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Research Article

POT-Based Estimation of the Renewal Function of Interoccurrence Times of Heavy-Tailed Risks

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Making use of the peaks over threshold (POT) estimation method, we propose a semiparametric estimator for the renewal function of interoccurrence times of heavy-tailed insurance claims with infinite variance. We prove that the proposed estimator is consistent and asymptotically normal, and we carry out a simulation study to compare its finite-sample behavior with respect to the nonparametric one. Our results provide actuaries with confidence bounds for the renewal function of dangerous risks.

1. Introduction

Let $X_1, X_2, \ldots$ be independent and identically distributed (iid) positive random variables (rvs), representing claim interoccurrence times of an insurance risk, with common distribution function (df) F having finite mean μ and variance σ^2. Let

$$\mathbb{S}_m := \begin{cases} X_1 + \cdots + X_m, & m = 1, 2, \ldots, \\ 0, & m = 0 \end{cases} \tag{1.1}$$

be the claim occurrence times, and define the number of claims recorded over the time interval $[0, t]$ by

$$\mathbb{N}(t) := \max\{m \geq 0,\ \mathbb{S}_m \leq t\}. \tag{1.2}$$

The corresponding renewal function is defined by

$$\mathbb{R}(t) := E[\mathbb{N}(t)] = \sum_{k=1}^{\infty} F^{(k)}(t), \quad t > 0, \tag{1.3}$$

where $F^{(k)}$ is the k-fold convolution of F for $k \geq 1$.

The renewal theory has proved to be a powerful tool in stochastic modeling in a wide variety of applications such as reliability theory, where a renewal process is used to model the successive repairs of a failed machine (see [1]), risk theory, where a renewal process is used to model the successive occurrences of risks (see [2, 3]), inventory theory, where a renewal process is used to model the successive times between demand points (see [4]), manpower planning, where a renewal process is used to model the sequence of resignations from a given job (see [5]), and warranty analysis, where a renewal process is used to model the successive purchases of a new item following the expiry of a free-replacement warranty (see [6]). Therefore, the need for renewal function estimates seems more than pressing in many practical problems. For a summary of renewal theory, one refers to Feller [7], Asmussen [8], and Resnick [9].

Statistical estimation of the renewal function has been considered in several ways. Using a nonparametric approach, Frees [10] introduced two estimators based on the empirical counterparts of F and $F^{(k)}$ by suitably truncating the sum in (1.3). Zhao and Subba Rao [11] proposed an estimation method based on the kernel estimate of the density and the renewal equation. A histogram-type estimator, resembling to the second estimator of Frees, was given by Markovich and Krieger [12].

When $E[X^2] = \infty$, Sgibnev [13] gave an asymptotic approximation of (1.3) as follows:

$$\mathbb{R}(t) - \frac{t}{\mu} \sim \frac{1}{E[X^2]} \int_0^t \left(\int_y^{\infty} \overline{F}(x)dx \right) dy, \tag{1.4}$$

with $\overline{F} := 1 - F$ being the tail of F.

By replacing F by its empirical counterpart F_n in (1.4), Bebbington et al. [14] recently proposed a nonparametric estimator for $\mathbb{R}(t)$ in the case where F is of infinite variance, given by

$$\widetilde{\mathbb{R}}_n(t) := \frac{t}{\widetilde{\mu}} + \frac{1}{\widetilde{\mu}_2} \int_0^t \left(\int_y^{\infty} \overline{F}_n(x)dx \right) dy, \tag{1.5}$$

where $\widetilde{\mu}$ and $\widetilde{\mu}_2$, respectively, represent the first and second sample moments of F. Their main result says that whenever F belongs to the domain of attraction of a stable law S_α with $1/2 < \alpha < 1$ (see, e.g., [15]), the df of $\widetilde{\mathbb{R}}_n(t)$ converges, for suitable normalizing constants, to S_α. This result provides confidence bounds for $\mathbb{R}(t)$ with respect to the quantiles of S_α.

In general, people prefer estimators having simple formulas and carrying some kind of asymptotic normality property in order to facilitate confidence interval construction. From this point of view, the estimator $\widetilde{\mathbb{R}}_n(t)$ may not be as satisfactory to the users as it should be. Then an alternative estimator to $\widetilde{\mathbb{R}}_n(t)$ would be more useful in practice. Our task is to use the extreme value theory tools to construct such an alternative estimator.

Indeed, an important class of models having infinite second-order moments is the set of heavy-tailed distributions (e.g., Pareto, Burr, Student, etc.). A df F is said to be heavy-tailed with tail index $\xi > 0$ if

$$\overline{F}(x) = cx^{-1/\xi}\left(1 + x^{-\delta}\mathbb{L}(x)\right), \quad \text{as } x \longrightarrow \infty, \tag{1.6}$$

for $\xi \in (0,1)$, $\delta > 0$, and some real constant c, with $\mathbb{L}$ a slowly varying function at infinity, that is, $\mathbb{L}(tx)/\mathbb{L}(x) \to 1$ as $x \to \infty$ for any $t > 0$. For details on these functions, see Chapter 0 in Resnick [16] or Seneta [17]. Notice that when $\xi \in (1/2, 1)$ we have $\mu < \infty$ and $E[X^2] = \infty$. In this case, an asymptotic approximation of the renewal function $\mathbb{R}(t)$ is given in (1.4).

Prior to Sgibnev [13], Teugels [18] obtained an approximation of $\mathbb{R}(t)$ when F is heavy-tailed with tail index $\xi \in (1/2, 1)$:

$$\mathbb{R}(t) - \frac{t}{\mu} \sim \frac{\xi^2 t^2 \overline{F}(t)}{\mu^2(1-\xi)(2\xi - 1)}, \quad \text{as } t \longrightarrow \infty. \tag{1.7}$$

Extreme value theory allows for an accurate modeling of the tails of any unknown distribution, making the (semiparametric) statistical inference more accurate for heavy-tailed distributions. Indeed, the semiparametric approach permits extrapolating beyond the largest value of a given sample while the nonparametric one does not since the empirical df vanishes outside the sample. This represents a big handicap for those dealing with heavy-tailed data.

