VOLTERRA EQUATIONS WITH FRACTIONAL STOCHASTIC INTEGRALS

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Some fractional stochastic systems of integral equations are studied. The fractional stochastic Skorohod integrals are also studied. The existence and uniquness of the considered stochastic fractional systems are established. An application of the fractional Black-Scholes is considered.

1. Introduction

We assume that a probability space (Ω,η,P) is given, where Ω denotes the space $C(\mathbb{R}_+,\mathbb{R}^k)$ equipped with the topology of uniform convergence on compact sets, η the Borel σ-field of Ω, and P a probability measure on Ω.

Let $\{W_t(\omega)=\omega(t),\ t\geq 0\}$ be a Wiener process. For any $t\geq 0$, we define $\eta_t=\sigma\{\omega(s); s<t\}\vee Z$, where Z denotes the class of the elements in η_t which have zero P-measure.

Pardoux and Protter discussed the existence and uniqueness of the solution of the stochastic integral equation of the form

$$X_t=X_0+\int_0^t F(t,s,X_s)\,ds+\sum_{i=1}^{k}\int_0^t G_i(H_t;t,s,X_s)\,dW_s^i, \tag{1.1}$$

whose solution $\{X_t\}$ should be $\mathbb{R}^d$-valued and η_t adapted process; $\{H_t\}$ is an $\mathbb{R}^p$-valued (see [16]). It is supposed that F maps $\{t,s;\ 0\leq s<t\}\times\mathbb{R}^d$ into $\mathbb{R}^d$ and G_i $(i=1,2,\dots,k)$ maps $\mathbb{R}^p\times\{t,s;\ 0\leq s<t\}\times\mathbb{R}^d$ into $\mathbb{R}^d$.

In the present work, we study the existence, uniqueness, and continuity of the solution of the fractional stochastic integral equation of the form

$$X_t=X_0+I_t^{\beta}F(t,s,X_s)+\sum_{i=1}^{k}W_t^{\beta}G_i(H_t;t,s,X_s), \tag{1.2}$$

where $0<\beta<1$, $I_t^{\beta}F(t,s,X_s)$, the fractional integral of $F(t,s,X_s)$, is defined by (see [22])

$$I_t^{\beta}F(t,s,X_s)=\frac{1}{\Gamma(\beta)}\int_0^t\frac{F(t,s,X_s)}{(t-s)^{1-\beta}}\,ds. \tag{1.3}$$

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The fractional Wiener process $W_t^\beta G_i(H_t;t,s,X_s)$ of $G_i(H_t;t,s,X_s)$ is defined by (see [7])

$$W_t^\beta G_i(H_t;t,s,X_s) = \frac{1}{\Gamma((\beta+1)/2)} \int_0^t \frac{G_i(H_t;t,s,X_s)}{(t-s)^{(1-\beta)/2}} dW_s^i. \tag{1.4}$$

Stochastic Volterra equations have been studied in several papers (see [2, 3, 4, 6, 16, 17, 18]). In this work, we will use the Skorohod integral (see [9, 10, 11, 12, 13, 14, 15, 19, 20]) to interpret (1.2) as in [16] since the integrands in the stochastic integrals are not adapted; therefore we cannot use, as usual, the Ito integral to interpret the equation. In Section 2, we state some results concerning the Skorohod integral, which will be used later together with the precise interpretation of (1.2). In Sections 3 and 4, we prove the existence and uniqueness of a solution to (1.2) in two steps. In Section 5, we establish, under additional assumptions, the existence of almost surely (a.s.) continuous modification of the solution process. In Section 6, we show the continuity of the solution of (1.2).

