Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2011, Article ID 595741, 19 pages
doi:10.1155/2011/595741

*Research Article*

# Drift and the Risk-Free Rate

**Anda Gadidov[1] and M. C. Spruill[2]**

[1] *Department of Mathematics and Statistics, Kennesaw State University, Kennesaw, GA 30144-5591, USA*
[2] *School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA*

Correspondence should be addressed to M. C. Spruill, spruill@math.gatech.edu

Received 5 May 2011; Revised 27 June 2011; Accepted 2 July 2011

Academic Editor: Arjun K. Gupta



It is proven, under a set of assumptions differing from the usual ones in the unboundedness of the time interval, that, in an economy in equilibrium consisting of a risk-free cash account and an equity whose price process is a geometric Brownian motion on $[0, \infty)$, the drift rate must be close to the risk-free rate; if the drift rate $\mu$ and the risk-free rate $r$ are constants, then $r = \mu$ and the price process is the same under both empirical and risk neutral measures. Contributing in some degree perhaps to interest in this mathematical curiosity is the fact, based on empirical data taken at various times over an assortment of equities and relatively short durations, that no tests of the hypothesis of equality are rejected.

## 1. Introduction

In the Black-Scholes model of a market with a single equity, its price $S_t$ is a geometric Brownian motion (GBM) satisfying for time $t \geq 0$ the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1.1}$$

where the volatility $\sigma$, the drift rate $\mu$, and the rate $r$ for the risk-free security are all constants. The stochastic process $B_t$, $0 \leq t$ is a standard Brownian motion. In the formulation of Harrison and Kreps [1] the process is on $t \in [0, T]$, $T < \infty$, and is defined on the probability space $(\Omega, \mathcal{F}_T, P_T)$, where the filtration $\mathcal{F}_t = \sigma(B_s,\ 0 \leq s \leq t)$, $t \geq 0$, is that generated by $B_t$. They show in this case that under equilibrium pricing for their securities market model, allowing only simple trading strategies, there is a measure $P_T^*$ equivalent to $P_T$ and prices can be expressed as expectations with respect to $P_T^*$. Furthermore, under $P_T^*$, $e^{-rt}S_t$ is a martingale on $[0, T]$ with respect to $\mathcal{F}_t$. There are three free parameters in the model, $\mu$, $\sigma$, and $r$. It is shown here that, if the equity's prices are given by (1.1) on $[0, \infty)$ and again only simple trading strategies are allowed on finite but arbitrary sets of nonrandom times, then there is an equivalent martingale measure and pricing with respect to it in the same manner represents a viable pricing system in the sense of Kreps [2] if and only if $r = \mu$. In this case, there are really only two free parameters.

Besides the results found in Lemma 4.5 relating to the Black-Scholes model, results of a somewhat more general nature in which $r$, $\mu$, and $\sigma$ depend upon time deterministically, can be found in Lemma 4.1.

The arguments given here are for an economy consisting of a single equity and a cash account. To the extent, therefore, that such models are pertinent to actual equities prices, an empirical investigation of $\mu = r$ for real market data is of interest. Assuming that the model is true for our empirical data consisting of daily closes of some selected equities, the hypothesis that $r = \mu$ is tested and in no case is the hypothesis of equality rejected by these optimal tests.

The organization of the paper is as follows. First the terms, definitions, and basic results of Harrison and Kreps [1] are recalled in the context of our model on $[0, \infty)$. Then connections are made between a presumed empirical GBM process with drift $\mu$ and the martingale arising from viability. It is not assumed but shown that the price process must also be a GBM under the equivalent measure; its drift is $r$ and its volatility agrees with that of the empirical GBM. It is shown also that the arguments used in [1] to obtain this result on $[0, T]$ cannot generally be used here. Next, the main result on $\mu = r$ is presented; namely, that under equilibrium the drift of the empirical GBM must be the risk-free rate. If the price process is a GBM under the empirical measure, then a consequence of viability is that it is also a GBM under an equivalent (risk-neutral) measure. Finally, the development and results of our hypothesis tests appear in Tables 1 and 2.

Proofs of technical details most pertinent to the main ideas appear in the body of the paper; proofs of more tangential ones have been placed in the appendix.

## 2. Viability, the Extension Property, and Equivalent Martingale Measures

The notation and assumptions are those of [1] except that here there is an infinite rather than a finite horizon. Thus, there is a linear space $X$ of functions $x : \Omega \to R$ which are random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. The points $\omega \in \Omega$ represent states of the world; the points $x \in X$ represent bundles of goods in some abstract economy. A subspace $M \subset X$ represents the space of bundles that can be constructed out of marketed bundles of goods. There is a bounded linear functional $\pi$ defined on $M$, with $\pi(m)$ representing the market price of $m \in M$ and a collection $A$ of agents represented by complete transitive binary relations $\succeq$ on the space $X$. The pair $(M, \pi)$ is viable (as a model of economic equilibrium) if there is an order $\succeq \in A$ of the above specifications and an $m^* \in M$ such that $\pi(m^*) \leq 0$ and $m^* \succeq m$ for all $m \in M$ such that $\pi(m) \leq 0$. Letting $\Psi$ denote the collection of positive bounded linear functionals on $X$, Kreps [2] proves the following lemma.

**Lemma 2.1.** *The price system $(M, \pi)$ is viable with respect to X if and only if there is a $\psi \in \Psi$ such that $\psi \mid M = \pi$.*

Here the securities market model of Harrison and Kreps [1] is extended to $[0, \infty)$ and involves, as there, a risk-free security with rate 0 and a security whose price at time $t$ under the state of the world $\omega \in \Omega$ is $Z_t(\omega)$, where the second-order stochastic process $Z_t$ is measurable

**Table 1:** $P$ values for testing $r = \mu$. None are significant at 0.05.

| Symbol | Dates | UMPU | LRT | Effect size |
|---|---|---|---|---|
| AAPL | 1-5-2001 to 12-6-2001 | 0.416 | 0.413 | $5.2 \times 10^{-2}$ |
| APD | 1-5-2001 to 12-6-2001 | 0.520 | 0.517 | $3.8 \times 10^{-2}$ |
| C | 1-1-1983 to 1-1-2000 | 0.059 | 0.059 | $3.1 \times 10^{-2}$ |
| CVS | 8-30-2001 to 11-30-2001 | 0.420 | 0.397 | $9.9 \times 10^{-2}$ |
| DAL | 11-03-1994 to 02-02-1995 | 0.914 | 0.913 | $-8.3 \times 10^{-3}$ |
| F | 11-03-1994 to 02-02-1995 | 0.258 | 0.226 | $-1.5 \times 10^{-1}$ |
| IBM | 3-29-1984 to 07-31-1984 | 0.618 | 0.611 | $4.8 \times 10^{-2}$ |
| K | 11-12-1999 to 02-10-2000 | 0.996 | 0.995 | $1.5 \times 10^{-1}$ |
| LMT | 1-5-2001 to 12-6-2001 | 0.237 | 0.277 | $5.1 \times 10^{-2}$ |
| PG | 11-12-1999 to 02-10-2000 | 0.457 | 0.445 | $9.2 \times 10^{-2}$ |

