}{\beta_1}\right)^{(d-1)/2}, \left(\frac{\beta_2}{\beta_1}\right)^{(d-2)/2}, \ldots, \left(\frac{\beta_2}{\beta_1}\right)^{1/2}, 1 \right), \\ R_n &= \left(\sqrt{\frac{\alpha_1}{\alpha_2}}\right)^n R_0, \quad n \geq 1. \end{aligned} \tag{3.4}$$

This implies the existence of a spectral measure Σ corresponding to Q. In order to determine the measure explicitly, note that the matrices in (2.23) have the form

$$\begin{aligned} D &:= D_n = -\sqrt{\alpha_1\alpha_2} I_d, \quad n \geq 1, \\ E_0 &= \begin{pmatrix} \gamma & -\sqrt{\beta_1\beta_2} & & & \\ -\sqrt{\beta_1\beta_2} & \gamma & -\sqrt{\beta_1\beta_2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\sqrt{\beta_1\beta_2} & \gamma & -\sqrt{\beta_1\beta_2} \\ & & & -\sqrt{\beta_1\beta_2} & \gamma \end{pmatrix}, \\ E &:= E_n = \begin{pmatrix} \delta & -\sqrt{\beta_1\beta_2} & & & \\ -\sqrt{\beta_1\beta_2} & \delta & -\sqrt{\beta_1\beta_2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\sqrt{\beta_1\beta_2} & \delta & -\sqrt{\beta_1\beta_2} \\ & & & -\sqrt{\beta_1\beta_2} & \delta \end{pmatrix}, \quad n \geq 1. \end{aligned} \tag{3.5}$$

The eigenvalues of the matrix E are given by

$$\lambda_k = \delta + 2\sqrt{\beta_1\beta_2}\cos\left(\frac{k\pi}{d+1}\right), \quad k = 1,\ldots,d, \tag{3.6}$$

with corresponding eigenvectors given by $u^{(k)} = (u_1^{(k)},\ldots,u_d^{(k)})^T$, where

$$u_j^{(k)} = \sqrt{\frac{2}{d+1}}\sin\left(\frac{kj\pi}{d+1}\right), \quad j,k = 1,\ldots,d. \tag{3.7}$$

With the notations $H := \operatorname{diag}(\lambda_1 - z,\ldots,\lambda_d - z)$ and $U := (u^{(1)},\ldots,u^{(d)})$, it follows that

$$E - zI_d = UHU^T, \qquad U^TU = I_d. \tag{3.8}$$

Let $\overline{Q}$ be the infinitesimal generator obtained from Q by replacing the first diagonal block B_0 by block B_1 (which coincides with all other blocks B_i, $i \geq 2$), and denote by $\overline{\Sigma}$ the spectral measure corresponding to $\overline{Q}$. From [23] we obtain for the Stieltjes transform $\overline{\Phi}(z)$ of the matrix measure $\overline{\Sigma}$

$$\begin{aligned}\overline{\Phi}(z) &= -\frac{1}{2}D^{-2}(E - zI_d)^{1/2}\left\{I_d + \left\{I_d - 4D^2(E - zI_d)^{-2}\right\}^{1/2}\right\}(E - zI_d)^{1/2}\\ &= -\frac{1}{2\alpha_1\alpha_2}UH^{1/2}\left\{I_d + \left\{I_d - 4\alpha_1\alpha_2H^{-2}\right\}^{1/2}\right\}H^{1/2}U^T,\end{aligned} \tag{3.9}$$

and Theorem 2.3 gives the Stieltjes transform $\Phi(z)$ of the measure Σ. Moreover, the results in [23, page 318] also show that the support of the spectral measure is given by

$$\begin{aligned}\operatorname{supp}(\Sigma) &= \left\{x \in \mathbb{R} : D^{-1/2}(xI_d - E)D^{-1/2} \text{ has an eigenvalue in } [-2,2]\right\}\\ &= \left[-2\sqrt{\alpha_1\alpha_2} + \delta + 2\sqrt{\beta_1\beta_2}\cos\left(\frac{\pi d}{d+1}\right), 2\sqrt{\alpha_1\alpha_2} + \delta + 2\sqrt{\beta_1\beta_2}\cos\left(\frac{\pi}{d+1}\right)\right].\end{aligned} \tag{3.10}$$

Note that $\operatorname{supp}(\Sigma) \subset [0,\infty)$ if $\delta \geq \alpha_1 + \alpha_2 + \beta_1 + \beta_2$.