Equation (1.2) has many important financial applications. These systems arise if we consider the fractional analog of a portfolio (see [1]). The fractional Black-Scholes market consists of a bank account or a bond and a stock. The price process A_t of the bond at time t is given by

$$A_t = \exp\left\{ \int_0^t r(s)ds \right\}, \tag{1.5}$$

where $r(s) \geq 0$, $s \in [0,t]$, is the interest rate. A portfolio is a pair (u_t, v_t) of random variables for fixed $t \in [0,T]$. The price X_t of the stock could be governed by a fractional Volterra equation of the form

$$X_t = X_0 + \frac{1}{\Gamma(\beta)} \int_0^t \frac{\mu(s)X_s}{(t-s)^{1-\beta}} ds + \frac{1}{\Gamma((\beta+1)/2)} \int_0^t \frac{\sigma(s)X_s}{(t-s)^{(1-\beta)/2}} dW_s. \tag{1.6}$$

Here the drift $\mu \geq 0$ and volatility $\sigma > 0$ are continuous functions on $[0,T]$. The numbers u_t and v_t are the bond and stock units, respectively (held by an investor). Hence, the corresponding value process is

$$V_t = u_t A_t + v_t X_t. \tag{1.7}$$

The process V_t could be governed by the equation

$$\begin{aligned} V_t = V_0 &+ \frac{1}{\Gamma(\beta)} \int_0^t \frac{r(s)A_s u_s}{(t-s)^{1-\beta}} ds + \frac{1}{\Gamma(\beta)} \int_0^t \frac{\mu(s)X_s v_s}{(t-s)^{1-\beta}} ds \\ &+ \frac{1}{\Gamma((1-\beta)/2)} \int_0^t \frac{\sigma(s)v_s X_s}{(t-s)^{(1-\beta)/2}} dW_s. \end{aligned} \tag{1.8}$$

2. The Skorohod integral

We will now define the Skorohod integral. Most of this section is a review of some basic notations and a few results from [4, 16].

Let, again, $\Omega = C(\mathbb{R}_+, \mathbb{R}^k)$, let η be its Borel field, and let P denote Wiener measure on (Ω, η),

$$W_t(\omega) = \omega(t). \tag{2.1}$$

Let $\eta_t^0 = \sigma\{W_s;\ 0 \le s < t\}$ and $\eta_t = \eta_t^0 \vee Z$, where Z denotes the class of sets which have zero P-measure of η.

For $h \in L^2(\mathbb{R}_+; \mathbb{R}^k)$, we denote the Wiener integral by

$$W(h) = \int_0^T (h(t), dW_t). \tag{2.2}$$

Let A denote the dense subset of $L^2(\Omega, \eta, P)$ consisting of those classes of random variables of the form

$$F = f\big(W(h_1), \dots, W(h_n)\big), \tag{2.3}$$

where $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of nonnegative integers), $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n \in L^2(\mathbb{R}_+; \mathbb{R}^k)$; C_b^∞ is the set of infinitely differentiable functions on $[0, b]$ whose derivatives of any order are null at b. If F has the form (2.3), we define its derivative in the direction i as the process $\{D_t^i F;\ t \ge 0\}$ defined by

$$D_t^i F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}\big(W(h_1), \dots, W(h_n)\big) h_k^i(t). \tag{2.4}$$

DF will stand for the k-dimensional process $\{D_t F = (D_t^1 F, \dots, D_t^k F);\ t \ge 0\}$.

Proposition 2.1. *The differential operators D^i, $i = 1, \dots, k$, are unbounded closable operators from $L^2(\Omega)$ into $L^2(\Omega \times \mathbb{R}_+)$.*

Let $D_i^{1,2}$ be the closure of A with respect to the norm

$$\|F\|_{i,1,2} = \|F\|_2 + \big\| \cdot \|D^i F\|_{L^2(\mathbb{R}_+)} \big\|_2, \tag{2.5}$$

where $\|F\|_2 = (E(F^2))^{1/2}$, $E(X)$ is the mathematical expectation of X, and

$$\|g\|^2_{L^2(\mathbb{R}_+)} = \int_0^\infty g^2(t)\,dt. \tag{2.6}$$

Similarly, the domain $D^{1,2} = \bigcap_{i=1}^k D_i^{1,2}$ is the closure of A with respect to the norm (see [10])

$$\|F\|_{1,2} = \|F\|_2 + \sum_{i=1}^k \big\| \cdot \|D^i F\|_{L^2(\mathbb{R}_+)} \big\|_2. \tag{2.7}$$

We identify D^i and D with their closed extensions ($D^{1,2}$ is the domain of $D : L^2(\Omega) \to L^2(\Omega \times \mathbb{R}_+;\mathbb{R}^k)$).