Extreme value theory has two aspects. The first one consists in approximating the tail distribution by the generalized extreme value (GEV) distribution, thanks to Fisher-Tippett theorem (see [19, 20]). The second aspect (commonly known as POT method) is based on Balkema-de Haan result which says that the distribution of the excesses over a fixed threshold is approximated by the generalized Pareto distribution (GPD) (see [21, 22]). Those interested in extreme value theory and its applications are referred to the textbooks of de Haan and Ferriera [23] and Embrechts et al. [24]. In our situation, we have a fixed threshold equal to the horizon $t = t_n$ (see Section 3). Therefore, the POT method would be the appropriate choice to derive an estimator for $\mathbb{R}(t)$ by exploiting the heavy-tail property of df F used in approximation (1.4). The asymptotic normality of our estimator is established under suitable assumptions.

The remainder of the paper is organized as follows. In Section 2, we introduce the GPD approximation, mostly known as the POT method. A new estimator of the renewal function $\mathbb{R}(t)$ is proposed in Section 3, along with two main results on its limiting behavior. Section 4 is devoted to a simulation study. The proofs are postponed until Section 5.

2. GPD Approximation

The distribution of the excesses, over a "fixed" threshold t, pertaining to df F is defined by

$$F_t(y) := P(X_1 - t \le y \mid X_1 > t), \quad \text{for } y > 0. \tag{2.1}$$

It is shown, in Balkema and de Haan [21] and Pickands [22], that F_t is approximated by a generalized Pareto distribution (GPD) function $\mathbb{G}_{\xi,\beta}$ with shape parameter $\xi \in \mathbb{R}$ and scale parameter $\beta = \beta(t) > 0$, in the following sense:

$$\sup_{y>0}\left|F_t(y) - \mathbb{G}_{\xi,\beta}(y)\right| = O\left(t^{-\delta}\mathbb{L}(t)\right), \quad \text{as } t \longrightarrow \infty, \tag{2.2}$$

where $t^{-\delta}\mathbb{L}(t) \to 0$ as $t \to \infty$ for any $\delta > 0$. The GPD function $\mathbb{G}_{\xi,\beta}$ is a two-parameter df defined by

$$\mathbb{G}_{\xi,\beta}(y) = \begin{cases} 1 - \left(1 + \xi\dfrac{y}{\beta}\right)^{-1/\xi}, & \xi \neq 0, \\ 1 - \exp\left(-\dfrac{y}{\beta}\right), & \xi = 0, \end{cases} \tag{2.3}$$

for $0 \le y < \infty$ if $\xi \ge 0$ and $0 \le y < -\beta/\xi$ if $\xi < 0$.

Let $Y_1, \ldots, Y_N$ be iid rvs with exact GPD $\mathbb{G}_{\xi,\beta}$. It is well known by standard arguments (see, e.g., [25, Chapter 9]) that there exists, with probability 1 as N tends to infinity, a local maximum $(\widehat{\xi}_N, \widehat{\beta}_N)$ for the Log-Likelihood of $\mathbb{G}_{\xi,\beta}$'s density based on the sample $(Y_1, \ldots, Y_N)$. In this case, by Theorem 3.7 page 447 in the work of Lehmann and Casella [26], we infer that $\widehat{\xi}_N$ and $\widehat{\beta}_N$ are consistent estimators of ξ and β. Moreover, these estimators are asymptotically normal provided that $\xi > -1/2$. The extension to $\xi \le -1/2$ was investigated by Smith [27].

Suppose now that $Y_1, \ldots, Y_N$ are drawn not from $\mathbb{G}_{\xi,\beta}$, but from F_t. In view of the asymptotic approximation (2.2), Smith [27] has proposed estimates for (ξ, β) via the Maximum Likelihood approach. The obtained estimators $(\widehat{\xi}_N, \widehat{\beta}_N)$ are solutions of the following system:

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{N}\log\left(1 + \xi\frac{y_i}{\beta}\right) &= \xi, \\ \frac{1}{N}\sum_{i=1}^{N}\frac{y_i/\beta}{1 + y_i/\beta} &= \frac{1}{1+\xi}, \end{aligned} \tag{2.4}$$

where $(y_1, \ldots, y_N)$ is a realization of $(Y_1, \ldots, Y_N)$.

Letting $t = t_N \to \infty$ as $N \to \infty$ and $\beta_N = t_N\xi$ and making use of (2.2), Smith [28] established, in Theorem 3.2, the asymptotic normality of $(\widehat{\xi}_N, \widehat{\beta}_N)$ as follows:

$$\sqrt{N}\begin{pmatrix} \dfrac{\widehat{\beta}_N}{\beta_N} - 1 \\ \widehat{\xi}_N - \xi \end{pmatrix} \xrightarrow{\mathfrak{D}} \mathcal{N}_2\left(0, \mathbb{Q}^{-1}\right) \quad \text{as } N \longrightarrow \infty, \tag{2.5}$$

where

$$\mathbb{Q}^{-1} = (1+\xi)\begin{pmatrix} 2 & -1 \\ -1 & 1+\xi \end{pmatrix}, \tag{2.6}$$

provided that $\sqrt{N}t_N^{-\delta}\mathbb{L}(t_N) \to 0$ as $N \to \infty$ and $x \mapsto x^{-\delta}\mathbb{L}(x)$ is nonincreasing near infinity. In the case $\sqrt{N}t_N^{-\delta}\mathbb{L}(t_N) \nrightarrow 0$, the limiting distribution in (2.5) is biased. Here $\xrightarrow{\mathfrak{D}}$ denotes convergence in distribution and $\mathcal{N}_2(\omega, \Sigma)$ stands for the bivariate normal distribution with mean vector ω and covariance matrix Σ.