4. α-Recurrence

The decay parameter of continuous-time quasi-birth-and-death processes was introduced by van Doorn [19]. To be precise assume that $(X_t)_{t\geq 0}$ is an irreducible quasi-birth-and-death process with state space (1.1) and infinitesimal generator Q defined by (1.2), where

$$B_0\mathbf{1} + A_0\mathbf{1} < \mathbf{0}. \tag{4.1}$$

Then the decay parameter α of the process $(X_t)_{t\ge 0}$ is defined by

$$\alpha = \sup\left\{ s \ge 0 : e_j^T \int_0^\infty e^{st} P_{ii'}(t) dt\, e_{j'} < \infty \right\}, \quad (i,j),(i',j') \in E. \tag{4.2}$$

A state $(i,\ell) \in E$ is called α-recurrent

$$e_\ell^T \int_0^\infty e^{\alpha t} P_{ii}(t) dt\, e_\ell = \infty, \tag{4.3}$$

where $e_\ell = (0,\ldots,0,1,0,\ldots,0)^T \in \mathbb{R}^d$ denotes the ℓth unit vector. The process $(X_t)_{t\ge 0}$ is called α-recurrent if and only if some state (and then all states in E) is α-recurrent. The process $(X_t)_{t\ge 0}$ is called α-positive if and only if for some state $(i,\ell) \in E$ (and then for all states in E)

$$e_\ell^T \lim_{t\to\infty} e^{\alpha t} P_{ii}(t) e_\ell > 0. \tag{4.4}$$

The following results characterize α-recurrence of the process $(X_t)_{t\ge 0}$ in terms of the spectral measure Σ, the corresponding orthogonal polynomials $Q_j(x)$, and the blocks of the infinitesimal generator. Throughout this section it will be assumed that condition (2.4) of Theorem 2.1 is satisfied.

Theorem 4.1. *Assume that the conditions of Theorem 2.1 are satisfied with a spectral measure supported in the interval* $[\alpha,\infty)$ *and that there exists a transition function, which satisfies the Kolmogorov forward differential equation* (2.9)*. The process* $(X_t)_{t\ge 0}$ *is* α*-recurrent if and only if for some state* $(i,\ell) \in E$ *(and then for all states in E)*

$$e_\ell^T \left(\int \frac{Q_i(x) d\Sigma(x) Q_i^T(x)}{x-\alpha} \right) \left(\int Q_i(x) d\Sigma(x) Q_i^T(x) \right)^{-1} e_\ell = \infty. \tag{4.5}$$

Proof. With representation (2.11) and Fubini's Theorem, condition (4.3) is equivalent to

$$e_\ell^T \left(\int \int_0^\infty e^{(\alpha-x)t} dt\, Q_i(x) d\Sigma(x) Q_i^T(x) \right) \left(\int Q_i(x) d\Sigma(x) Q_i^T(x) \right)^{-1} e_\ell = \infty, \tag{4.6}$$

which implies (4.5). □

In the following we define for a matrix measure Σ with existing moments the $d \times d$ matrices $\zeta_0 = 0$ and $\zeta_k = (S_{k-1} - S_{k-1}^-)^{-1}(S_k - S_k^-) \in \mathbb{R}^{d\times d}$, where $S_{2n} - S_{2n}^-$ and $S_{2n-1} - S_{2n-1}^-$ denote the Schur complement of S_{2n} and S_{2n-1} in the matrix $\underline{H}_{2n}$ and

$$\underline{H}_{2n-1} = \begin{pmatrix} S_1 & \cdots & S_n \\ \vdots & & \vdots \\ S_n & \ldots & S_{2n-1} \end{pmatrix}, \tag{4.7}$$

respectively (see [24]). The next result gives a representation of the Stieltjes transform of the spectral measure Σ in terms of the quantities ζ_j and the blocks of the generator matrix (1.2).