**Table 2:** $P$ values for testing $r = \mu$, March 1, 2007, to August 30, 2007.

| Symbol | UMPU | LRT | Effect size |
|---|---|---|---|
| AAPL | 0.086 | 0.068 | $1.6 \times 10^{-1}$ |
| APD | 0.272 | 0.261 | $1.0 \times 10^{-1}$ |
| C | 0.596 | 0.593 | $-4.7 \times 10^{-2}$ |
| CVS | 0.257 | 0.244 | $1.0 \times 10^{-1}$ |
| F | 0.953 | 0.953 | $-5.3 \times 10^{-3}$ |
| GE | 0.528 | 0.525 | $5.6 \times 10^{-2}$ |
| GM | 0.939 | 0.939 | $-6.8 \times 10^{-3}$ |
| IBM | 0.121 | 0.103 | $1.5 \times 10^{-1}$ |
| K | 0.389 | 0.382 | $7.8 \times 10^{-2}$ |
| LMT | 0.974 | 0.974 | $2.9 \times 10^{-3}$ |
| PG | 0.911 | 0.911 | $1.0 \times 10^{-2}$ |

with respect to a filtration $\mathcal{F}_t \subset \mathcal{F}_\infty = \sigma(\cup_{s\geq 0}\mathcal{F}_s) = \mathcal{F}$. Only simple trades are allowed. Simple trading strategies are denoted by $\theta$ and implicit in $\theta$'s description is a finite set of nonrandom trading times $0 \leq t_1 < t_2 < \cdots < t_k$. The 2-vector of functions $\theta(t) = (\theta_R(t), \theta_E(t))$ has elements which indicate the units held in the risk-free asset and the equity so that the value of the portfolio at time $t$ is $\theta(t) \cdot V(t)$, the ordinary inner product of the vector $\theta$ with $V(t,\omega) = (1, Z_t(\omega))$. The function $\theta$ is $\mathcal{F}_t$-adapted, and, for each $\omega, \theta(t)$ is constant on $t_{i-1} \leq t < t_i$. As in [1] simple trades involve finite arbitrary collections of nonrandom points of time at which trades occur but, in contrast, here there is no fixed "consumption time." Instead, if the trading strategy has its last trading time at $t_k$ as above, then at the last time all is placed in the risk-free cash account so that, at times $t \geq t_k$, $\theta(t) = (\theta(t_k) \cdot V(t_k), 0)$. The subspace $M$ is the linear span of the random variables $\theta \cdot V$, where $\theta$ is a simple trading strategy. It is implicit that $E_P[\|\theta(t_j)\|^2] < \infty$ for each $j = 1, 2, \ldots, k$, an assumption made throughout. A simple trading strategy $\theta$ is self-financing if $\theta(t_{j-1}) \cdot V(t_j) = \theta(t_j) \cdot V(t_j)$ for each $j = 1, 2, \ldots, k$. If a security market model $(M, \pi)$ is viable and if $\theta$ is self-financing, then

$\theta(0) \cdot V(0) = \pi(m)$ for $m = \lim_{t\to\infty}\theta(t) \cdot V(t) = \theta(t_k) \cdot V(t_k)$. The existence of an equivalent martingale measure is asserted in Lemma 2.2 and its proof can be carried out as by Harrison and Kreps.

**Lemma 2.2.** *If $(M, \pi)$ is viable with respect to $X = L_2(\Omega, \mathcal{F}, P)$, then there is an equivalent measure $P^*$ with $dP^*/dP \in X$ and under this measure $Z_t$ is a martingale with respect to $\mathcal{F}_t$.*

The risk-free security of concern here has instantaneous rate $r(t)$ at time $t$, the price process $S_t$ solves (3.1), and the corresponding trades relative to the process of real interest here, $\widetilde{V}_t = (e^{\int_0^t r(s)ds}, S_t)$, can be obtained by taking $Z_t = e^{-\int_0^t r(s)ds}S_t$ and $V_t = e^{-\int_0^t r(s)ds}\widetilde{V}_t$ (see [1, Section 7]). Under a viable pricing system, it follows that $e^{-\int_0^t r(s)ds}S_t$ is a martingale with respect to $\mathcal{F}_t$. Thus, pricing for a final transaction time $T$ is given by $\pi(m) = E_{P^*}[e^{-\int_0^T r(s)ds}m]$.

## 3. Price Process under $P$ and $P^*$

It is assumed that under $P$ the price process $S_t(\omega)$, $t \ge 0$ solves the SDE

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dB_t \tag{3.1}$$

analogous to (1.1) but with deterministically varying $r(\cdot)$, $\mu(\cdot)$, and $\sigma(\cdot)$ subject to the following assumptions:

(A1) the functions $\mu(\cdot)$ and $r(\cdot)$ are continuous on $[0, \infty)$ and $\sigma(\cdot)$ is absolutely continuous with a derivative bounded on compact intervals,

(A2) for some $0 < \sigma_L < \sigma_U < \infty$ and all $s \ge 0$, $\sigma_L \le \sigma(s) \le \sigma_U$ holds,

(A3) the risk premium $\rho(s) = \sigma^{-1}(s)(\mu(s) - r(s))$ is uniformly bounded: for some $\rho_U < \infty$ and all $s > 0$, $|\rho(s)| \le \rho_U$,

(A4) the risk free rate is uniformly bounded: for some $0 < r_U < \infty$ and all $s \ge 0$, $|r(s)| \le r_U$.

The solution $S$ to (3.1) at time $T$ given its value at time $0 \le t < T$ can be written explicitly as

$$S_T = S_t \exp\left\{\int_t^T \mu(s)ds - \frac{1}{2}\int_t^T \sigma^2(s)ds + \int_t^T \sigma(s)dB_s\right\}. \tag{3.2}$$

Under the measure $P^*$ the continuous discounted value process

$$X_t = S_t \exp\left\{-\int_0^t r(s)ds\right\} \tag{3.3}$$

is a martingale, so by Theorems 4.2 (Chapter 3) of Karatzas and Shreve [3] and IX.5.3 of Doob [4], for example, provided the quadratic variation process $\langle X\rangle_t(\omega)$ is an absolutely continuous function of $t$ for $P^*$-almost every $\omega$ with a nonzero derivative, there exists

$W_t$ a Wiener process under $P^*$ and a measurable $\mathcal{F}_t$-adapted process $\Phi(t,\omega)$ such that $d(e^{-\int_0^t r(s)ds}S_t(\omega)) = \Phi(t,\omega)dW_t(\omega)$. Moreover $\langle X\rangle_t(\omega) = \int_0^t \Phi^2(s,\omega)ds$. It also follows by Lemma 3.1 that under the equivalence of $P$ and $P^*$ one has $\Phi^2(s,\omega) = X_s^2(\omega)\sigma^2(s)$.

**Lemma 3.1.** *Let conditions (A) be satisfied, and let $S_t$ satisfy (3.1), where $B_t$ is a standard Brownian motion under $P$. Suppose that $X_t = e^{-\int_0^t r(s)ds}S_t$ is a martingale under the equivalent measure $P^*$. Then, under the measure $P^*$, $\langle X\rangle_t(\omega) = \int_0^t X_s^2(\omega)\sigma^2(s)ds$. See proof in the Appendix.*

By Ito's formula, $d(e^{-\int_0^t r(s)ds}S_t) = -r(t)e^{-\int_0^t r(s)ds}S_t dt + e^{-\int_0^t r(s)ds}dS_t$, so

$$dS_t = r(t)S_t dt + e^{\int_0^t r(s)ds}\Phi(t,S_t)dW_t. \tag{3.4}$$

It follows from Lemma 3.1 that, under the equivalence of $P$ and $P^*$, one has

$$\Phi(t,\omega) = e^{-\int_0^t r(s)ds}S_t(\omega)\sigma(t) \tag{3.5}$$

and hence that

$$dS_t = r(t)S_t dt + \sigma(t)S_t dW_t. \tag{3.6}$$

Thus, if $T > t$, then under $P^*$

$$S_T = S_t \exp\left\{\int_t^T r(s)ds - \frac{1}{2}\int_t^T \sigma^2(s)ds + \int_t^T \sigma(s)dW_s\right\}. \tag{3.7}$$

It has been shown that under the equivalent martingale measure $P^*$ the price process satisfies on $[0,\infty)$ an SDE with a standard Brownian motion $W_t$, $t \geq 0$, the same volatility as the empirical one, and a drift coincident with the risk-free rate. How does this result on $[0,\infty)$ relate to the results of [1] on $[0,T]$, $0 < T < \infty$?