We denote by $D^{1,2}_{i,\text{loc}}$ the set of measurable F's which are such that there exists a sequence $\{(\Omega_n, F_n);\ n \in \mathbb{N}\} \subset \eta \times D^{1,2}_i$ with the two properties

(i) $\Omega_n \uparrow \Omega$ a.s., $n \to \infty$,
(ii) $F_n|\Omega_n = F|_{\Omega_n}$, $n \in \mathbb{N}$.

For F $\in D^{1,2}_{i,\text{loc}}$, we define without ambiguity $D^i_t F = D^i_t F_n$ on $\Omega_n \times \mathbb{R}_+$ for all $n \in \mathbb{N}$; $D^{1,2}_{\text{loc}}$ is defined similarly.

For $i = 1,\ldots,k$, we define δ_i, the Skorohod integral with respect to W^i_t, as the adjoint of D^i, that is, $\operatorname{Dom}\delta_i$ (the set of adapted processes for the Skorohod integral) is the set of $u \in L^2(\Omega \times \mathbb{R}_+)$ which are such that there exists a constant c with

$$\left| E\int_0^\infty D^i_t F u_t\, dt \right| \leq c\|F\|_2, \quad \forall F \in A. \tag{2.8}$$

If $u \in \operatorname{Dom}\delta_i$, $\delta_i(u)$ is defined as the unique element of $L^2(\Omega)$ which satisfies

$$E(\delta_i(u)F) = E\int_0^\infty D^i_t F u_t\, dt, \quad \forall F \in A. \tag{2.9}$$

Let $L^{1,2}_i = L^2(\mathbb{R}_+;D^{1,2}_i)$. We have that $L^{1,2}_i \subset \operatorname{Dom}\delta_i$, and, for $u \in L^{1,2}_i$,

$$E[\delta_i(u)^2] = E\int_0^\infty u_t^2\, dt + E\int_0^\infty \int_0^\infty D^i_s u_t D^i_t u_s\, ds\, dt. \tag{2.10}$$

Note that if $u \in L^2_{\text{loc}}(\mathbb{R}_+;D^{1,2}_i)$, then $u1_{[0,T]} \in L^{1,2}_i$ for any $T > 0$ and we can write

$$\int_0^T u_t dW^i_t = \delta_i(u1_{[0,T]}). \tag{2.11}$$

The Skorohod integral is a local operation on $L^2_{\text{loc}}(\mathbb{R}_+;D^{1,2}_i)$ in the sense that if $u,v \in L^2_{\text{loc}}(\mathbb{R}_+;D^{1,2}_i)$, then $\int_0^t u_s dW^i_s = \int_0^t v_s dW^i_s$ a.s. on $\{\omega;\ u_s(\omega) = v_s(\omega)$, for almost all $s < t\}$.

Let $L^{1,2}_{i,\text{loc}}$ denote the set of measurable processes u which are such that, for any $T > 0$, there exists a sequence

$$\{(\Omega^T_n, u^T_n);\ n \in \mathbb{N}\} \subset \eta \times L^{1,2}_i \tag{2.12}$$

such that

(i) $\Omega^T_n \uparrow \Omega$ a.s., as $n \to \infty$,
(ii) $u = u^T_n\, dP \times dt$ a.e. on $\Omega^T_n \times [0,T]$, $n \in \mathbb{N}$.

For $u \in L^{1,2}_{i,\mathrm{loc}}$, we can define its Skorohod integral with respect to W^i_t by

$$\int_0^t u_s dW^i_s = \int_0^t u^T_{n,s} dW^i_s \quad \text{on } \Omega^T_n \times [0,T]. \tag{2.13}$$

Finally, $L^{1,2} = \bigcap_{i=1}^k L^{1,2}_i$, and $L^{1,2}_{\mathrm{loc}}$ is defined similarly as $L^{1,2}_{i,\mathrm{loc}}$.

We now introduce the particular class of integrands which we will use below.

Let $u : \mathbb{R}_+ \times \Omega \times \mathbb{R}^P \to \mathbb{R}$ satisfy the following.

(i) For all $x \in \mathbb{R}^p$, $(t,\omega) \to (t,\omega,x)$ is η_t progressively measurable.

(ii) For all $(t,\omega) \in \mathbb{R}_+ \times \Omega$, $u(t,\omega,\cdot) \in C^1(\mathbb{R}^p)$.