3. Estimating the Renewal Function in Infinite Time

Since we are interested in the renewal function in infinite time, we must assume that time t is large enough and for asymptotic considerations, we will assume that t depends on the sample size n. That is, $t = t_n$, with $t_n \to \infty$ as $n \to \infty$. Relation (1.7) suggests that in order to construct an estimator of $\mathbb{R}(t_n)$, we need to estimate μ, ξ and $\overline{F}(t_n)$. Let $n = n(t)$ be the number of X_is, which are observed on horizon t_n and denoted by

$$N_{t_n} := \operatorname{card}(\{X_i > t_n : 1 \le i \le n\}), \tag{3.1}$$

the number of exceedances over t_n, with $\operatorname{card}(K)$ being the cardinality of set K. Notice that N_{t_n} is a binomial rv with parameters n and $p_n := \overline{F}(t_n)$ for which the natural estimator is $\widehat{p}_n := N_t/n$.

Select, from the sample $(X_1, \ldots, X_n)$, only those observations $X_{i_1}, \ldots, X_{i_{N_{t_n}}}$ that exceed t_n. The N_t excesses

$$E_{j:n} := X_{i_j} - t_n, \quad j = 1, \ldots, N_{t_n} \tag{3.2}$$

are iid rvs with common df F_{t_n}. As seen in Section 2, the maximum likelihood estimators $(\widehat{\xi}_n, \widehat{\beta}_n)$ are solutions of the following system:

$$\begin{gathered} \frac{1}{v_n}\sum_{j1}^{v_n}\log\left(1+\xi\frac{e_{j:n}}{\beta}\right) = \xi, \\ \frac{1}{v}\sum_{j=1}^{v_n}\frac{e_{j:n}/\beta}{1+e_{j:n}/\beta} = \frac{1}{1+\xi}, \end{gathered} \tag{3.3}$$

where v_n is an observation of N_{t_n} and the vector $(e_{1:n}, \ldots, e_{v_n:n})$ a realization of $(E_{1:n}, \ldots, E_{N_{t_n}:n})$. Regarding the distribution mean $\mu = E[X_1]$, we know that, for $\xi \in (0, 1/2]$, X_1 has finite variance and therefore μ could naturally be estimated by the sample mean $\overline{X} := n^{-1}S_n$ which, by the Central Limit Theorem (CLT), is asymptotically normal. Whereas for $\xi \in (1/2, 1)$, X_1 has infinite variance, in which case the CLT is no longer valid. This case is frequently met in real insurance data (see, e.g., [29]). Using the GPD approximation, Johansson [30] has proposed an alternative estimator for $\mu = \int_0^\infty x dF(x)$. For each $n \geq 1$, we write μ as the sum of two components:

$$\mu_n^* := \int_0^{t_n} x dF(x), \qquad \tau_n := \int_{t_n}^{\infty} x dF(x) = -\int_0^{\infty} (t_n + s) d\overline{F}(t_n + s). \tag{3.4}$$

Johansson [30] defined his estimator of μ, by estimating both $F(x)$ and $\overline{F}(t_n + s)$, as follows:

$$\widehat{\mu}_n^{(J)} := \int_0^{t_n} x dF_n(x) - \int_0^{\infty} (t_n + s) d\widehat{\overline{F}}(t_n + s), \tag{3.5}$$

where F_n is the empirical df based on the sample $(X_1, \ldots, X_n)$ and $\widehat{\overline{F}}(t_n + s)$ is an estimate of $\overline{F}(t_n + s)$ obtained from the relation

$$\overline{F}_{t_n}(s) = \frac{\overline{F}(t_n + s)}{\overline{F}(t_n)}, \quad s > 0, \tag{3.6}$$

which implies that $\overline{F}(t_n + s) = p_n \overline{F}_{t_n}(s),\ s > 0$. Approximation (2.2) motivates us to estimate $\overline{F}_{t_n}(s)$ by $\widehat{\overline{F}}_{t_n}(s) := \overline{\mathbb{G}}_{\widehat{\xi}_n, \widehat{\beta}_n}(s),\ s > 0$. Hence, an estimate of $\overline{F}(t_n + s)$ is

$$\widehat{\overline{F}}(t_n + s) := \widehat{p}_n \overline{\mathbb{G}}_{\widehat{\xi}_n, \widehat{\beta}_n}(s), \quad s > 0. \tag{3.7}$$

By integrating (3.5), we get

$$\begin{aligned} \widehat{\mu}_n^{(J)} &= \frac{1}{n} \sum_{i=1}^{n} X_i \mathbf{1}_{\{X_i \leq t_n\}} + \widehat{p}_n \left(t_n + \frac{\widehat{\beta}_n}{1 - \widehat{\xi}_n} \right) \\ &=: \widehat{\mu}_n^* + \widehat{\tau}_n, \end{aligned} \tag{3.8}$$

with $\widehat{\xi}_n \in (0, 1)$ with large probability. Here, $\mathbf{1}_K$ denotes the indicator function of set K. Respectively, substituting $\widehat{\mu}_n^{(J)}$, $\widehat{\xi}_n$, and $\widehat{p}_n$ for μ, ξ and $\overline{F}(t_n)$ in (1.7) yields the following estimator for the renewal function $\mathbb{R}(t_n)$

$$\widehat{\mathbb{R}}_n(t_n) := \frac{t_n}{\widehat{\mu}_n^{(J)}} + \frac{\widehat{\xi}_n^2 t_n^2 \widehat{p}_n}{\widehat{\mu}_n^{(J)2} \left(1 - \widehat{\xi}_n\right) \left(2\widehat{\xi}_n - 1\right)}. \tag{3.9}$$

The asymptotic behavior of $\widehat{\mathbb{R}}_n(t_n)$ is given by the following two theorems.