Harrison and Kreps show that there is a probability measure $P_T^*$ equivalent to $P_T$ under which $e^{-\int_0^t r(s)ds}S_t$, $t \leq T$ is a martingale, where $P_T$ is $P$ restricted to $\mathcal{F}_T$. Karatzas and Shreve [3, Section 3.5A] (see also [5, Section 1.7, Proposition 7.4]), show that there is a probability measure $\widetilde{P}^*$ on $(\Omega,\mathcal{F}_\infty)$ with the property that for every $T > 0$ the measure restricted to $\mathcal{F}_T$ is $P_T^*$. They point out, however, that generally $\widetilde{P}^*$ need not be equivalent to $P$. That is, in fact the case if $\int_0^\infty \rho^2(s)ds = \infty$ as can be seen from Proposition 3.2 (see also the remark preceding Example 7.6, Chapter 1, of [5]). In that case, not only is our measure $P^*$ not obtained from the arguments of Harrison and Kreps, it is singular with respect to one that is. The scope of our main result below would be more limited if the finiteness of the integral were assumed.

**Proposition 3.2.** *If the measure $\widetilde{P}^*$ satisfies $\zeta = d\widetilde{P}^*/dP \in L_2(\Omega,\mathcal{F},P)$ and, for each $t > 0$, $\widetilde{P}^*|_{\mathcal{F}_t} = P_t^*$ then $\widetilde{P}^* \perp P^*$ if $\int_0^\infty \rho^2(s)ds = \infty$. See proof in the Appendix.*

## 4. Relationship between $\mu$ and $r$

Let $\alpha > 0$ be arbitrary. Suppose that there is an essentially disjoint collection $\mathcal{C}_\alpha = \{I_j\}_{j\geq 1}$ of subintervals of the real line satisfying

$$\bigcup_{j\geq 1} I_j \subset \{s \geq 0 : |\mu(s) - r(s)| \geq \alpha\}. \tag{4.1}$$

Under the assumption that the functions $\mu$ and $r$ are continuous observe that, unless $\mu(s) = r(s)$ for all $s > 0$, for $\alpha > 0$ sufficiently small, the collections $\mathcal{C}_\alpha$ are nonempty. Denote the Lebesgue measure of an interval $I$ by $m(I)$ and fix $\alpha > 0$ for which $\mathcal{C}_\alpha$ is nonempty. The equities price process under the actual probability measure $P$ is given in (3.2) and if the pricing system is viable then under $P^*$ the same process is given by (3.7). Lemmas 4.2, 4.3, and 4.4 are used in the proof of Lemma 4.1, the key to our main result.

**Lemma 4.1.** *Under the assumptions (A) and $P[S_0 = s_0] = 1$ for some $s_0 > 0$, if $(M, \pi)$ is a viable pricing system for $M$, the class of marketable claims under simple trading strategies, then for every $\gamma > 0$, the set of indices $j$ for which $m(I_j) \geq \gamma$ is at most finite.*

*Proof.* Suppose that $(M, \pi)$ is viable and the claim is not true. Then, for some $\gamma > 0$, there is an infinite collection of such intervals $[a_j, b_j] \subset I_{i_j}$, $j = 1, 2, \ldots$ with $b_j - a_j \geq \gamma$. Writing

$$|\mu(s) - r(s)| = (\mu(s) - r(s))_+ - (\mu(s) - r(s))_-, \tag{4.2}$$

assume without loss of generality that thereon $|\mu(s) - r(s)| = (\mu(s) - r(s))_+ \geq \alpha$. Then, also without loss of generality, one can assume that there are points $d_j < u_j$, where $a_{i_j} \leq d_j < u_j \leq b_{i_j}$, and $\int_{d_j}^{u_j} (\mu(s) - r(s))ds = \beta/2 = c$, $j = 1, 2, \ldots$, where $\beta = \gamma\alpha$.

Consider a sequence of trading strategies $\theta_n$. Under the simple strategy $\theta_n$, buy $(nS_{d_1})^{-1} e^{\int_0^{u_1} r(s)ds}$ shares of the equity at time $d_1$ to spend $n^{-1} e^{\int_0^{u_1} r(s)ds}$ units. At this time also sell $n^{-1} e^{\int_0^{u_1} r(s)ds}$ of the risk free security to spend $-n^{-1} e^{\int_0^{u_1} r(s)ds}$ at time $d_1$. Cash flow at time $d_1$ is then 0. At time $t = u_1$, sell the equity to spend $-e^{\int_0^{u_1} r(s)ds} S_{u_1}/nS_{d_1}$ and redeem the bond to spend $n^{-1} e^{\int_0^{u_1} r(s)ds} e^{\int_{d_1}^{u_1} r(s)ds}$. "Cash" on hand at this stage is

$$\frac{e^{\int_0^{u_1} r(s)ds}}{n} \left( \frac{S_{u_1}}{S_{d_1}} - e^{\int_{d_1}^{u_1} r(s)ds} \right). \tag{4.3}$$

Invest it in the risk-free security. At times $d_2, d_3, \ldots, d_n$, repeat this, buying at time $d_j$, $(nS_{d_j})^{-1} e^{\int_0^{u_j} r(s)ds}$ shares of the equity and selling $n^{-1} e^{\int_0^{u_j} r(s)ds}$ of the risk-free security to spend a total of 0. At time $u_j$, sell the shares to obtain $-e^{\int_0^{u_j} r(s)ds} S_{u_j}/nS_{d_j}$ and redeem the bond to spend $n^{-1} e^{\int_0^{u_j} r(s)ds} e^{\int_{d_j}^{u_1} r(s)ds}$ and invest the "cash" in the risk-free security. At time $u_n$, there will be an amount

$$C_n = n^{-1} \sum_{j=1}^{n} e^{\int_{u_j}^{u_n} r(s)ds} \times e^{\int_0^{u_j} r(s)ds} \left( \frac{S_{u_j}}{S_{d_j}} - e^{\int_{d_j}^{u_j} r(s)ds} \right). \tag{4.4}$$

The price of this will be

$$\pi(C_n) = E_{P^*}\left[e^{-\int_0^{u_n} r(s)ds} C_n\right] = E_{P^*}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{S_{u_j}}{S_{d_j}} - e^{\int_{d_j}^{u_j} r(s)ds}\right)\right]. \tag{4.5}$$

Under the geometric Brownian motion model (3.1), the term inside the expectation is $n^{-1}\sum_{j=1}^{n} Y_j$, where

$$Y_j = e^{\int_{d_j}^{u_j}(\mu(s)-\sigma^2(s)/2)ds+\int_{d_j}^{u_j}\sigma(s)dB_s} - e^{\int_{d_j}^{u_j} r(s)ds}, \tag{4.6}$$

an average of independent, but not identically distributed, random variables. By Lemma 4.2, there is a subsequence $n_k$ and an $L > 0$ such that

$$\frac{1}{n_k}\sum_{j=1}^{n_k} e^{\int_{d_j}^{u_j} r(s)ds} \longrightarrow L \tag{4.7}$$

and, under $P$,

$$J_k = V_{n_k} = \frac{1}{n_k}\sum_{i=1}^{n_k} Y_i \xrightarrow{p} L(e^c - 1). \tag{4.8}$$