(iii) For some increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $|u(t,\omega,x)| + |u'(t,\omega,x)| \le \phi(|x|)$ for all $(t,\omega,x) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^p$, where $u'(t,x)$ stands for the gradient $(\partial u/\partial x)(t,x)$.

Let θ be a p-dimensional random vector such that

(iv) $\theta^j \in D^{1,2}_i \bigcap L^\infty(\Omega)$, $j = 1,\dots,p$.

We fix $T > 0$ and consider

$$I^i(x) = \int_0^T \frac{u(t,x)}{(T-t)^{(1-\beta)/2}} dW^i_t, \quad 0 < \beta < 1. \tag{2.14}$$

Define, moreover, $v_t = u(t,\theta)$.

Under conditions (i), (ii), (iii), and (iv), the following proposition holds.

It is proved in [10] that

$$\int_0^T u(t,\theta) dW^i_t = \int_0^T u(t,x) dW^i_t|_{x=\theta} - \int_0^T u'(t,\theta) D^i_t \theta \, dt. \tag{2.15}$$

The same relation is proved in [16] under slightly different conditions. Equation (2.15) is used to define the Skorohod integral $\int_0^T u(t,\theta) dW^i_t$.

Proposition 2.2. *The random field $\{I^i(x);\, x \in \mathbb{R}^p\}$ defined above possesses an a.s. continuous modification so that the random variable $I^i(\theta)$ can be defined, $v \in \mathrm{Dom}\,\delta_i$ (see [10, 16]).*

Condition (iv) can be replaced by (iv′): $\theta^j \in D^{1,2}_{i,\mathrm{loc}}$, $j = 1,\dots,p$.

Under conditions (i), (ii), (iii), and (iv′), $v \in (\mathrm{Dom}\,\delta_i)_{\mathrm{loc}}$ in the sense that there exists a sequence $\{(\Omega_n, v_n);\, n \in \mathbb{N}\} \subset \eta \times \mathrm{Dom}\,\delta_i$ such that $\Omega_n \uparrow \Omega$ a.s. and $v_n|\Omega_n = v|\Omega_n$.

Indeed, let $\{(\Omega'_n, \theta_n)\}$ be a localizing sequence for θ in $(D^{1,2}_i)^p$, and let $\{\psi_n;\, n \in \mathbb{N}\} \subset C^\infty_c(\mathbb{R}^p; \mathbb{R}^p)$ satisfy $\psi_n(x) = x$ whenever $|x| \le n$.

Define $v_n(t) = u(t, \psi_n(\theta_n))$, $\Omega_n = \Omega'_n \bigcap \{|\theta| \le n\}$; then $\{(\Omega_n, v_n);\, n \in \mathbb{N}\}$ satisfies the above conditions.

It is then natural to define, for every $\epsilon > 0$, the Skorohod integral $\int_0^{T-\epsilon} (v_t/(T-t)^{(1-\beta)/2})\, dW^i_t$ by formula (2.15) and the latter coincides with $\int_0^{T-\epsilon} (v_n(t)/(T-t)^{(1-\beta)/2}) dW^i_t$ on Ω_n.

It is clear that $\lim_{\epsilon \to 0} \int_0^{T-\epsilon} (v_t/(T-t)^{(1-\beta)/2}) dW^i_t$ exists in the mean by using the norm $\|\cdot\|_2$.

3. Statement of the problem: interpretation of (1.2)

Our aim is to study the equation

$$X_t = X_0 + a\int_0^t \frac{F(t,s,X_s)}{(t-s)^{1-\beta}}ds + b\sum_{i=1}^k \int_0^t \frac{G_i(H_t;t,s,X_s)}{(t-s)^{(1-\beta)/2}}dW_s^i,$$
$$0<\beta<1, \quad a = \frac{1}{\Gamma(\beta)}, \quad b = \frac{1}{\Gamma((\beta+1)/2)}. \tag{3.1}$$

We define $D = \{(t,s)\in\mathbb{R}_+^2;\ 0\le s<t\}$.

The coefficients F and G are given as follows: $F: D\times\mathbb{R}^d \to \mathbb{R}^d$ is measurable and, for each $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$, $F(\cdot,s,x)$ is η_t progressively measurable on $\Omega\times[s,+\infty)$.