Theorem 3.1. *Let F be a df fulfilling* (1.6) *with $\xi \in (1/2, 1)$. Suppose that $\mathbb{L}$ is locally bounded in $[x_0, +\infty)$ for $x_0 \geq 0$ and $x \mapsto x^{-\delta}\mathbb{L}(x)$ is nonincreasing near infinity, for some $\delta > 0$. Then, for any $t_n = O(n^{\alpha\xi/4})$ with $\alpha \in (0,1)$, one has*

$$\widehat{\mathbb{R}}_n(t_n) - \mathbb{R}(t_n) = O_{\mathbb{P}}\left(n^{(\alpha/2)(\xi-1/4)-1/2}\right), \quad \text{as } n \longrightarrow \infty. \tag{3.10}$$

Theorem 3.2. *Let F be as in Theorem 3.1. Then for any $t_n = O(n^{\alpha\xi/4})$ with $\alpha \in (4/(1+2\xi\delta), 1)$, we have*

$$\frac{\sqrt{n}}{s_n t_n}\left(\widehat{\mathbb{R}}_n(t_n) - \mathbb{R}(t_n)\right) \xrightarrow{\mathfrak{D}} \mathcal{N}(0,1), \quad \text{as } n \longrightarrow \infty, \tag{3.11}$$

where

$$\begin{aligned} s_n^2 := \theta_1^2 &+ \frac{p_n(1-p_n)}{\gamma_n^2}\left(\theta_2 + \theta_1\left(t_n + \frac{\beta_n}{1-\xi}\right)\right)^2 + \frac{p_n}{\gamma_n^2}\left(\theta_3 + \frac{\theta_1 p_n \beta_n}{(1-\xi)^2}\right)^2 \\ &+ \frac{\theta_1^2 \beta_n^2 p_n^3}{\gamma_n^2 (1-\xi)^2} - \frac{\theta_1 \beta_n p_n^2}{\gamma_n^2}\left(\theta_3 + \frac{\theta_1 p_n \beta_n}{(1-\xi)^2}\right) \end{aligned} \tag{3.12}$$

with

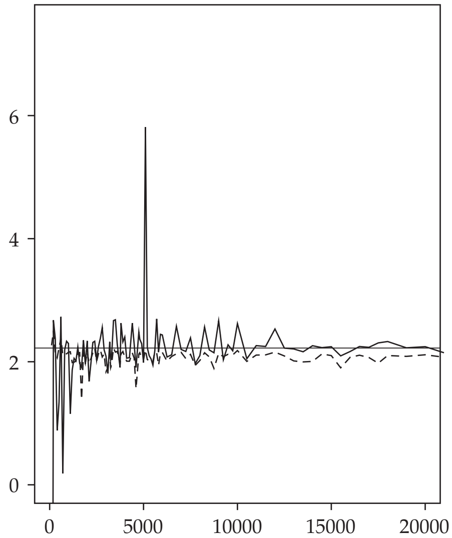
$$\begin{aligned} \theta_1 &:= -\frac{1}{\mu^2} - \frac{2\xi^2 t_n p_n}{\mu^3(1-\xi)(2\xi-1)}, \\ \theta_2 &:= \frac{\xi^2 t_n}{\mu^2(1-\xi)(2\xi-1)}, \\ \theta_3 &:= \frac{t_n p_n}{\mu^2(1-\xi)(2\xi-1)}\left(2\xi + \frac{4\xi^3 - 3\xi^2}{(1-\xi)(2\xi-1)}\right), \end{aligned} \tag{3.13}$$

$p_n := \overline{F}(t_n)$, $\beta_n := t_n\xi$, *and* $\gamma_n^2 := \mathrm{Var}(X_1 \mathbf{1}_{\{X_1 \leq t_n\}})$.

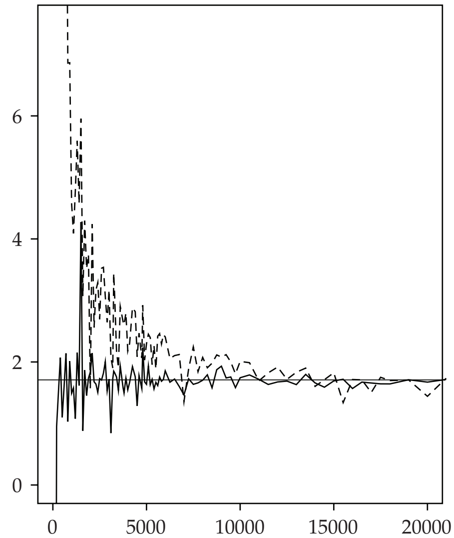
4. Simulation Study

In this section, we carry out a simulation study (by means of the statistical software **R**, see [31]) to illustrate the performance of our estimation procedure, through its application to sets of samples taken from two distinct Pareto distributions $F(x) = 1 - x^{-1/\xi}$, $x > 1$ (with tail indices $\xi = 3/4$ and $\xi = 2/3$). We fix the threshold at 4, which is a value above the intermediate statistic corresponding to the optimal fraction of upper-order statistics in each sample. The latter is obtained by applying the algorithm of Cheng and Peng [32]. For each sample size, we generate 200 independent replicates. Our overall results are then taken as the empirical means of the values in the 200 repetitions.

A comparison with the nonparametric estimator is done as well. In the graphical illustration, we plot both estimators versus the sample size ranging from 1000 to 20000.



(a)



(b)

Figure 1: Plots of the new and sample estimators of the renewal function, of interoccurrence times of Pareto-distributed claims with tail indices 2/3 (a) and 3/4 (b), versus the sample size. The horizontal line represents the true value of the renewal function $\mathbb{R}(t)$ evaluated at $t = 4$.