Since a sequence converging in probability has a subsequence which converges almost surely, there is a subsequence $k'$ such that $J_{k'} \overset{\text{a.s.}\,P}{\longrightarrow} L(e^c - 1)$. By equivalence of $P$ and $P^*$, that convergence is also almost surely $P^*$. By Lemma 4.4, the sequence $J_k$ is uniformly integrable under $P^*$, so expectations converge (see [6, Theorem 5.4]) and one concludes that $E_{P^*}[J_{k'}] \longrightarrow L(e^c - 1)$. On the other hand, by Lemma 4.3, $E_{P^*}[(n_k)^{-1}\sum_{i=1}^{n_k} Y_i] = 0$ for every $k$. It follows that $c = 0$, a contradiction. □

**Lemma 4.2.** *Under conditions (A), there is a subsequence $n_k$ and $L > 0$ such that*

$$\begin{aligned} &\frac{1}{n_k}\sum_{j=1}^{n_k} e^{\int_{d_j}^{u_j} r(s)ds} \longrightarrow L, \\ &\frac{1}{n_k}\sum_{j=1}^{n_k} Y_j \xrightarrow{p} L(e^c - 1) \end{aligned} \tag{4.9}$$

*(see proof in the Appendix).*

**Lemma 4.3.** $E_{P^*}[Y_j] = 0$ *(see proof in the Appendix).*

**Lemma 4.4.** *Under conditions (A) the sequence $n^{-1}\sum_{j=1}^{n} Y_j$ is uniformly integrable under $P^*$ (see proof in the Appendix).*

**Lemma 4.5.** *If $\mu(\cdot)$ and $r(\cdot)$ are constant and if the model is viable, then $r = \mu$.*

*Proof.* Suppose that $|r - \mu| = d > 0$. Then, for $0 < \alpha < d$, and any $\gamma > 0$, essentially disjoint intervals $I$ in $\mathcal{C}_\alpha$ can be chosen in such a way that there is an infinite collection satisfying $m(I) \geq \gamma$. By Lemma 4.1 this violates the viability of the model.

The next most interesting case is when

$$\mu(s) - r(s) = \rho\sigma(s), \tag{4.10}$$

where $\rho$, the possibly varying risk premium, is assumed constant. □

**Lemma 4.6.** *Under conditions (A), if the pricing system $(M, \pi)$ is viable and if (4.10) holds for all $s \geq 0$, then $\rho = 0$.*

*Proof.* Assume that $\rho \neq 0$, and let $0 < \alpha < |\rho|\sigma_L$. Then, for any $\gamma > 0$ essentially disjoint intervals in $\mathcal{C}_\alpha$ can be chosen in such a way that there is an infinite collection, contradicting Lemma 4.1 unless $\rho = 0$.

Returning to the market expressed in terms of $V(t)$, define the functional $\psi$ on $X = L_2(\Omega, \mathcal{F}, P)$, where $P = P_\mu$ is the empirical measure under which $S_t$ satisfies (3.1), by $\psi(x) = \int x(\omega)dP(\omega)$. Our interest in the following theorem centers on the pricing system defined on $M$ by $\pi(m) = \int m(\omega)dP_r(\omega)$, and $P_r$ is the measure corresponding to the process solving (3.6). □

**Theorem 4.7.** *For the Black-Scholes model on $[0, \infty)$ ($r = 0, \mu$, and $\sigma$ are constants), the pricing system $(M, \pi)$ is viable with respect to X if and only if $\mu = 0$.*

*Proof.* By Lemma 4.5 it is known that, if the system $(M, \pi)$ is viable then $r = \mu$. It suffices to show that, if $r = \mu$, then the pricing system given by $\pi$ is viable. According to Lemma 2.1, it suffices to show, as it plainly does here, that $\psi$ extends $\pi$. □

## 5. Empirical Considerations

For these considerations to have relevance to real data, equities prices should be adequately modeled as solutions to SDE (3.1). For roughly a century, models in agreement with (3.1) [7–10] have appeared in the literature, and we will assume this model here. Statistical tests of $r = \mu$, assuming the Black-Scholes model over a suitably brief time span, are then employed on a small set of data. The results found in Tables 1 and 2 are consistent with the truth of the hypothesis of equality.

### *5.1. UMPU Test*

Assuming the model (3.1) with $\mu$ and $r$ constant, some tests of $\mu = r$ are developed here based on readily available daily log return data, $R_t = \ln(S_{t+1}/S_t)$, where $R_t$ are i.i.d. $N(\mu - \xi^2/2, \xi^2)$, and applied to different underlying equities at various historical times.

Let $X_i = R_i - r$. Setting $\eta = E[X_i]$ and $\xi^2 = \mathrm{Var}(X_i)$, the hypothesis pair $H_0' : \mu = r$ versus $H_a' : \mu \neq r$ becomes $H_0 : \eta + \xi^2/2 = 0$ versus $H_a : \eta + \xi^2/2 \neq 0$, which will be tested based on observing $X_1, \ldots, X_n$ i.i.d. $N(\eta, \xi^2)$. Writing

$$f\left(\mathbf{x} \mid \eta, \xi^2\right) = \frac{1}{(2\pi\xi^2)^{n/2}} \exp\left\{-\frac{1}{2\xi^2}\sum_{i=1}^{n}(x_i - \eta)^2\right\}, \tag{5.1}$$

one has that $(\sum X_i^2, \sum X_i)$ are joint sufficient statistics for $(-1/2\xi^2, \eta/\xi^2) = (\theta_1, \theta_2)$. It can be seen, based upon the theory of tests of a single parameter from an exponential family (see [11]), that a uniformly most powerful unbiased (UMPU) test $\phi_\alpha$ of size $\alpha$ exists for testing the hypothesis of interest here,

$$H_0 : \theta_2 = -\frac{1}{2} \text{ versus } H_a : \theta_2 \neq -\frac{1}{2}. \tag{5.2}$$

Furthermore, denoting by $P_\alpha(\theta)$ the power function of the test $\phi_\alpha$, one has the following result.

**Lemma 5.1.** *The test which rejects $H_0$ if $|\tau_n| > z_{\alpha/2}$, where*

$$\tau_n = \frac{\sqrt{n}\left(\overline{X}_n/S_n^2 + 1/2\right)}{\sqrt{1/S_n^2 + 3/2 + S_n^2/4}}, \tag{5.3}$$

*satisfies for all $\theta$ in $H_0$*

$$\lim_{n\to\infty} P[|\tau_n| > z_{\alpha/2} \mid \theta] = P_\alpha(\theta). \tag{5.4}$$

*See proof in the Appendix.*

The results of testing hypothesis (5.2) for various equities at varying times are found in Table 1 and for a fixed set of times in Table 2. The risk-free rates were determined from the US treasury for the corresponding time spans at each initial time and, in the latter case, reveal that none of the drift rates differ significantly (at $\alpha = 0.05$) from $r = 2.0113 \times 10^{-4}$, the daily rate for 26-week treasury bills during that stretch. In the tables, the entry Effect Size is $\widehat{\eta}/\widehat{\sigma} + \widehat{\sigma}/2$, a rough estimate of $\mu - r$ in terms of the volatility.