For $i=1,\dots,k$, $G_i:\mathbb{R}^p\times D\times\mathbb{R}^d\to\mathbb{R}^d$ is measurable, for each (h,t,x), $G_i(h;t,\cdot,x)$ is η_s progressively measurable on $\Omega\times[0,t]$, and for each (ω,t,s,x), $G_i(\cdot;t,s,x)$ is of class C^1. $\{H_t\}$ is a given progressively measurable p-dimensional process. It will follow from these hypotheses that we will be able to construct a progressively measurable solution $\{X_t\}$.

Therefore, for each t, the process $\{G_i(h;t,s,X_s);\ s\in[0,T]\}$ is of the form $v_s = u(s,\theta)$ with $u(s,h) = G_i(h;t,s,X_s)$ and $\theta = H_t$. We will impose below conditions on G, $\{H_t\}$, and the solution $\{X_t\}$ so as to satisfy requirements (i), (ii), (iii), and (iv′) of Section 2.

In particular, we will consider only nonanticipating solutions. Therefore, the stochastic integrals in (3.1) will be interpreted according to (2.15), that is,

$$\int_0^t \frac{G_i(H_t;t,s,X_s)}{(t-s)^{(1-\beta)/2}}dW_s^i = \int_0^t \frac{G_i(h;t,s,X_s)}{(t-s)^{(1-\beta)/2}}dW_s^i|_{h=H_t} - \int_0^t \frac{G_i'(H_t;t,s,X_s)}{(t-s)^{(1-\beta)/2}}D_s^iH_t\,ds. \tag{3.2}$$

In other words, we can rewrite (3.1) as

$$X_t = X_0 + \int_0^t \tilde{F}(t,s,X_s)ds + b\sum_{i=1}^k \int_0^t \frac{G_i(h;t,s,X_s)}{(t-s)^{(1-\beta)/2}}dW_s^i|_{h=H_t}, \tag{3.3}$$

where

$$\tilde{F}(t,s,x) = a\frac{F(t,s,x)}{(t-s)^{1-\beta}} - b\sum_{i=1}^k \frac{G_i'(H_t;t,s,x)D_s^iH_t}{(t-s)^{(1-\beta)/2}}, \tag{3.4}$$

and the stochastic integrals are now the usual Itô integrals.

We will show below that (3.3) makes sense for any progressively measurable process X which satisfies $X\in\bigcap_{t>0}L^q(0,t)$ a.s., for some $q>p$. We will find such a solution to (3.3); it will then follow from (3.2) that it is a solution to (3.1). Similarly, uniqueness for (3.1) in the above class will follow from uniqueness for (3.3) in that class.

4. Existence and uniqueness under strong hypotheses

We formulate a set of further hypotheses (those stated in Section 3 are assumed to hold throughout the paper) under which we will establish a first result of the existence and uniqueness of a solution of (1.2).

Let B be an open bounded subset of $\mathbb{R}^p$, $K > 0$, and $q > p$ such that

(H1) $X_0 \in L^q(\Omega, \eta_0, P; \mathbb{R}^d)$,
(H2) $P(H_t \in B, \forall t \geq 0) = 1$,
(H3) $H \in (L^{1,2})^p$, $|D_s H_t| \leq K$ a.s., $0 \leq s < t$,
(H4) $|F(t,s,x)| + \sum_{i=1}^k |G_i(h;t,s,x)| + \sum_{i=1}^k |G_i'(h;t,s,x)| \leq K(1+|x|)$, for any $0 \leq s < t$, $h \in B$, $x \in \mathbb{R}^d$,
(H5) $|F(t,s,x) - F(t,s,y)| + \sum_{i=1}^k |G_i(h;t,s,x) - G_i(h;t,s,y)| + \sum_{i=1}^k |G_i'(h;t,s,x) - G_i'(h;t,s,y)| \leq K|x-y|$, for any $0 \leq s < t$, $h \in B$, and $x, y \in \mathbb{R}^d$ a.s.