Table 1: Semiparametric and nonparametric estimates of the renewal function of interoccurrence times of Pareto-distributed claims with shape parameter 3/4. Simulations are repeated 200 times for different sample sizes.

	True value $R = 1.708$					
Sample size	Semiparametric $\widehat{R}$			Nonparametric $\widetilde{R}$		
	Mean	Bias	RMSE	Mean	Bias	RMSE
1000	1.696	−0.013	0.250	2.141	0.433	0.553
2000	1.719	0.011	0.183	1.908	0.199	0.288
5000	1.705	−0.003	0.119	1.686	−0.022	0.168

Figure 1 clearly shows that the new estimator is consistent and that it is always better than the nonparametric one. For the numerical investigation, we take samples of sizes 1000, 2000 and 5000. In each case, we compute the semiparametric estimate $\widehat{R}$ as well as the nonparametric estimate $\widetilde{R}$. We also provide the bias and the root mean squared error (rmse).

The results are summarized in Tables 1 and 2 for $\xi = 3/4$ and $\xi = 2/3$ respectively. We notice that, regardless of the tail index value and the sample size, the semiparametric estimation procedure is more accurate than the nonparametric one.

5. Proofs

The following tools will be instrumental for our needs.

Table 2: Semiparametric and nonparametric estimates of the renewal function of interoccurence times of Pareto-distributed claims with shape parameter 2/3. Simulations are repeated 200 times for different sample sizes.

	True value $R = 2.222$					
Sample size	Semiparametric $\widehat{R}$			Nonparametric $\widetilde{R}$		
	Mean	Bias	RMSE	Mean	Bias	RMSE
1000	2.265	0.042	0.185	2.416	0.193	0.229
2000	2.247	0.024	0.157	2.054	−0.167	0.223
5000	2.223	0.001	0.129	2.073	−0.149	0.192

Proposition 5.1. *Let F be a df fulfilling* (1.6) *with* $\xi \in (1/2, 1)$, $\delta > 0$, *and some real* c. *Suppose that* $\mathbb{L}$ *is locally bounded in* $[x_0, +\infty)$ *for* $x_0 \geq 0$. *Then for* n *large enough and for any* $t_n = O(n^{\alpha\xi/4})$, $\alpha \in (0, 1)$, *one has*

$$\begin{gathered} p_n = c(1 + o(1))n^{-\alpha/4}, \\ \gamma_n^2 = O\left(n^{(\alpha/2)(\xi-1/2)}\right), \\ s_n^2 = O\left(n^{(\alpha/2)(\xi-1/2)}\right), \\ \sqrt{np_n} t_n^{-\delta} \mathbb{L}(t_n) = O\left(n^{-\alpha/8-\alpha\xi\delta/4+1/2}\right), \end{gathered} \tag{5.1}$$

where p_n, γ_n^2, *and* s_n^2 *are those defined in Theorem 3.2.*

Lemma 5.2. *Under the assumptions of Theorem 3.2, one has, for any real numbers* u_1, u_2, u_3 *and* u_4,

$$\begin{aligned} &E\left[\exp\left\{iu_1 \frac{\sqrt{n}}{\gamma_n}(\widehat{\mu}_n^* - \mu_n^*) + i\sqrt{np_n}(u_2, u_3)\begin{pmatrix} \dfrac{\widehat{\beta}_n}{\beta_n} - 1 \\ \widehat{\xi}_n - \xi \end{pmatrix} + iu_4 \frac{\sqrt{n}(\widehat{p}_n - p_n)}{\sqrt{p_n(1-p_n)}}\right\}\right] \\ &\longrightarrow \exp\left\{-\frac{u_1^2}{2} - \frac{1}{2}(u_2, u_3)\mathbb{Q}^{-1}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} - \frac{u_4^2}{2}\right\}, \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{5.2}$$

where $i^2 = -1$.

Proof of the Proposition. We will only prove the second result, the other ones are straightforward from (1.6). Let $x_0 > 0$ be such that $\overline{F}(x) = cx^{-1/\xi}\ (1 + x^{-\delta}\mathbb{L}(x))$, for $x > x_0$. Then for n large enough, we have

$$E\left[X_1 \mathbf{1}_{\{X_1 \leq t_n\}}\right] = \int_0^{t_n} x dF(x) = \int_0^{x_0} x dF(x) + \int_{x_0}^{t_n} x dF(x). \tag{5.3}$$

Recall that $\mu < \infty$, hence $\int_0^{x_0} x dF(x) < \infty$. Making use of the proposition assumptions, we get $E[X_1 \mathbf{1}_{\{X_1 \leq t_n\}}] = O(1)$ and $E[X_1^2 \mathbf{1}_{\{X_1 \leq t_n\}}] = O(t_n^{2-1/\xi})$ and therefore $\gamma_n^2 = O(n^{\alpha/2(\xi/2-1)})$. □

Proof of Lemma 5.2. See Johansson [30]. □

Proof of Theorem 3.1. We may readily check that for all large n,

$$\frac{\left(\widehat{\mathbb{R}}_n(t_n) - \mathbb{R}_n(t_n)\right)}{t_n} \sim A_n + B_n + C_n, \tag{5.4}$$

where

$$\begin{aligned} A_n &:= \left(-\frac{1}{\widehat{\mu}_n^{(J)}\mu} - \frac{\xi^2 t_n p_n \left(\widehat{\mu}_n^{(J)} + \mu\right)}{\widehat{\mu}_n^{(J)2}\mu^2(1-\xi)(2\xi-1)} \right)\left(\widehat{\mu}_n^{(J)} - \mu\right), \\ B_n &:= \frac{\widehat{\xi}_n^2 t_n}{\widehat{\mu}_n^{(J)2}\left(1-\widehat{\xi}_n\right)\left(2\widehat{\xi}_n - 1\right)}(\widehat{p}_n - p_n), \\ C_n &:= \frac{t_n p_n}{\widehat{\mu}_n^{(J)2}\left(1-\widehat{\xi}_n\right)\left(2\widehat{\xi}_n - 1\right)} \times \left(\widehat{\xi}_n + \xi + \frac{2\xi^2\left(\widehat{\xi}_n + \xi\right) - 3\xi^2}{(1-\xi)(2\xi-1)} \right)\left(\widehat{\xi}_n - \xi\right). \end{aligned} \tag{5.5}$$