It is perhaps surprising that there were no significances especially in the case of C in Table 1 and AAPL in Table 2, the former because of the long time span and the approximation assuming a fixed risk-free rate in a world in which it is constantly changing, and the latter because on June 29, 2007 Apple introduced its first iPhone a major milestone in the company's rising fortunes. The former is in line with the sample size and observed effect size while the latter is consistent with a fundamental change.

### 5.2. Likelihood Ratio Tests

Likelihood ratio tests present an alternative possibility. They are known to be optimal (see, e.g., [12]) in large samples. According to Wilks' theorem, the null hypothesis should be rejected when $-2\ln\lambda(x)$ is too large. Setting

$$w^* = 2\left[\sqrt{1 + \frac{\sum_{j=1}^{n} x_j^2}{n}} - 1\right], \tag{5.5}$$

one has the following.

**Lemma 5.2.** *The likelihood ratio test of* (5.2) *of size* $\alpha$ *rejects* $H_0$ *if the test statistic*

$$\tau = n(\ln w^* - 1) + \frac{1}{w^*}\sum_{j=1}^{n}\left(x_j + \frac{w^*}{2}\right)^2 - n\ln\left(\frac{\sum_{j=1}^{n}(x_j - \overline{x})^2}{n}\right) \tag{5.6}$$

*exceeds* $\chi_\alpha^2$. *See proof in the Appendix.*

$P$ values are found in Tables 1 and 2 both for the UMPU test and for the LRT and one observes that they are quite close and, again, there are no significances at $\alpha = 0.05$.

## Appendix

*Proof of Lemma 3.1.* Fix $t > 0$ and $0 < t_1 < \cdots < t_n = t$ an equidistant partition of the interval $[0,t]$. To ease notation let us denote $X_j := X_{t_j}$, $A_t := \int_0^t (\mu(s) - r(s) - (1/2)\sigma^2(s))ds$, $M_t := \int_0^t \sigma(s)dB_s$, and correspondingly $A_j := A_{t_j}$, $M_j := M_{t_j}$. Also, denote $\Delta A_j := A_j - A_{j-1}$, $\Delta M_j := M_j - M_{j-1}$.

Notice that, under $P$, $M_t$ is a normal variable with mean zero and variance $\int_0^t \sigma^2(s)ds$. Let us evaluate

$$\begin{aligned}
\left|\sum_{j=1}^{n}|X_j - X_{j-1}|^2 - \int_0^t X_s^2\sigma^2(s)ds\right| &= \left|\sum_{j=1}^{n}\left(X_{j-1}^2\left[e^{\Delta A_j + \Delta M_j} - 1\right]^2 - \int_{t_{j-1}}^{t_j} X_s^2\sigma^2(s)ds\right)\right| \\
&\le \left|\sum_{j=1}^{n}\left(X_{j-1}^2\left[e^{\Delta A_j + \Delta M_j} - 1\right]^2 - X_{j-1}^2(\Delta A_j + \Delta M_j)^2\right)\right| \\
&\quad + \left|\sum_{j=1}^{n}\left(X_{j-1}^2(\Delta A_j + \Delta M_j)^2 - \int_{t_{j-1}}^{t_j} X_s^2\sigma^2(s)ds\right)\right| \\
&\le \left|\sum_{j=1}^{n}\left(X_{j-1}^2\left[e^{\Delta A_j + \Delta M_j} - 1\right]^2 - X_{j-1}^2(\Delta A_j + \Delta M_j)^2\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\sum_{j=1}^{n} X_{j-1}^2 (\Delta A_j)^2\right| + \left|\sum_{j=1}^{n} 2X_{j-1}^2 \Delta A_j \cdot \Delta M_j\right| \\
&+\left|\sum_{j=1}^{n}\left( X_{j-1}^2 (\Delta M_j)^2 - \int_{t_{j-1}}^{t_j} X_s^2 \sigma^2(s)ds\right)\right| \\
:=\ & T_{1,n} + T_{2,n} + T_{3,n} + T_{4,n}.
\end{aligned}
\tag{A.1}
$$

Since $\lim_{x\to 0} x^{-3}[(e^x-1)^2 - x^2] = 1$, for $x$ sufficiently close to zero $|(e^x-1)^2 - x^2| \le 2|x^3|$. Moreover, $\max_{j\le n}|\Delta A_j| \le (t/n)(|\mu_U| + |r_U| + (1/2)\sigma_U^2) := (t/n)C(\mu, r, \sigma) \to 0$, and since $\max_{j\le n} E_P(\Delta M_j)^2 = \max_{j\le n}\int_{t_{j-1}}^{t_j}\sigma^2(s)ds \le (t/n)\sigma_U^2 \to 0$, there exists a subsequence $n_k$ on which convergence holds almost surely. In order to simplify notation, we assume that $n_k = n$. Then, for $n$ large enough and for almost all $\omega$ we have

$$
\begin{aligned}
T_{1,n} &\le 2\sum_{j=1}^{n} X_{j-1}^2 \left|\Delta A_j + \Delta M_j\right|^3 \\
&\le 2\sum_{j=1}^{n} X_{j-1}^2 \left(\left|\Delta A_j\right|^3 + 3\left|\Delta A_j\right|^2\left|\Delta M_j\right| + 3\left|\Delta A_j\right|\left|\Delta M_j\right|^2 + \left|\Delta M_j\right|^3\right) \\
&\le 2\sum_{j=1}^{n} X_{j-1}^2 \Big(C(\mu, r, \sigma)^3\left(t^3/n^3\right) + 3C(\mu, r, \sigma)^2\left(t^2/n^2\right)\left|\Delta M_j\right| \\
&\qquad + 3C(\mu, r, \sigma)(t/n)\left|\Delta M_j\right|^2 + \left|\Delta M_j\right|^3\Big).
\end{aligned}
\tag{A.2}
$$

We have

$$
E_P\left|\Delta M_j\right| = \sqrt{\frac{2}{\pi}}\sqrt{\int_{t_{j-1}}^{t_j}\sigma^2(s)ds} \le \sqrt{\frac{2}{\pi}}\sigma_U\frac{t^{1/2}}{n^{1/2}},
\tag{A.3}
$$

$E_P(|\Delta M_j|^2) \le (t/n)\sigma_U^2$, and $E_P(|\Delta M_j|^3) = (2\sqrt{2}/\sqrt{\pi})(\int_{t_{j-1}}^{t_j}\sigma^2(s)ds)^{3/2} \le (t^{3/2}/n^{3/2})\sigma_U^2$. From conditions (A) it follows that there exists $K > 0$ such that

$$
E_P X_{j-1}^2 = S_0^2 e^{2A_{j-1}} e^{2\int_0^{t_{j-1}}\sigma^2(s)ds} \le K.
\tag{A.4}
$$

Moreover, using that $X_{j-1}$ is independent of $\Delta M_j$, we have

$$E_P(T_{1,n}) \leq 2K\left(C(\mu,r,\sigma)^3\frac{t^3}{n^2}+C_1(\mu,r,\sigma)\frac{t^{5/2}}{n^{3/2}}+C_2(\mu,r,\sigma)\frac{t^2}{n}+C_3(\mu,r,\sigma)\frac{t^{3/2}}{n^{1/2}}\right)\longrightarrow 0, \tag{A.5}$$

and it follows that $T_{1,n}$ has a subsequence which converges $P$-almost surely to zero.