Note that q will be a fixed real number such that $q > p$, and $L^q_{\text{prog}}(\Omega \times (0,t))$ will stand for the space $L^q(\Omega \times (0,t), \xi_t, P \times \lambda)$, where ξ_t denotes the σ-algebra of progressively measurable subsets of $\Omega \times (0,t)$ and λ denotes the Lebesgue measure on $(0,t)$, and set $1-\beta = \alpha/q$, $0 < \alpha < 1$.

Lemma 4.1. *Let $X \in \bigcap_{t>0} L^q_{\text{prog}}(\Omega \times (0,t))$, where $q > p$, and suppose that (H.4) is in force. Then for any $T \geq t > 0$ and $i \in \{1,\dots,k\}$, the random field*

$$\left\{ \int_0^t \frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}} dW_s^i;\ h \in B \right\} \tag{4.1}$$

possesses an a.s. continuous modification.

Proof. Using Burkholder-Gundy and Hölder's inequalities together with (H.4), we obtain

$$\begin{aligned}
&E\left(\left| \int_0^t \frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}} dW_s^i - \int_0^t \frac{G_i(k;t,s,X_s)}{(t-s)^{\alpha/2q}} dW_s^i \right|^q \right) \\
&\quad \leq c_1 E \int_0^t \frac{|G_i(h;t,s,X_s) - G_i(k;t,s,X_s)|^q}{(t-s)^{\alpha/2}} ds \\
&\quad \leq c_1 K^q |h-k|^q \int_0^t \frac{E(1+|X_s|^q)}{(t-s)^{\alpha/2}} ds \\
&\quad \leq c_2 K^q |h-k|^q \int_0^t (t-s)^{-\alpha/2} ds \\
&\quad \leq c_3 |h-k|^q t^{1-\alpha/2},
\end{aligned} \tag{4.2}$$

where c_1, c_2, and c_3 are positive constants. The result now follows from the multidimensional generalization of Kolmogrov's lemma (see [1, 8]).

We can now assume that, for fixed t, the random field (4.1) is a.s. continuous in h, provided $X \in L^q_{\text{prog}}(\Omega \times (0,t))$.

From $X \in \bigcap_{t>0} L^q_{\text{prog}}(\Omega \times (0,t))$, define

$$\begin{aligned}
I_t(X,h) &= b \sum_{i=1}^k \int_0^t \frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}} dW_s^i, \quad h \in \mathbb{R}^p,\ t > 0, \\
J_t(X) &= \int_0^t \tilde{F}(t,s,X_s) ds + I_t(X,H_t).
\end{aligned} \tag{4.3}$$

□

Lemma 4.2. *For any $t > 0$, there is a constant $c > 0$ such that*

$$E(|J_t(X)|^q) \leq c\left(\int_0^t E(1+|X_s|^q)\left\{\frac{1}{(t-s)^\alpha}+\frac{1}{(t-s)^{\alpha/2}}\right\}ds\right), \quad 0<\alpha<1. \tag{4.4}$$

Proof.

$$\begin{aligned}\left|\int_0^t \tilde{F}(t,s,X_s)\,ds\right|^q &\leq c\int_0^t \frac{|F(t,s,X_s)|^q}{(t-s)^\alpha}+\sum_{i=1}^k \frac{|G_i'(h;t,s,X_s)|^q_{h=H_t}|D_s^i H_t|^q}{(t-s)^{\alpha/2}}ds \\ &\leq \tilde{c}\int_0^t (1+|X_s|^q)\left\{\frac{1}{(t-s)^\alpha}+\frac{1}{(t-s)^{\alpha/2}}\right\}ds,\end{aligned} \tag{4.5}$$

where we have used (H.2), (H.3), and (H.4).

$$|I_t(X,H_t)| \leq \sup_{h\in B}|I_t(X,h)| \tag{4.6}$$

(c, $\tilde{c}$ are positive constants).