Note that $\operatorname{supp}(\Sigma) \subset [\alpha, \infty)$ is crucial for our approach and in general difficult to check. Consider for example the case of recurrence (i.e., $\alpha = 0$), then it follows from the results of Duran and Lopez-Rodriguez [25] that the spectral measure Σ can be found as weak accumulation points of a sequence of discrete measures with support precisely on

$$\Delta_n = \{x \mid \det Q_n(x) = 0\}. \tag{4.8}$$

A straightforward calculation shows that the set Δ_n coincides with the eigenvalues of the matrix

$$-\begin{pmatrix} B_0 & A_0 & & & \\ C_1^T & B_1 & A_1 & & \\ & \ddots & \ddots & \ddots & \\ & & & C_{n-1}^T & B_1 \end{pmatrix} \tag{4.9}$$

and consequently all bounds on eigenvalues of these matrices will yield bounds on the support of spectral measure.

Corollary 4.2. *Assume that conditions* (2.4) *of Theorem 2.1 are satisfied. Let* $\{Q_n(x)\}_{n\geq 0}$ *denote the corresponding orthogonal matrix polynomials defined by recursion* (1.3). *Assume that the corresponding spectral measure* Σ *is supported in the interval* $[0, \infty)$ *and that it is determined by its moments. Then the Stieltjes transform of the measure* Σ *can be represented as*

$$\int \frac{d\Sigma(x)}{z-x} = \lim_{n\to\infty} \left\{ zI_d - \left\{ I_d - \left\{ zI_d - \cdots - \left\{ zI_d - \zeta_{2n+1}^T \right\}^{-1} \zeta_{2n}^T \right\}^{-1} \cdots \right\}^{-1} \zeta_2^T \right\}^{-1} \zeta_1^T \right\}^{-1} S_0. \tag{4.10}$$

In particular, the following representations hold:

$$\int \frac{d\Sigma(x)}{x} = \lim_{n\to\infty} \sum_{j=0}^{n+1} \left(\zeta_{2j+1}^T \zeta_{2j-1}^T \cdots \zeta_1^T \right)^{-1} \left(\zeta_{2j}^T \zeta_{2j-2}^T \cdots \zeta_2^T \right) S_0 \tag{4.11}$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n+1} T_{j+1}^{-1} A_j^{-1} C_j^T T_{j-1} T_j^{-1} A_{j-1}^{-1} C_{j-1}^T T_{j-2} T_{j-1}^{-1} \cdots T_0 T_1^{-1} A_0^{-1} T_0 S_0, \tag{4.12}$$

where $T_j = Q_j(0)$, $j \geq 0$.

Proof. From Lemma 3.3 in [24] it follows that the monic orthogonal matrix polynomials $\{\underline{P}_n(x)\}_{n\geq 0}$ with respect to a matrix measure Σ supported in $[0,\infty)$ satisfy the recursive relation

$$x\underline{P}_{\mathrm{n}}(x) = \underline{P}_{n+1}(x) + \left(\zeta_{2n+1}^T + \zeta_{2n}^T\right)\underline{P}_n(x) + \zeta_{2n}^T\zeta_{2n-1}^T\underline{P}_{n-1}(x), \tag{4.13}$$

with $\underline{P}_{-1}(x) = 0, \underline{P}_0(x) = I_d, \zeta_0 = 0$, and $\zeta_k = (S_{k-1} - S_{k-1}^-)^{-1}(S_k - S_k^-)$, where the matrices

$$\Delta_{2n} := \langle \underline{P}_n, \underline{P}_n \rangle = (S_0\zeta_1 \cdots \zeta_{2n})^T \tag{4.14}$$

are positive definite. Then the polynomials

$$P_n(x) := \Delta_{2n}^{-1/2}\underline{P}_n(x), \quad n \geq 0, \tag{4.15}$$

are orthonormal with respect to the matrix measure Σ and satisfy the recursion

$$xP_n(x) = A_{n+1}P_{n+1}(x) + B_nP_n(x) + A_n^TP_{n-1}(x) \tag{4.16}$$

with $P_{-1}(x) = 0, P_0(x) = S_0^{-1/2}$, and

$$\begin{aligned} A_{n+1} &= \Delta_{2n}^{-1/2}\Delta_{2n+2}^{1/2}, \\ B_n &= \Delta_{2n}^{-1/2}\left(\zeta_{2n}^T + \zeta_{2n+1}^T\right)\Delta_{2n}^{1/2}, \\ A_n^T &= \Delta_{2n}^{-1/2}\zeta_{2n}^T\zeta_{2n-1}^T\Delta_{2n-2}^{1/2}. \end{aligned} \tag{4.17}$$