Let us now evaluate

$$E_PT_{2,n} \leq \sum_{j=1}^{n}KC^2(\mu,r,\sigma)\frac{t^2}{n^2}=KC^2(\mu,r,\sigma)\frac{t^2}{n}\longrightarrow 0, \tag{A.6}$$

therefore $T_{2,n}$ has a subsequence that converges $P$-almost surely to zero.

$$E_PT_{3,n} \leq \sum_{j=1}^{n}C_4(\mu,r,\sigma)\frac{t^{3/2}}{n^{3/2}}=C_4(\mu,r,\sigma)\frac{t^{3/2}}{n^{1/2}}\longrightarrow 0, \tag{A.7}$$

and therefore $T_{3,n}$ has a subsequence convergent to zero $P$-almost surely

For the last term $T_4$, we write

$$\begin{aligned}|T_{4,n}| &\leq \left|\sum_{j=1}^{n}\left\{\left(\int_{t_{j-1}}^{t_j}X_{j-1}dM_s\right)^2-\left(\int_{t_{j-1}}^{t_j}X_sdM_s\right)^2\right\}\right|\\ &\quad+\left|\sum_{j=1}^{n}\left\{\left(\int_{t_{j-1}}^{t_j}X_sdM_s\right)^2-\int_{t_{j-1}}^{t_j}X_s^2\sigma^2(s)ds\right\}\right|\\ &:= T_{5,n}+T_{6,n}.\end{aligned} \tag{A.8}$$

Since $\int_0^t X_sdM_s$ is a continuous $P$ square integrable martingale, by Theorem 5.8 of [3, Chapter 1] it follows that $\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}X_sdM_s)^2 \to \langle\int_0^t X_sdM_s\rangle$ in probability $P$. By [13, Proposition 2.3, Chapter 2], $\langle\int_0^t X_sdM_s\rangle=\int_0^t X_s^2\sigma^2(s)ds$; therefore, $T_{6,n}$ converges to zero $P$-almost surely on a subsequence.

As for $T_{5,n}$, from Hölder's inequality we have

$$E|T_{5,n}| \leq \sum_{j=1}^{n}\left[E\left(\int_{t_{j-1}}^{t_j}(X_{j-1}-X_s)dM_s\right)^2\right]^{1/2}\left[E\left(\int_{t_{j-1}}^{t_j}(X_{j-1}+X_s)dM_s\right)^2\right]^{1/2}. \tag{A.9}$$

By Itô's isometry,

$$\begin{aligned}
E\left(\int_{t_{j-1}}^{t_j}(X_{j-1}-X_s)dM_s\right)^2 &= E\left(\int_{t_{j-1}}^{t_j}(X_{j-1}-X_s)^2\sigma^2(s)ds\right)\\
&= \int_{t_{j-1}}^{t_j} E(X_{j-1}-X_s)^2\sigma^2(s)ds\\
&= \int_{t_{j-1}}^{t_j} E(X_{j-1})^2 E\left(\frac{X_s}{X_{j-1}}-1\right)^2\sigma^2(s)ds\\
&= S_0^2\int_{t_{j-1}}^{s} e^{2A_{j-1}+2\int_0^{t_j}\sigma^2(u)du}\left[e^{2A_s-2A_{j-1}+2\int_{t_{j-1}}^{t_j}\sigma^2(u)du}\right.\\
&\qquad\left.-2e^{A_s-A_{j-1}+(1/2)\int_{t_{j-1}}^{s}\sigma^2(u)du}+1\right]ds\\
&\le K\int_{t_{j-1}}^{t_j}\left[e^{A_s-A_{j-1}+(1/2)\int_{t_{j-1}}^{s}\sigma^2(u)du}-1\right]^2 ds.
\end{aligned} \tag{A.10}$$

For $n$ large enough

$$\begin{aligned}
\left[e^{A_s-A_{j-1}+(1/2)\int_{t_{j-1}}^{s}\sigma^2(u)du}-1\right]^2 &\sim \left(A_s-A_{j-1}+\frac{1}{2}\int_{t_{j-1}}^{s}\sigma^2(u)du\right)^2\\
&\le C^2(\mu,r,\sigma)(s-t_{j-1})^2.
\end{aligned} \tag{A.11}$$

By a similar argument, $[E(\int_{t_{j-1}}^{t_j}(X_{j-1}+X_s)dM_s)^2]^{1/2} \le C$ for some constant depending on $\mu$, $r$, $\sigma$, and $t$. Then,

$$E|T_{5,n}| \le C\sum_{j=1}^{n}(t_j-t_{j-1})^{3/2} \longrightarrow 0, \tag{A.12}$$

and therefore $T_{5,n}$ has a subsequence converging to zero $P$-almost surely It has been shown that

$$\left|\sum_{j=1}^{n}|X_j-X_{j-1}|^2-\int_0^t X_s^2\sigma^2(s)ds\right| \longrightarrow 0 \tag{A.13}$$

almost surely $P$ on a subsequence, and, since $P\sim P^*$, convergence holds almost surely in $P^*$ as well. Thus, $|\sum_{j=1}^{n}|X_j-X_{j-1}|^2-\langle X\rangle(t)| \to 0$ in $P^*$, and it follows that $\langle X\rangle_t(\omega) = \int_0^t X_s^2(\omega)\sigma^2(s)ds$. □

*Proof of Lemma 4.1.* The random variables $\zeta_t = E[\zeta \mid \mathcal{F}_t]$ constitute a martingale and $\zeta_t$ is the Radon-Nikodym (RN) derivative of the probability measure $\widetilde{P}^*|_{\mathcal{F}_t}$ with respect to $P|_{\mathcal{F}_t}$. By Theorem 4.1 of [4, page 319] since $\zeta_t$ are nonnegative and $E[\zeta_t] = 1$ for all $t$, one has that $\lim_{t\to\infty}\zeta_t = \zeta$ with $P$-probability 1. One has from [1, Theorem 3], that for each $t > 0$, $\zeta_t = e^{\int_0^t \rho(s)dB_s-(1/2)\int_0^t \rho_s^2 ds}$. Then since $\int_0^t \rho(s)dB_s \sim N(0, \int_0^t \rho_s^2 ds)$, one has for any $a$ such that $e^a$ is a continuity point of the distribution of the random variable $\zeta$ that

$$\begin{aligned}P[\zeta_t \geq e^a] &= P\left[\int_0^t \rho(s)dB_s \geq \frac{1}{2}\int_0^t \rho_s^2 ds + a\right]\\ &= P\left[Z \geq \frac{1}{2}\sqrt{\int_0^t \rho_s^2 ds} + \frac{a}{\sqrt{\int_0^t \rho_s^2 ds}}\right].\end{aligned} \tag{A.14}$$

So, unless $\int_0^\infty \rho_s^2 ds < \infty$, one has $P[\zeta \geq e^a] = 0$. But then $P[\zeta < e^a] = 1$ so that choosing $a < 0$ shows that $\zeta$ cannot be the RN derivative of a probability measure absolutely continuous with respect to $P$. □