Johansson [30] proved that there exists a bounded sequence k_n such that

$$\widehat{\mu}_n^{(J)} - \mu = O_{\mathbb{P}}\left(\gamma_n\sqrt{\frac{k_n}{n}}\right), \tag{5.6}$$

hence $\widehat{\mu}_n^{(J)} - \mu = O_{\mathbb{P}}(n^{(\alpha/4)(\xi-1/2)-1/2})$. The first result of the proposition yields that

$$t_n p_n\left(\widehat{\mu}_n^{(J)} - \mu\right) = O_{\mathbb{P}}\left(n^{(\alpha/4)(2\xi-3/2)-1/2}\right). \tag{5.7}$$

Since $(\alpha/4)(2\xi - 3/2) - 1/2 < 0$, then $t_n p_n\ (\widehat{\mu}_n^{(J)} - \mu) = o_{\mathbb{P}}(1)$. On the other hand, by the CLT we have

$$\widehat{p}_n - p_n = O_{\mathbb{P}}\left(\sqrt{\frac{p_n}{n}}\right), \tag{5.8}$$

then $t_n(\widehat{p}_n - p_n) = O_{\mathbb{P}}(n^{(\alpha/4)(\xi-1/2)-1/2}) = o_{\mathbb{P}}(1)$. On the other hand, Smith [28], yields

$$\widehat{\xi}_n - \xi = O_{\mathbb{P}} t_n^{-\delta}\mathbb{L}(t_n), \tag{5.9}$$

it follows that, $\widehat{\xi}_n^2 t_n\ (\widehat{p}_n - p_n) = O_{\mathbb{P}}(n^{(\alpha/4)(\xi(1-2\delta)-1/2)-1/2}) = o_{\mathbb{P}}(1)$, therefore

$$\frac{\widehat{\xi}_n^2 t_n(\widehat{p}_n - p_n)}{\widehat{\mu}_n^{(J)2}\left(1-\widehat{\xi}_n\right)\left(2\widehat{\xi}_n - 1\right)} = o_{\mathbb{P}}(1),$$
$$t_n p_n\left(\widehat{\xi}_n - \xi\right) = O_{\mathbb{P}}\left(n^{(\alpha/4)(\xi(1-\delta)-1)}\right) = o_{\mathbb{P}}(1),$$
$$\widehat{\xi}_n t_n p_n\left(\widehat{\xi}_n - \xi\right) = O_{\mathbb{P}}\left(n^{(\alpha/4)(\xi(1-2\delta)-1)}\right) = o_{\mathbb{P}}(1),$$
$$p_n\left(\widehat{\xi}_n - \xi\right) = O_{\mathbb{P}}\left(n^{(-\alpha/4)((1+\xi\delta))}\right) = o_{\mathbb{P}}(1). \tag{5.10}$$

Thus,

$$\frac{\left(\widehat{\xi}_n + \xi\right)}{\widehat{\mu}_n^{(J)2}\left(1-\widehat{\xi}_n\right)\left(2\widehat{\xi}_n - 1\right)} t_n p_n\left(\widehat{\xi}_n - \xi\right) \xrightarrow{\mathbb{P}} 0,$$
$$\frac{t_n p_n\left(2\xi^2\left(\widehat{\xi}_n + \xi\right) - 3\xi^2\right)}{\widehat{\mu}_n^{(J)2}\left(1-\widehat{\xi}_n\right)\left(2\widehat{\xi}_n - 1\right)(1-\xi)(2\xi-1)}\left(\widehat{\xi}_n - \xi\right) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \longrightarrow \infty. \tag{5.11}$$

Therefore for all large n, we get $\widehat{\mathbb{R}}(t_n) - \mathbb{R}(t_n) = O_{\mathbb{P}}(n^{(\alpha/2)(\xi-1/4)-1/2})$, as sought. □

Proof of Theorem 3.2. From the proof of Theorem 3.1, for all large n, it is easy to verify that

$$\frac{\left(\widehat{\mathbb{R}}_n(t_n) - \mathbb{R}_n(t_n)\right)}{t_n} = \theta_1(1+o_{\mathbb{P}}(1))\left(\widehat{\mu}_n^{(J)} - \mu\right)$$
$$+ \theta_2(1+o_{\mathbb{P}}(1))(\widehat{p}_n - p_n)$$
$$+ \theta_3(1+o_{\mathbb{P}}(1))\left(\widehat{\xi}_n - \xi\right), \tag{5.12}$$

where

$$\theta_1 = -\frac{1}{\mu^2} - \frac{2\xi^2 t_n p_n}{\mu^3(1-\xi)(2\xi-1)},$$
$$\theta_2 = \frac{\xi^2 t_n}{\mu^2(1-\xi)(2\xi-1)},$$
$$\theta_3 = \frac{t_n p_n}{\mu^2(1-\xi)(2\xi-1)}\left(2\xi + \frac{4\xi^3 - 3\xi^2}{(1-\xi)(2\xi-1)}\right). \tag{5.13}$$