From Theorem 1.2 in [20] it follows that

$$\begin{aligned} F_n(z) &= (P_{n+1}(z))^{-1}\widetilde{P}_{n+1}^{(1)}(z) \\ &= S_0^{1/2}\left\{zI_d - B_0 - A_1\left\{zI_d - B_1 - A_2\left\{zI_d - B_2 - \cdots\right.\right.\right. \\ &\qquad \left.\left.\left.\cdots - A_n\{zI_d - B_n\}^{-1}A_n^T\right\}^{-1}\right\}^{-1}\cdots A_1^T\right\}^{-1} S_0^{1/2}, \end{aligned} \tag{4.18}$$

where $\widetilde{P}_n^{(1)}(z)$ denote the first associated polynomials for $P_n(z)$ defined by recursion (4.13) with initial conditions $\widetilde{P}_0^{(1)}(z) = \mathbf{0}$, $\widetilde{P}_1^{(1)}(z) = \zeta_1^T$. An application of Markov's Theorem (see [26]), (4.17), and (4.18) now yields

$$\begin{aligned}
&\int \frac{d\Sigma(x)}{z-x} \\
&= \lim_{n\to\infty} F_n(z) \\
&= \lim_{n\to\infty} \Bigg\{ zI_d - \zeta_1^T - \Bigg\{ zI_d - \zeta_2^T - \zeta_3^T - \Bigg\{ zI_d - \zeta_4^T - \zeta_5^T \cdots \\
&\qquad \cdots - \left\{ zI_d - \zeta_{2n}^T - \zeta_{2n+1}^T \right\}^{-1} \zeta_{2n}^T \zeta_{2n-1}^T \Bigg\}^{-1} \cdots \zeta_4^T \zeta_3^T \Bigg\}^{-1} \zeta_2^T \zeta_1^T \Bigg\}^{-1} S_0 \\
&= \lim_{n\to\infty} \Bigg\{ zI_d - \Bigg\{ I_d - \Bigg\{ zI_d - \cdots - \left\{ zI_d - \zeta_{2n+1}^T \right\}^{-1} \zeta_{2n}^T \Bigg\}^{-1} \cdots \Bigg\}^{-1} \zeta_2^T \Bigg\}^{-1} \zeta_1^T \Bigg\}^{-1} S_0.
\end{aligned} \tag{4.19}$$

If $z = 0$, then we obtain from (4.19) and (1.3) in [27]

$$\int \frac{d\Sigma(x)}{-x} = -\lim_{n\to\infty} \sum_{j=0}^{n+1} X_{j+1}^{-1} \zeta_{2j}^T \zeta_{2j-1}^T X_{j-1} X_j^{-1} \zeta_{2j-2}^T \zeta_{2j-3}^T X_{j-2} X_{j-1}^{-1} \cdots X_1 X_2^{-1} \zeta_2^T S_0, \tag{4.20}$$

where $X_0 = I_d$, $X_1 = -\zeta_1^T$, and

$$X_{n+1} = -\left(\zeta_{2n+1}^T + \zeta_{2n}^T\right) X_n - \zeta_{2n}^T \zeta_{2n-1}^T X_{n-1}, \quad n \geq 1. \tag{4.21}$$

An induction argument yields $X_n = (-1)^n \zeta_{2n-1}^T \zeta_{2n-3}^T \cdots \zeta_1^T, n \geq 1$, and the first representation in (4.11) follows. For the second part we note that the polynomials $\underline{Q}_n(x) := (-1)^n A_0 \cdots A_{n-1} Q_n(x), n \geq 0$, have leading coefficient I_d and because of (1.3) they satisfy the recursion

$$\underline{Q}_{n+1}(x) = x\underline{Q}_n(x) + A_0 \cdots A_{n-1} B_n A_{n-1}^{-1} \cdots A_0^{-1} \underline{Q}_n(x) - A_0 \cdots A_{n-1} C_n^T A_{n-2}^{-1} \cdots A_0^{-1} \underline{Q}_{n-1}(x). \tag{4.22}$$