*Proof of Lemma 4.2.* Existence of the subsequence and nonzero limit $L$ is obvious from assumptions (A) and the sequential compactness of the real line. Since under $P$

$$\int_{d_j}^{u_j} \sigma(s)dB_s \sim N\left(0, \int_{d_j}^{u_j} \sigma^2(s)ds\right) \tag{A.15}$$

and $E_P[e^{\int_{d_j}^{u_j}\sigma(s)dB_s}]$ is just the mgf $\phi(u)$ of this random variable evaluated at $u = 1$, it follows that

$$X_j = e^{\int_{d_j}^{u_j}\mu(s)ds}\left(e^{\int_{d_j}^{u_j}\sigma(s)dB_s - \int_{d_j}^{u_j}\sigma^2(s)/2ds} - 1\right) \tag{A.16}$$

is zero mean under $P$ and that, defining $X_{n,j} = (1/n)X_j$, one has $(1/n)\sum_{j=1}^n (Y_j - E_P[Y_j]) = \sum_{j=1}^n X_{n,j}$, where $Y_j$ are given in (4.6). Under $P$, $X_{n,j}$ are independent across $j$ for each $n$. Let $V_n = (1/n)\sum_{j=1}^n Y_j$, $W_n = (1/n)\sum_{j=1}^n E_P[Y_j]$, and $W_0 = L(e^c - 1)$. Then, $W_n$ are constants converging to $W_0$ along a subsequence and $E_P[(V_n - W_n)^2] = b_n^2$, where $b_n^2 = \sum_{j=1}^n \sigma_{n,j}^2$ and $\sigma_{n,j}^2 = \mathrm{Var}(X_{n,j})$. Setting $\eta_j = \int_{d_j}^{u_j}\sigma^2(s)ds$,

$$E_P\left[e^{k\int_{d_j}^{u_j}\sigma(s)dB_s - k\int_{d_j}^{u_j}\sigma^2(s)/2ds}\right] = e^{((k^2-k)/2)\eta_j}, \tag{A.17}$$

so one has

$$\sigma_{n,j}^2 = \frac{e^{2\int_{d_j}^{u_j}\mu(s)ds}(e^{\eta_j} - 1)}{n^2}. \tag{A.18}$$

In case assumptions (A) hold, then since

$$c = \int_{d_j}^{u_j} \rho(s)\sigma(s)ds \geq \alpha\left(u_j - d_j\right) \tag{A.19}$$

so that $\eta_j \leq \sigma_U^2(u_j - d_j) \leq \sigma_U^2 c/\alpha < \infty$ for all $j$, it follows that $b_n^2$ is $O(n^{-1})$. Therefore

$$\|V_n - W_0\|_P^2 = b_n^2 + 2 \cdot 0 \cdot (W_n - W_0) + (W_n - W_0)^2 \longrightarrow 0 \tag{A.20}$$

on the subsequence for which $W_n \rightarrow W_0$. This convergence implies convergence in probability. □

*Proof of Lemma 4.3.* Under $P^*$, $e^{-\int_0^t r(s)ds}S_t$ is a martingale so that for $0 \leq t < T$ one has

$$e^{-\int_0^t r(s)ds}S_t = E_{P^*}\left[e^{-\int_0^T r(s)ds}S_T \mid S_t\right] = e^{-\int_0^T r(s)ds}S_t e^{\int_t^T \mu(s)ds - \int_t^T \sigma^2(s)/2ds} E_{P^*}\left[e^{\int_t^T \sigma(s)dB_s}\right]. \tag{A.21}$$

Therefore,

$$E_{P^*}\left[e^{\int_t^T \sigma(s)dB_s}\right] = e^{\int_t^T (r(s)-\mu(s))ds} e^{\int_t^T \sigma^2(s)/2ds}. \tag{A.22}$$

Thus,

$$\begin{aligned} E_{P^*}\left[Y_j\right] = \exp&\left\{\int_{d_j}^{u_j} \mu(s)ds - \int_{d_j}^{u_j} \frac{\sigma^2(s)}{2}ds + \int_{d_j}^{u_j} (r(s) - \mu(s))ds + \int_{d_j}^{u_j} \frac{\sigma^2(s)}{2}ds\right\} \\ &- \exp\left\{\int_{d_j}^{u_j} r(s)ds\right\} \end{aligned} \tag{A.23}$$

and the claim follows. □

*Proof of Lemma 4.4.* First consider the distribution of the random variable $V_\sigma = \int_t^T \sigma(s)dB_s$ under $P^*$. Under $P^*$,

$$\int_t^T \sigma(s)dB_s \sim N\left(-\int_t^T \rho(s)\sigma(s)ds, \int_t^T \sigma^2(s)ds\right). \tag{A.24}$$

To see this, compute the MGF of $V_\sigma$ as $\varphi_\sigma(u) = E[e^{uV_\sigma}]$. Since $uV_\sigma = V_{u\sigma}$, by (A.22) one has

$$\varphi_\sigma(u) = \varphi_{u\sigma}(1) = e^{\int_t^T (ur(s)-u\mu(s))ds} e^{\int_t^T (u\sigma(s))^2/2ds} \tag{A.25}$$

and the claim has been established. Next the claim is that

$$E_{P^*}\left[Y_j^2\right] = e^{2\int_{d_j}^{u_j} r(s)ds}\left(e^{\eta_j} - 1\right). \tag{A.26}$$

To see this, set $V_{\sigma,j} = \int_{d_j}^{u_j} \sigma(s)dB_s$ and recall that $\int_{d_j}^{u_j} \rho(s)\sigma(s)ds = c$ and $\int_{d_j}^{u_j} \sigma^2(s)ds = \eta_j$, so that

$$Y_j^2 = \left(e^{\int_{d_j}^{u_j} \mu(s)ds + V_{\sigma,j} - \eta_j/2} - e^{\int_{d_j}^{u_j} r(s)ds}\right)^2. \tag{A.27}$$

Therefore, one has

$$\begin{aligned}
E_{P^*}\left[Y_j^2\right] &= e^{2\int_{d_j}^{u_j} \mu(s)ds - \eta_j} e^{-2c+2\eta_j} - 2e^{\int_{d_j}^{u_j} \mu(s)ds + \int_{d_j}^{u_j} r(s)ds - \eta_j/2} e^{-c+\eta_j/2} + e^{2\int_{d_j}^{u_j} r(s)ds} \\
&= e^{2\int_{d_j}^{u_j} r(s)ds + 2c + \eta_j - 2c} - 2e^{2\int_{d_j}^{u_j} r(s)ds} + e^{2\int_{d_j}^{u_j} r(s)ds} \\
&= e^{2\int_{d_j}^{u_j} r(s)ds}\left(e^{\eta_j} - 1\right).
\end{aligned} \tag{A.28}$$

We do not know that $B_{u_j} - B_{d_j}$ are independent under $P^*$ but by Jensen's inequality

$$\left(\frac{1}{n}\sum_{j=1}^{n} w_j\right)^2 \leq \frac{1}{n}\sum_{j=1}^{n} w_j^2 \tag{A.29}$$

so that

$$E_{P^*}\left[\left(\frac{1}{n}\sum_{j=1}^{n} Y_j\right)^2\right] \leq \frac{1}{n}\sum_{j=1}^{n}\left(e^{\eta_j} - 1\right)\max_{1\leq j\leq n} e^{2\int_{d_j}^{u_j} r(s)ds}, \tag{A.30}$$

and under (A) this is uniformly bounded in $n$. Therefore, the sequence $(1/n)\sum_{j=1}^{n} Y_j$ is uniformly integrable under $P^*$. □

*Proof of Lemma 5.1.* As it is well known, the test $\phi = \phi_\alpha$ will satisfy the derivative condition for exponential class densities

$$E_{\theta_2=-1/2}\left[S_2\phi(S_2, s) \mid S_1 = s\right] = \alpha E_{\theta_2=-1/2}\left[S_2 \mid S_1 = s\right] \tag{A.31}$$

and the size condition

$$E_{\theta_2=-1/2}\left[\phi(S_2, s) \mid S_1 = s\right] = \alpha, \tag{A.32}$$

where the expectations refer to the conditional distribution of $S_2 = \sum X_i$ given $S_1 = \sum X_i^2$. By the theory of Lehmann [11], the UMPU test can be based upon the statistic

$$\frac{\overline{X}_n}{S_n^2} + \frac{1}{2} \tag{A.33}$$

since it is a monotonic function of $\sum X_i$ given $\sum X_i^2$. The latter follows simply from

$$\frac{\partial}{\partial v}\frac{v}{nt - v^2} = \frac{nt + v^2}{\left(nt - v^2\right)^2} > 0. \tag{A.34}$$

There is no closed-form expression for the conditional distribution of $\sum X_i$ given $\sum X_i^2$ under the condition $\theta_2 = -1/2$ but a large sample approximation can be made as follows.