It is easy to show, using, in particular, (H4) and Lebesgue's dominated convergence theorem, that the mapping

$$h \longrightarrow I_t(X,h) \tag{4.7}$$

from $\mathbb{R}^p$ into $L^q(\Omega)$ is differentiable and that

$$\frac{\partial I_t(X,h)}{\partial h_j} = b\int_0^t \frac{\partial}{\partial h_j}\sum_{i=1}^k\left(\frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}}\right)dW_s^i. \tag{4.8}$$

Since $q > p$, we can infer from Sobolev's embedding theorem (see [13]) that

$$E\left(\sup_{h\in B}|I_t(X,h)|^q\right) \leq cE\int_B\left(|I_t(X,h)|^q+\sum_{j=1}^p\left|\frac{\partial I_t}{\partial h_j}(X,h)\right|^q\right)dh. \tag{4.9}$$

It then follows from the Burkholder-Gundy inequality that

$$\begin{aligned}E\left(\sup_{h\in B}|I_t(X,h)|^q\right) &\leq cE\int_B\int_0^t\sum_{i=1}^k\left(\frac{|G_i(h;t,s,X_s)|^q}{(t-s)^{\alpha/2}}+\sum_{j=1}^p\left|\frac{\partial}{\partial h_j}\frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}}\right|^q\right)ds\,dh \\ &\leq c\left(\int_B dh\right)\left(E\int_0^t\frac{(1+|X_s|^q)}{(t-s)^{\alpha/2}}ds\right) \\ &\leq \tilde{c}\left\{\int_0^t\frac{E(1+|X_s|^q)}{(t-s)^{\alpha/2}}ds\right\},\end{aligned} \tag{4.10}$$

where we have used (H4) and B is bounded. From (4.5) and (4.10), the proof is complete. □

A similar argument, using (H5) instead of (H4), yields the following lemma.

Lemma 4.3. *For any* $0 < t \le T$, *there exists a constant* $c > 0$ *such that*

$$E(|J_t(X) - J_t(Y)|^q) \le c\left\{\int_0^t E|X_s - Y_s|^q\left\{\frac{1}{(t-s)^\alpha} + \frac{1}{(t-s)^{\alpha/2}}\right\}ds\right\}. \tag{4.11}$$

We are now in a position to prove the main result of this section.

Theorem 4.4. *Under conditions (H.1), (H.2), (H.3), (H.4), and (H.5), there exists a unique element* $X \in \bigcap_{t>0} L^q_{\text{prog}}(\Omega \times (0,t))$, *which solves (3.1). Moreover, if* τ *is a stopping time, uniqueness holds on the random interval* $[0,\tau]$.

Proof. Equation (3.1) can be rewritten as

$$X_t = X_0 + J_t(X), \quad t \ge 0. \tag{4.12}$$

Uniqueness. Let $X, Y \in \bigcap_{t>0} L^q_{\text{prog}}(\Omega \times (0,t))$ and let τ be a stopping time such that

$$X_t = X_0 + J_t(X), \quad Y_t = X_0 + J_t(Y), \quad 0 \le t \le \tau. \tag{4.13}$$

From Lemma 4.3,

$$\begin{aligned} E(|X_t - Y_t|^q) &= E(|J_t(X) - J_t(Y)|^q) \\ &\le c\int_0^t E|X_s - Y_s|^q\left\{\frac{1}{(t-s)^\alpha} + \frac{1}{(t-s)^{\alpha/2}}\right\}ds. \end{aligned} \tag{4.14}$$

Set $(E|X_t - Y_t|^q) = \gamma_t$. It is easy to see that

$$\gamma_t \le 2c\eta(t)\int_0^t \frac{\gamma_s}{(t-s)^\alpha}ds, \tag{4.15}$$

where $\eta(t) = \text{Max}(1, t^\alpha)$ and c is a positive constant. Thus, (see [5, 21])

$$\gamma_t \le \frac{M^2(\Gamma(t-\alpha))^2}{\Gamma(2(1-\alpha))}\int_0^t \frac{\gamma_s}{(t-s)^{2\alpha-1}}ds, \tag{4.16}$$

and, by a classical argument, we get

$$\gamma_t \le \frac{M^n(\Gamma(t-\alpha))^n}{\Gamma(n(1-\alpha))}\int_0^t \frac{\gamma_s}{(t-s)^{n\alpha-(n-1)}}ds, \tag{4.17}$$

where $M = 2c\eta(T)$. Thus for sufficiently large n, $n > 1/(1-\alpha)$, we get, from (4.17),

$$\gamma_t \le \frac{K^n}{\Gamma(n(1-\alpha))}\int_0^t \gamma_s\, ds, \tag{4.18}$$

where K is a positive constant.

Taking the limit as $n \to \infty$, we find that (4.18) leads to $\gamma_t = 0$.