A comparison with the polynomials $\underline{P}_n(x)$ in (4.13) now yields

$$\begin{aligned}
A_0 \cdots A_{n-1} B_n A_{n-1}^{-1} \cdots A_0^{-1} &= -\left(\zeta_{2n}^T + \zeta_{2n+1}^T\right), \\
A_0 \cdots A_{n-1} C_n^T A_{n-2} \cdots A_0 &= \zeta_{2n}^T \zeta_{2n-1}^T.
\end{aligned} \tag{4.23}$$

Define $T_n := Q_n(0), n \geq 0$. Then (4.23) imply

$$T_n = A_{n-1}^{-1} \cdots A_0^{-1} \zeta_{2n-1}^T \zeta_{2n-3}^T \cdots \zeta_1^T, \quad \forall n \geq 0. \tag{4.24}$$

Therefore, we can define the polynomials $\widehat{Q}_n(x) := T_n^{-1} Q_n(x)$. From (1.3) it follows that these polynomials satisfy the recurrence relation

$$x\widehat{Q}_n(x) = \widehat{A}_n \widehat{Q}_{n+1}(x) + \widehat{B}_n \widehat{Q}_n(x) + \widehat{C}_n^T \widehat{Q}_{n-1}(x) \tag{4.25}$$

with

$$\widehat{A}_n = T_n^{-1} A_n T_{n+1}, \qquad \widehat{B}_n = T_n^{-1} B_n T_n, \qquad \widehat{C}_n^T = T_n^{-1} C_n^T T_{n-1}, \tag{4.26}$$

and $\widehat{A}_n + \widehat{B}_n + \widehat{C}_n^T = 0$. Consequently we obtain from (4.23) that

$$\begin{aligned} \widehat{A}_0 \cdots \widehat{A}_{n-1} \widehat{B}_n \widehat{A}_{n-1}^{-1} \cdots \widehat{A}_0^{-1} &= -\left(\zeta_{2n}^T + \zeta_{2n+1}^T\right), \\ \widehat{A}_0 \cdots \widehat{A}_{n-1} \widehat{C}_n^T \widehat{A}_{n-2}^{-1} \cdots \widehat{A}_0^{-1} &= \zeta_{2n}^T \zeta_{2n-1}^T, \end{aligned} \tag{4.27}$$

and hence

$$\begin{aligned} \zeta_{2n+1}^T &= \widehat{A}_0 \cdots \widehat{A}_n \widehat{A}_{n-1}^{-1} \cdots \widehat{A}_0^{-1}, \\ \zeta_{2n}^T &= \widehat{A}_0 \cdots \widehat{A}_{n-1} \widehat{C}_n^T \widehat{A}_{n-1}^{-1} \cdots \widehat{A}_0^{-1}. \end{aligned} \tag{4.28}$$

Equation (4.11) finally yields

$$\begin{aligned} \int \frac{d\Sigma(x)}{x} &= \lim_{n\to\infty} \sum_{j=0}^{n+1} \widehat{A}_j^{-1} \widehat{C}_j^T \widehat{A}_{j-1}^{-1} \cdots \widehat{A}_1^{-1} \widehat{C}_1^T \widehat{A}_0^{-1} S_0 \\ &= \lim_{n\to\infty} \sum_{j=0}^{n+1} T_{j+1}^{-1} A_j^{-1} C_j^T T_{j-1} T_j^{-1} A_{j-1}^{-1} C_{j-1}^T T_{j-2} T_{j-1}^{-1} \cdots T_0 T_1^{-1} A_0^{-1} T_0 S_0, \end{aligned} \tag{4.29}$$

which completes the proof of the theorem. □

In the following, the α-recurrence condition will be represented in terms of properties of the spectral measure, the corresponding orthogonal matrix polynomials, and the blocks of the infinitesimal generator (1.2). For this purpose, consider the process $(X_{t,\alpha})_{t\geq 0}$ with state space E defined in (1.1) and infinitesimal generator matrix

$$Q_\alpha = \begin{pmatrix} B_{0,\alpha} & A_{0,\alpha} & & & 0 \\ C_{1,\alpha}^T & B_{1,\alpha} & A_{1,\alpha} & & \\ & C_{2,\alpha}^T & B_{2,\alpha} & A_{2,\alpha} & \\ & & C_{3,\alpha}^T & B_{3,\alpha} & A_{3,\alpha} \\ 0 & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4.30}$$