Multiplying by $\sqrt{n}/\gamma_n$ and using the proposition and the lemma together with the continuous mapping theorem, we find that

$$\begin{aligned}\frac{\sqrt{n}}{\gamma_n t_n}\left(\widehat{\mathbb{R}}_n(t_n)-\mathbb{R}_n(t_n)\right) &= \theta_1(1+o_{\mathbb{P}}(1))\frac{\sqrt{n}}{\gamma_n}\left(\widehat{\mu}_n^{(J)}-\mu\right)\\ &\quad+\theta_2(1+o_{\mathbb{P}}(1))\frac{\sqrt{n}}{\gamma_n}(\widehat{p}_n-p_n)\\ &\quad+\theta_3(1+o_{\mathbb{P}}(1))\frac{\sqrt{n}}{\gamma_n}\left(\widehat{\xi}_n-\xi\right).\end{aligned} \tag{5.14}$$

On the other hand, from Johansson [30], we have for all large n

$$\begin{aligned}\frac{\sqrt{n}}{\gamma_n}\left(\widehat{\mu}_n^{(J)}-\mu\right) &= \frac{\sqrt{n}}{\gamma_n}(\widehat{\mu}_n^*-\mu_n^*)+\left(t_n+\frac{\beta_n}{1-\xi_n}\right)\frac{\sqrt{n}}{\gamma_n}(\widehat{p}_n-p_n)\\ &\quad+\frac{p_n\beta_n}{(1-\xi)^2}\frac{\sqrt{n}}{\gamma_n}\left(\widehat{\xi}_n-\xi\right)+\frac{p_n}{1-\xi}\frac{\sqrt{n}}{\gamma_n}\left(\widehat{\beta}_n-\beta_n\right)+o_{\mathbb{P}}(1).\end{aligned} \tag{5.15}$$

This enables us to rewrite $(\sqrt{n}/\gamma_n t_n)(\widehat{\mathbb{R}}_n(t_n)-\mathbb{R}_n(t_n))$ into

$$\begin{aligned}&\theta_1\frac{\sqrt{n}}{\gamma_n}(\widehat{\mu}_n^*-\mu_n^*)+\frac{\sqrt{p_n(1-p_n)}}{\gamma_n}\left(\theta_2+\theta_1\left(t_n+\frac{\beta_n}{1-\xi}\right)\right)\frac{\sqrt{n}(\widehat{p}_n-p_n)}{\sqrt{p_n(1-p_n)}}\\ &\quad+\theta_1\frac{\beta_n p_n\sqrt{p_n}}{\gamma_n(1-\xi)}\sqrt{\frac{n}{p_n}}\left(\frac{\widehat{\beta}_n}{\beta_n}-1\right)\\ &\quad+\frac{\sqrt{p_n}}{\gamma_n}\left(\theta_3+\theta_1\frac{p_n\beta_n}{(1-\xi)^2}\right)\sqrt{\frac{n}{p_n}}\left(\widehat{\xi}_n-\xi\right)+o_{\mathbb{P}}(1),\\ &\qquad\mathbb{Q}^{-1}=(1+\xi)\begin{pmatrix}2 & -1\\ -1 & 1+\xi\end{pmatrix}.\end{aligned} \tag{5.16}$$

In view of Lemma 5.2, we infer that for all large n, the previous quantity is

$$\begin{aligned}&\theta_1\mathcal{W}_1+\frac{\sqrt{p_n(1-p_n)}}{\gamma_n}\left(\theta_2+\theta_1\left(t_n+\frac{\beta_n}{1-\xi}\right)\right)\mathcal{W}_2\\ &\quad+\frac{\sqrt{2(1+\xi)}\theta_1\beta_n p_n\sqrt{p_n}}{\gamma_n(1-\xi)}\mathcal{W}_3+\frac{(1+\xi)\sqrt{p_n}}{\gamma_n}\left(\theta_3+\frac{\theta_1 p_n\beta_n}{(1-\xi)^2}\right)\mathcal{W}_4+o_{\mathbb{P}}(1),\end{aligned} \tag{5.17}$$

where $(\mathcal{W}_i)_{i=1,4}$ are standard normal rvs with $E[W_iW_j] = 0$ for every $i, j = 1, \ldots, 4$ with $i \neq j$, except for

$$\begin{aligned} E[W_3W_4] &= E\left[\sqrt{2(1+\xi)}\sqrt{\frac{n}{p_n}}\left(\frac{\widehat{\beta}_n}{\beta_n}-1\right)(1+\xi)\sqrt{\frac{n}{p_n}}\left(\widehat{\xi}_n-\xi\right)\right] \\ &= \sqrt{2(1+\xi)}(1+\xi)E\left[\sqrt{\frac{n}{p_n}}\left(\frac{\widehat{\beta}_n}{\beta_n}-1\right)\sqrt{\frac{n}{p_n}}\left(\widehat{\xi}_n-\xi\right)\right] \\ &= -\sqrt{2(1+\xi)}(1+\xi)^2. \end{aligned} \tag{5.18}$$

Therefore, the rv $(\sqrt{n}/\gamma_n t_n)(\widehat{\mathbb{R}}_n(t_n) - \mathbb{R}_n(t_n))$ is Gaussian with mean zero with asymptotic variance

$$\begin{aligned} K_n^2 &:= \theta_1^2 + \frac{p_n(1-p_n)}{\gamma_n^2}\left(\theta_2+\theta_1\left(t_n+\frac{\beta_n}{1-\xi}\right)\right)^2 + \frac{2(1+\xi)\theta_1^2\beta_n^2p_n^3}{\gamma_n^2(1-\xi)^2} \\ &\quad + \frac{(1+\xi)^2p_n}{\gamma_n^2}\left(\theta_3+\frac{\theta_1p_n\beta_n}{(1-\xi)^2}\right)^2 - \frac{2\theta_1\beta_np_n^2(1+\xi)^4}{(1-\xi)\gamma_n^2}\left(\theta_3+\frac{\theta_1p_n\beta_n}{(1-\xi)^2}\right) + o_{\mathbb{P}}(1). \end{aligned} \tag{5.19}$$

Observe now that $K_n^2 = s_n^2 + o_{\mathbb{P}}(1)$, where s_n^2 is that in (3.12), this completes the proof of Theorem 3.2. □

6. Conclusion

In this paper, we have proposed a new estimator for the renewal function of heavy-tailed claim interoccurence times, via a semiparametric approach. Our considerations are based on one aspect of the extreme value theory, namely, the POT method. We have proved that our estimator is consistent and asymptotically normal. Moreover, simulations show that it is more accurate than the nonparametric estimator given by Bebbington et al. [14].

