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ON EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE EQUATION OF MOTION FOR CONSTRAINED MECHANICAL SYSTEMS

FIRDAUS E. UDWADIA*

Mechanical Engineering, Civil Engineering and Decision Systems, University of Southern California, Los Angeles, CA 90089-1453 USA

ROBERT E. KALABA

Electrical Engineering, Biomedical Engineering and Economics, University of Southern California, Los Angeles, CA 90089 USA

M. ZUHAIR NASHED

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716 USA

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In this paper we provide sufficient conditions for the existence and uniqueness of the solution of the newly obtained equation of motion for constrained mechanical systems.

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Consider an unconstrained mechanical system consisting of n masses $m_1, m_2, \ldots, m_n$. Its motion in an inertial Cartesian rectangular coordinate system is governed by the system of differential equations

$$M\ddot{\mathbf{x}} = \mathbf{F}\,(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{1}$$

where the 3n by 3n constant diagonal matrix M has the masses m_i in sets of three along its diagonal, and the 3n-vector $\mathbf{F}$ is the vector containing the components of the "given" or "impressed" forces in the three coordinate directions. Consider the point $\mathbf{x}(0) = \mathbf{x}_0$, $\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$. We shall assume that $\mathbf{F}$ has continuous partial derivatives in a closed bounded domain G around the point $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$ and for values of t in an interval $-a \leq t \leq a$. Then the solution of equation (1) exists and is unique, locally. Hence we assume that the solution of the unconstrained equation of motion leads, locally, to a unique solution.

*To whom correspondence should be addressed.

We now impose further, a set of m smooth (actually C^2 is sufficient) constraints of the form

$$\varphi_i(\mathbf{x}\,(t), \dot{\mathbf{x}}\,(t)\,, t) = 0, \quad i = 1, 2, \ldots, m, \tag{2}$$

which, upon differentiation with respect to t, yield the equation

$$A\,(\mathbf{x}, \dot{\mathbf{x}}, t)\,\ddot{\mathbf{x}} = \mathbf{b}\,(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{3}$$

where the matrix A and the vector $\mathbf{b}$ are again continuous functions of their arguments. As is usual in mechanics, we shall assume that the initial conditions $\mathbf{x}_0$, $\dot{\mathbf{x}}_0$ at time $t = 0$ for the constrained system are such that equation set (2) is satisfied. Then the equation of motion of the constrained mechanical system was derived in [1] and [2], and can be expressed as

$$M\ddot{\mathbf{x}} = \mathbf{F}\,(\mathbf{x}, \dot{\mathbf{x}}, t) + M^{1/2}\,(AM^{-1/2})^{+}\,[\,\mathbf{b}\,(\mathbf{x}, \dot{\mathbf{x}}, t) - A(\mathbf{x}, \dot{\mathbf{x}}, t)\,M^{-1}\,\mathbf{F}]. \tag{4}$$

We present here two useful results related to the existence and uniqueness of the solutions of equation (4).

THEOREM 1 *Let G be a closed bounded domain in the 6n-dimensional space* $(\mathbf{x}, \dot{\mathbf{x}})$. *Consider now the closed bounded domain* G_1 *in the (6n+1)-dimensional space determined by G and values of t in the interval* $-b \leq t \leq b$. *Let the point* $(\mathbf{x}_0, \dot{\mathbf{x}}_0, 0)$ *be an interior point of* G_1. *Furthermore, let*

(1) *A, **b** and **F** be defined, and continuous functions of their arguments, in* G_1, *and,*
(2) *the rank of A remain the same throughout* G_1.

Then a solution $\mathbf{x}$(t) of (4) passing through $(\mathbf{x}_0, \dot{\mathbf{x}}_0, 0)$ exists and is defined in the interval $(-h,h)$, where,

$$h = \frac{\min\,(D,b)}{(1 + \mu\sqrt{6n})}. \tag{5}$$

Here D is the minimum Euclidean distance from the point $(\mathbf{x}_0, \dot{\mathbf{x}}_0, 0)$ to the boundary of G_1, and μ is the maximum absolute value in G_1 among the components of the right-hand-side vector when equation (4) is expressed as a system of first order differential equations.

Proof. Since $AM^{-1/2}$ is a continuous function of $\mathbf{x}$, $\dot{\mathbf{x}}$, and t, and its rank is constant in G_1, $(AM^{-1/2})^{+}$ is a continuous function in the same domain and hence the right-hand side of eq. (4) becomes a continuous function of its arguments. Hence the result (see, e.g., [3]).

THEOREM 2 *If A, b and F are continuous and differentiable with respect to their arguments in* G_1 *and if the rank of* A *is constant in* G_1 *then eq. (4) has a unique solution in a neighborhood of the point* $(\mathbf{x}_0, \dot{\mathbf{x}}_0, 0)$, *and passing through it.*

Proof Since the rank of $AM^{-1/2}$ is constant in G_1, it is continuous in G_1. Hence $(AM^{-1/2})^{+}$ is differentiable in G_1 [4]. The result then follows, (*see* [3]).

Changes in the rank of the matrix A occur infrequently in practical, well-modeled problems in mechanics. When they do, they can usually be averted by the use of alternative, yet equivalent, ways of specifying the constraints, and by a reparametrization of the problem through the choice of a different set of Lagrangian coordinates. Most practical problems which arise in the dynamics of mechanical systems with bilateral nonholonomic constraints (such as those illustrated, for example, in [5] and [6]) thus satisfy the conditions of Theorem 2, and therefore yield unique motions (trajectories), at least locally.

If the matrix A is not of a constant rank for all t in $[a, b]$, then there exists a collection of open intervals (a_i, b_i) whose union is dense in $[a, b]$ such that $A^+(t)$ is continuous on each of the subintervals (a_i, b_i). The situation for generalized inverses of operator-valued functions is more complicated; continuity and perturbation results for generalized inverses of linear operators are given in [7] and the references cited therein. The physical interpretation and implications of the case when the rank of the matrix $A(t)$ is not constant in constrained mechanical systems lead to some interesting questions in the modeling and analysis of such systems.

References

1. F. E. Udwadia, and R. E. Kalaba, A New Perspective on Constrained Motion, *Proc. Roy. Soc. of Lon.*, **439**, 407–410, (1992).
2. R. E. Kalaba and F. E. Udwadia, Lagrangian Mechanics, Gauss's Principle, Quadratic Programming, and Generalized Inverses: New Equations for Nonholonomically Constrained Discrete Mechanical Systems, *Quarterly of Applied Math.*, **52**, **2**, 229–241, (1994).
3. V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equations*, Dover, pp. 8 & 11, (1989).
4. G. Golub and V. Pareyra, The Differentiation of Pseudo-Inverses and Nonlinear Least Squares Problems Whose Variables Separate, *SIAM Journ. Numer. Anal.*, **10**, **2**, 413–432, (1973).
5. L. A. Pars *A Treatise on Analytical Mechanics*, Ox Bow Press, (1979).
6. E. T. Whittaker, *A Treatise on Analytical Dynamics of Particles and Rigid Bodies*, Dover, (1944).
7. M. Z. Nashed, ed. *Generalized Inverses and Applications*, Academic Press, New York, (1976).