where

$$\begin{aligned} A_{n,\alpha} &:= Q_n^{-1}(\alpha) A_n Q_{n+1}(\alpha), \quad n \geq 0, \\ B_{n,\alpha} &:= Q_n^{-1}(\alpha) B_n Q_n(\alpha), \quad n \geq 0, \\ C_{n,\alpha}^T &:= Q_n^{-1}(\alpha) C_n^T Q_{n-1}(\alpha), \quad n \geq 1. \end{aligned} \tag{4.31}$$

The corresponding sequence $\{Q_{n,\alpha}(x)\}_{n\geq 0}$ of matrix polynomials satisfies the recurrence relation

$$-xQ_{n,\alpha}(x) = A_{n+1,\alpha}Q_{n+1,\alpha}(x) + B_{n,\alpha}Q_{n,\alpha}(x) + C_{n,\alpha}^T Q_{n-1,\alpha}(x) \tag{4.32}$$

with initial conditions $Q_{-1,\alpha}(x) = 0$, $Q_{0,\alpha}(x) = I_d$. If conditions (2.4) of Theorem 2.1 are satisfied, then the matrix Q_α can be symmetrized with the matrices

$$R_{n,\alpha} = R_n Q_n(\alpha), \quad n \geq 0. \tag{4.33}$$

An induction argument shows the representation

$$Q_{n,\alpha}(x) = Q_n^{-1}(\alpha) Q_n(x+\alpha), \quad n \geq 0, \tag{4.34}$$

and therefore

$$\int Q_{n,\alpha}(x) d\Sigma_\alpha(x) Q_{m,\alpha}^T(x) = 0, \quad n \neq m, \tag{4.35}$$

where the matrix measure Σ_α is defined by

$$\Sigma_\alpha(0, x] = \Sigma(\alpha, \alpha + x]. \tag{4.36}$$

If representation (2.11) holds, it is easy to see that

$$e^{\alpha t} P_{00}(t) = \int e^{-tx} d\Sigma_{\alpha}(x) S_0^{-1}, \tag{4.37}$$

and the following remark is a consequence of Theorem 4.1.

Remark 4.3. Assume that the conditions of Theorem 4.1 are satisfied and that Σ is a corresponding spectral measure supported in the interval $[\alpha, \infty)$. The process $(X_t)_{t\geq 0}$ is α-recurrent if and only if

$$e_j^T \int_0^{\infty} \frac{d\Sigma_{\alpha}(x)}{x} S_0^{-1} e_j = e_j^T \int_{\alpha}^{\infty} \frac{d\Sigma(x)}{x-\alpha} S_0^{-1} e_j = \infty \tag{4.38}$$

for some $j \in \{1, \ldots, d\}$. The process is α-positive if

$$e_\ell^T \lim_{t\to\infty} e^{\alpha t} P_{00}(t) e_\ell > 0 \tag{4.39}$$

for some $\ell \in \{1, \ldots, d\}$. This is the case if and only if the measure $e_\ell^T d\Sigma(x) S_0^{-1} e_\ell$ has a jump in the point $x = \alpha$.

Theorem 4.4. *Assume that the conditions of Theorem 2.1 are satisfied and that the corresponding matrix measure Σ is supported in the interval $[\alpha, \infty)$ and determined by its moments. The process $(X_t)_{t\geq 0}$ is α-recurrent if and only if for some state $(0, \ell) \in E$ (and then for all states in $(0, k) \in E$)*

$$e_\ell^T \sum_{j=0}^{\infty} H_{j+1}^{-1} A_j^{-1} C_j^T H_{j-1} H_j^{-1} A_{j-1}^{-1} C_{j-1}^T H_{j-2} \cdots C_1^T H_1^{-1} A_0^{-1} H_0 S_0 e_\ell = \infty, \tag{4.40}$$

where $H_j := Q_j(\alpha)$, $j \geq 0$.