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References

[1] D. R. Cox, *Renewal Theory*, Methuen, London, UK, 1962.
[2] P. Embrechts, M. Maejima, and E. Omey, "Some limit theorems for generalized renewal measures," *Journal of the London Mathematical Society*, vol. 31, no. 1, pp. 184–192, 1985.
[3] P. Embrechts, M. Maejima, and J. Teugels, "Asymptotic behavior of compound distributions," *Astin Bulletin*, vol. 15, pp. 45–48, 1985.
[4] S. M. Ross, *Stochastic Processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, New York, NY, USA, 1983.

[5] D. J. Bartholomew and A. F. Forbes, *Statistical Techniques for Man-Power Planning*, Wiley, Chichester, UK, 1979.

[6] W. R. Blischke and E. M. Scheuer, "Applications of renewal theory in analysis of the free-replacement warranty," *Naval Research Logistics Quarterly*, vol. 28, no. 2, pp. 193–205, 1981.

[7] W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 2, Wiley, New York, NY, USA, 2nd edition, 1971.

[8] S. Asmussen, *Applied Probability and Queues*, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, Chichester, UK, 1987.

[9] S. Resnick, *Adventures in Stochastic Processes*, Birkhäuser, Boston, Mass, USA, 1992.

[10] E. W. Frees, "Nonparametric renewal function estimation," *The Annals of Statistics*, vol. 14, no. 4, pp. 1366–1378, 1986.

[11] Q. Zhao and S. Subba Rao, "Nonparametric renewal function estimation based on estimated densities," *Asia-Pacific Journal of Operational Research*, vol. 14, no. 1, pp. 115–126, 1997.

[12] N. M. Markovich and U. R. Krieger, "Nonparametric estimation of the renewal function by empirical data," *Stochastic Models*, vol. 22, no. 2, pp. 175–199, 2006.

[13] M. S. Sgibnev, "On the renewal theorem in the case of infinite variance," *Sibirskiĭ Matematicheskiĭ Zhurnal*, vol. 22, no. 5, pp. 178–189, 1981.

[14] M. Bebbington, Y. Davydov, and R. Zitikis, "Estimating the renewal function when the second moment is infinite," *Stochastic Models*, vol. 23, no. 1, pp. 27–48, 2007.

[15] V. M. Zolotarev, *One-Dimensional Stable Distributions*, vol. 65 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1986.

[16] S. I. Resnick, *Extreme Values, Regular Variation, and Point Processes*, vol. 4 of *Applied Probability. A Series of the Applied Probability Trust*, Springer, New York, NY, USA, 1987.

[17] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics, Vol. 508, Springer, Berlin, Germany, 1976.

[18] J. L. Teugels, "Renewal theorems when the first or the second moment is infinite," *Annals of Mathematical Statistics*, vol. 39, pp. 1210–1219, 1968.

[19] R. A. Fisher and L. H. C. Tippett, "Limiting forms of the frequency distribution of the largest or smallest member of a sample," *Proceedings of the Cambridge Philosophical Society*, vol. 24, pp. 180–190, 1928.

[20] B. Gnedenko, "Sur la distribution limite du terme maximum d'une série aléatoire," *Annals of Mathematics*, vol. 44, pp. 423–453, 1943.

[21] A. A. Balkema and L. de Haan, "Residual life time at great age," *Annals of Probability*, vol. 2, pp. 792–804, 1974.

[22] J. Pickands III, "Statistical inference using extreme order statistics," *The Annals of Statistics*, vol. 3, pp. 119–131, 1975.

[23] L. de Haan and A. Ferreira, *Extreme Value Theory: An Introduction*, Springer Series in Operations Research and Financial Engineering, Springer, New York, NY, USA, 2006.

[24] P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, vol. 33 of *Applications of Mathematics*, Springer, Berlin, Germany, 1997.

[25] D. R. Cox and D. V. Hinkley, *Theoretical Statistics*, Chapman and Hall, London, UK, 1974.

[26] E. L. Lehmann and G. Casella, *Theory of Point Estimation*, Springer Texts in Statistics, Springer, New York, NY, USA, 2nd edition, 1998.

[27] R. L. Smith, "Maximum likelihood estimation in a class of nonregular cases," *Biometrika*, vol. 72, no. 1, pp. 67–90, 1985.

[28] R. L. Smith, "Estimating tails of probability distributions," *The Annals of Statistics*, vol. 15, no. 3, pp. 1174–1207, 1987.

[29] J. Beierlant, G. Matthys, and G. Dierckx, "Heavy-tailed distributions and rating," *Astin Bulletin*, vol. 31, no. 1, pp. 37–58, 2001.

[30] J. Johansson, "Estimating the mean of heavy-tailed distributions," *Extremes*, vol. 6, no. 2, pp. 91–109, 2003.

[31] R. Ihaka and R. Gentleman, "R: a language for data analysis and graphics," *Journal of Computational and Graphical Statistics*, vol. 5, no. 3, pp. 299–314, 1996.

[32] S. Cheng and L. Peng, "Confidence intervals for the tail index," *Bernoulli*, vol. 7, no. 5, pp. 751–760, 2001.