It is shown below that under the hypothesis $H_0 : \eta/\xi^2 = -1/2$, as the sample size $n \to \infty$,

$$\sqrt{n}\left(\frac{\overline{X}_n}{S_n^2} + \frac{1}{2}\right) \xrightarrow{L} N(0, v), \tag{A.35}$$

where $v = (\xi^4 + 6\eta^2\xi^2 + 4\eta^4)/\xi^6$. Therefore, by Slutsky's theorems, a test suggests itself; namely, reject $H_0$ if

$$\frac{\sqrt{n}\left(\overline{X}_n / S_n^2 + 1/2\right)}{\sqrt{1/S_n^2 + 6\overline{X}^2/S_n^4 + 4\overline{X}^4/S_n^6}} \tag{A.36}$$

exceeds in absolute value the upper $\alpha/2$ cutoff of the standard normal. That test, which is more complicated than the one given in (5.3), would have asymptotically the same power function as the UMPU test, but it is sufficient here to operate under the null hypothesis. Under the null hypothesis one has $v = 1/\xi^2 + 3/2 + \xi^2/4$ so a suitable test statistic is given by (5.3). The claimed asymptotic distribution in (A.35) is verified. It is well known that with

$$\Sigma = \begin{pmatrix} \xi^2 & 0 \\ 0 & 2\xi^4 \end{pmatrix} \tag{A.37}$$

one has

$$\sqrt{n}\begin{bmatrix} m_{1,n} - \eta_1 \\ m_{2,n} - \eta_2 \end{bmatrix} \xrightarrow{L} N_2(0, \Sigma). \tag{A.38}$$

Here $m_{i,n} = (1/n)\sum_{j=1}^{n} X_j^i$ and $\eta_i = E[X^i]$. With $f(u, v) = u/(v - u^2) + 1/2$, $f(\eta_1, \eta_2) = (\eta/\xi^2) + (1/2)$ and the latter, under the null hypothesis, is 0. Hence, by the $\delta$-method and under the null hypothesis

$$\sqrt{n}\left(f(m_{1,n}, m_{2,n}) - 0\right) \xrightarrow{L} N\left(0, \nabla f' \Sigma \nabla f\right), \tag{A.39}$$

where

$$\nabla f' = \left(f_u(\eta_1, \eta_2), f_v(\eta_1, \eta_2)\right) = \left(\frac{\eta_2 + \eta_1^2}{\xi^4}, -\frac{\eta_1}{\xi^4}\right). \tag{A.40}$$

Upon simplification, the claim has been verified. □

*Proof of Lemma 5.2.* Write the maximum of the density under $H_0$ as $N(u, v)$, denote the denominator $D$ which is the unrestricted maximum similarly, and find

$$\lambda(x) = \frac{\sup_{u+w/2=0} f(x \mid u, w)}{\sup_{u,w>0} f(x \mid u, w)} = \frac{N}{D}. \tag{A.41}$$

Since for $b > 0$ and $v > 0$ it is the case that $\max_{t>0} t^{-b} e^{-v/t} = (v/b)^{-b} e^{-b}$ attained at $t = v/b$, one has

$$D = \left[e 2\pi \frac{\sum_{j=1}^{n} (x_j - \overline{x})^2}{n}\right]^{-n/2}. \tag{A.42}$$

For the numerator $\max_{u+w/2=0} \ln f(x \mid u, w)$ is sought as is the location of the maximum. The location $(u^*, w^*)$ is therefore where

$$\max_{u+w/2=0} \left[-\frac{n}{2} \ln w - \frac{1}{2w} \sum_{j=1}^{n} (x_j - u)^2\right] \tag{A.43}$$

is attained. More simply, $u^* = -w^*/2$, where $w^*$ maximizes

$$h(w) = -\frac{n}{2} \ln w - \frac{1}{2w} \sum_{j=1}^{n} \left(x_j + \frac{w}{2}\right)^2. \tag{A.44}$$

Thus,

$$\frac{\partial h}{\partial w} = \frac{-n}{2w} + \frac{\sum_{j=1}^{n} x_j^2}{2w^2} - \frac{n}{8} \tag{A.45}$$

and the derivative is zero at

$$w^* = \frac{n \pm \sqrt{n^2 + n \sum_{j=1}^{n} x_j^2}}{-n/2} \tag{A.46}$$

so that taking account of the proper sign and that the derivative is zero at the maximum, (5.5), the LRT rejects $H_0$ if

$$-2\left[-\frac{n}{2}\ln w^{*}-\frac{1}{2w^{*}}\sum_{j=1}^{n}\left(x_j+\frac{w^{*}}{2}\right)^{2}+\frac{n}{2}\left(\ln e+\ln\left(\frac{\sum_{j=1}^{n}\left(x_j-\overline{x}\right)^{2}}{n}\right)\right)\right] \tag{A.47}$$

exceeds the upper $\alpha$ cutoff of a chi square with 1 degree of freedom. Simplifying, the test statistic is that in (5.6). □

## References


[1] J. M. Harrison and D. M. Kreps, "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory*, vol. 20, no. 3, pp. 381–408, 1979.

[2] D. M. Kreps, "Arbitrage and equilibrium in economies with infinitely many commodities," *Journal of Mathematical Economics*, vol. 8, no. 1, pp. 15–35, 1981.

[3] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, vol. 113 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1991.

[4] J. L. Doob, *Stochastic Processes*, John Wiley & Sons, New York, NY, USA, 1953.

[5] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, vol. 39 of *Applications of Mathematics*, Springe, New York, NY, USA, 1998.

[6] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, NY, USA, 1968.

[7] P. A. Samuelson, "Proof that properly anticipated prices fluctuate randomly," *Industrial Management Review*, vol. 6, pp. 41–49, 1956.

[8] L. Bachelier, *Théorie de la Spéculation. Théorie mathématique du jeu. [Mathematical theory of games]*, Les Grands Classiques Gauthier- Villars. [Gauthier-Villars Great Classics], Jacques Gabay, Sceaux, France, 1995.

[9] M. F. M. Osborne, "Brownian motion in the stock market," *Operations Research*, vol. 7, pp. 145–173, 1959.

[10] P. A. Samuelson, "Proof that properly discounted present values of assets vibrate randomly," *The Bell Journal of Economics and Management Science*, vol. 4, no. 2, pp. 369–374, 1973.

[11] E. L. Lehmann, *Testing Statistical Hypotheses*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, New York, NY, USA, 2nd edition, 1986.

[12] P. J. Bickel and K. A. Doksum, *Mathematical Statistics: Basic Ideas and Selected Topics*, vol. I, Prentice Hall, Upper Saddle River, NJ, USA, 2nd edition, 2000.

[13] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, vol. 24 of *North-Holland Mathematical Library*, North-Holland Publishing, Amsterdam, The Netherlands, 2nd edition, 1989.