Proof. Because condition (2.4) holds for the polynomials $\{Q_n(x)\}_{n\geq 0}$, this condition is also fulfilled for the polynomials $\{Q_{n,\alpha}\}_{n\geq 0}$ with $R_{n,\alpha} := R_n Q_n(\alpha), n \geq 0$. From (4.34) it follows that $Q_{j,\alpha}(0) = I_d$ for all $j \geq 0$. Therefore we obtain with (4.12)

$$\int \frac{d\Sigma_{\alpha}(x)}{x} = \sum_{j=0}^{\infty} A_{j,\alpha}^{-1} C_{j,\alpha}^T A_{j-1,\alpha}^{-1} C_{j-1,\alpha}^T \cdots C_{1,\alpha}^T A_{0,\alpha}^{-1} S_0. \tag{4.41}$$

From the representation $A_{j,\alpha}^{-1} C_{j,\alpha}^T = Q_{j+1}(\alpha) A_j^{-1} C_j^T Q_{j-1}(\alpha)$, $j \geq 0$, it follows from Remark 4.3 that the state $(0, \ell)$ is α-recurrent if and only if

$$e_\ell^T \sum_{j=0}^{\infty} H_{j+1}^{-1} A_j^{-1} C_j^T H_{j-1} H_j^{-1} A_{j-1}^{-1} C_{j-1}^T H_{j-2} \cdots C_1^T H_1^{-1} A_0^{-1} H_0 S_0 e_\ell = \infty, \tag{4.42}$$

where $H_j = Q_j(\alpha), j \geq 0$. □

Remark 4.5. In the case $d = 1$, the results of Theorems 4.1 and 4.4 reduce to known results in the scalar case (see Theorem 5.2(ii), (iii), (vii) in [10]).

Remark 4.6. Assume that the conditions of Theorem 4.4 are satisfied, and let $\Sigma^{(1)}$ be a spectral measure corresponding to the sequence of associated matrix polynomials $\{Q_n^{(1)}(x)\}_{n\geq 0}$.

(1) The state $(0,\ell) \in E$ is α-recurrent if and only if

$$e_\ell^T \int \frac{d\Sigma(x)}{x-\alpha} S_0^{-1} e_\ell = e_\ell^T \left\{ -\alpha I_d - B_0 - A_0 \int \frac{d\Sigma^{(1)}(x)}{x-\alpha} R_1^T R_1 C_1^T \right\}^{-1} e_\ell = \infty. \tag{4.43}$$

(2) The state $(0,\ell) \in E$ is α-positive if and only if

$$\begin{aligned} e_\ell^T \lim_{t\to\infty} e^{\alpha t} P_{00}(t) e_\ell &= \lim_{z\to 0} z e_\ell^T \int \frac{d\Sigma(x)}{(z+\alpha)-x} S_0^{-1} e_\ell \\ &= e_\ell^T \lim_{z\to 0} \left\{ \frac{z+\alpha}{z} I_d + \frac{1}{z}\left(B_0 - A_0 \int \frac{d\Sigma^{(1)}(x)}{(z+\alpha)-x} R_1^T R_1 C_1^T \right) \right\}^{-1} > 0. \end{aligned} \tag{4.44}$$

Note that conditions (4.3) and (4.4) reduce to recurrence and positive recurrence if $\alpha = 0$. Therefore, with Theorem 4.2 we obtain the following conditions for recurrence and positive recurrence of a quasi-birth-and-death process.

Corollary 4.7. *Assume that the conditions of Theorem 2.1 are satisfied and that the corresponding matrix measure Σ is supported in the interval $[0,\infty)$ and determined by its moments. The following statements hold.*

(1) *The state $(i,\ell) \in E$ is recurrent if and only if*

$$e_\ell^T \left(\int \frac{Q_i(x) d\Sigma(x) Q_i^T(x)}{x} \right) \left(\int Q_i(x) d\Sigma(x) Q_i^T(x) \right)^{-1} e_\ell = \infty, \tag{4.45}$$

where $e_\ell = (0,\dots,0,1,0,\dots,0)^T$. In particular, the state $(0,\ell) \in E$ is recurrent if and only if

$$e_\ell^T \int_0^\infty \frac{d\Sigma(x)}{x} S_0^{-1} e_\ell = \infty. \tag{4.46}$$

(2) *The state $(0,\ell)$ is recurrent if and only if*

$$e_\ell^T \sum_{j=0}^\infty T_{j+1}^{-1} A_j^{-1} C_j^T T_{j-1} T_j^{-1} A_{j-1}^{-1} C_{j-1}^T T_{j-2} T_{j-1}^{-1} \cdots T_0 T_1^{-1} A_0^{-1} T_0 S_0 e_\ell = \infty \tag{4.47}$$

with $T_j = Q_j(0)$, $j \geq 0$.

(3) *The state* $(0,\ell)$ *is positive recurrent if and only if the matrix measure* $e_\ell^T d\Sigma(x) S_0^{-1} e_\ell$ *has a jump in the point* $x = 0$.

Remark 4.8. (1) Let $\Sigma^{(1)}$ be a spectral measure supported in $[0,\infty)$ corresponding to the associated polynomials $\{Q_n^{(1)}(x)\}_{n\geq 0}$ introduced in Theorem 2.4. Then, a combination of Theorem 2.4 and Corollary 4.7 shows that the state $(0,\ell) \in E$ is recurrent if and only if

$$\begin{aligned} e_\ell^T \int \frac{d\Sigma(x)}{x} S_0^{-1} e_\ell &= -\lim_{z\to 0} e_\ell^T \int \frac{d\Sigma(x)}{z-x} R_0^T R_0 e_\ell \\ &= e_\ell^T \left\{ -B_0 - A_0 \int \frac{d\Sigma^{(1)}(x)}{x} R_1^T R_1 C_1^T \right\}^{-1} e_\ell = \infty. \end{aligned} \tag{4.48}$$

An induction argument shows that

$$Q_n^{(1)}(x) = -\widetilde{Q}_{n+1}^{(1)}(x) S_0^{-1} A_0, \quad n \geq 0, \tag{4.49}$$

where $\widetilde{Q}_n^{(1)}(x)$ are the first associated polynomials corresponding to $Q_n^{(1)}(x)$, and $Q_n^{(1)}(x)$ are the associated polynomials of order $k = 1$ corresponding to $Q_n(x)$. Therefore it follows for the Stieltjes transform of the spectral measure corresponding to the associated orthogonal polynomials that

$$\begin{aligned} \int \frac{d\Sigma^{(1)}(x)}{x} = \lim_{n\to\infty} \sum_{j=0}^{n+1} A_0^{-1} S_0 Z_{j+1}^{-1} A_{j+1}^{-1} C_{j+1}^T Z_{j-1} Z_j^{-1} A_j^{-1} \cdots \\ \cdots A_2^{-1} C_2^T Z_1^{-1} A_1^{-1} Z_0 \left(R_1^T R_1 \right)^{-1}, \end{aligned} \tag{4.50}$$

where $Z_j := \widetilde{Q}_{j+1}^{(1)}(0)$.

(2) A straightforward calculation yields

$$e_i^T \Sigma(\{0\}) e_j = \lim_{z\to 0} z e_i^T \Phi(z) e_j. \tag{4.51}$$

From Theorem 2.4 it follows that the state $(0,\ell) \in E$ is positive recurrent if the condition

$$\begin{aligned} e_\ell^T \lim_{t\to\infty} P_{00}(t) e_\ell &= e_\ell^T \lim_{z\to 0} z \int \frac{d\Sigma(x)}{z-x} S_0^{-1} e_\ell \\ &= e_\ell^T \lim_{z\to 0} z R_0^{-1} \left\{ z I_d + R_0 B_0 R_0^{-1} - R_0 A_0 \int \frac{d\Sigma^{(1)}(x)}{z-x} R_1^T R_1 C_1^T R_0^{-1} \right\}^{-1} R_0 e_\ell \\ &= e_\ell^T \lim_{z\to 0} \left\{ I_d + \frac{1}{z} (B_0 - A_0 \int \frac{d\Sigma^{(1)}(x)}{z-x} R_1^T R_1 C_1^T \right\}^{-1} e_\ell > 0 \end{aligned} \tag{4.52}$$

holds.

Acknowledgments

The work of the authors was supported by the Deutsche Forschungsgemeinschaft (De 502/22-1/2). The authors would like to thank Martina Stein, who typed parts of this paper with considerable technical expertise. They are also grateful to two anonymous referees for their constructive comments on an earlier version of this paper.

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