To prove the existence, we define a sequence $\{X_t^n, 0 \le t \le T, n = 0, 1, 2, \ldots\}$ as follows:

$$X_t^o = X_o, \quad X_t^{n+1} = X_o + J_t(X^n), \quad T \le t \le 0. \tag{4.19}$$

Using Lemma 4.2, we can see that

$$X^n \in \bigcap_{t>0} L^q_{\text{prog}}(\Omega \times [0,T]). \tag{4.20}$$

It then follows from Lemma 4.3 that

$$g_{n+1}(t) \leq 2c\eta(t) \int_0^t \frac{g_n(s)}{(t-s)^\alpha} ds, \tag{4.21}$$

where

$$g_{n+1}(t) = E(|X_t^{n+1} - X_t^n|^q). \tag{4.22}$$

By using a similar argument, we can write

$$g_n \leq \frac{K^n}{\Gamma(n(1-\alpha))} \int_0^t g_o(s) ds. \tag{4.23}$$

The last estimation implies that X^n is a Cauchy sequence in $L^q_{\text{prog}}(\Omega \times [0,T])$. Then there exists X such that $X^n \to X$ in $\bigcap_{t>0} L^q_{\text{prog}}(\Omega \times [0,T])$, and again using Lemma 4.3, we can pass to the limit in (4.20), yielding that X solves (4.12). □

5. An existence and uniqueness result under weaker assumptions

We formulate a new set of weaker hypotheses.

(H1′) X_0 is η_0 measurable.
(H2′) $H \in (L^{1,2}_{\text{loc}})^p$, $\{H_t\}$ is a progressively measurable process which can be localized in $(L^{1,2})^p$ by a progressively measurable sequence.

We assume that there exists an increasing progressively measurable process $\{U_t,\ t \geq 0\}$ with values in $\mathbb{R}_+$ such that

(H3′) $|H_t| + \sum_{i=1}^k |D_s^i H_t| \leq U_t$ a.s., $0 \leq s < t$.

Finally, we suppose that for any $\mathbb{N} > 0$, there exists an increasing progressively measurable process $\{V_t^N;\ t \geq 0\}$ with values in $\mathbb{R}_+$ such that

(H4′) $|F(t,s,x)| + \sum_{i=1}^k |G_i(h;t,s,x)| + \sum_{i=1}^k |G_i'(h;t,s,x)| \leq V_t^N(1+|x|)$, for all $|h| \leq N$, $0 \leq s < t$, and $x \in \mathbb{R}^d$;
(H5′) $|F(t,s,x) - F(t,s,y)| + \sum_{i=1}^k |G_i(h;t,s,x) - G_i(h;t,s,y)| + \sum_{i=1}^k |G_i'(h;t,s,x) - G_i'(h;t,s,y)| \leq V_t^N |x-y|$, for all $|h| \leq N$, $0 \leq s < t$, $x, y \in \mathbb{R}^d$.

Let, again, q be a fixed real number, with $q > p$, and set $1 - \beta = \alpha/q$, $0 < \alpha < 1$. We have the following theorem.

Theorem 5.1. *Equation (3.1) has a unique solution in the class of progressively measurable processes which satisfy*

$$X \in \bigcap_{t>0} L^q(0,t) \quad a.s. \tag{5.1}$$

Proof. (a) We first see how (3.1) makes sense if $X \in \bigcap_{t>0} L^q(0,t)$ a.s.

That is, we have to show that for fixed $t > 0$,

$$\left\{ \int_0^t \frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}} dW_s^i;\ h \in \mathbb{R}^P \right\} \tag{5.2}$$

is a well-defined random field which possesses an a.s. continuous version.

For that sake, we define

$$\tau_n = \inf \left\{ t;\ \int_0^t \frac{|X_s|^q}{(t-s)^\alpha} ds \geq n \text{ or } V_t^N \geq n \right\}. \tag{5.3}$$

The argument of Lemma 4.1 can be used to show that

$$h \longrightarrow \int_0^{t\wedge\tau_n} \frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}} dW_s^i \tag{5.4}$$

possesses an a.s. continuous modification on $\{|h| \leq N\}$. Since this is true for any n and N, and $\cup_n\{\tau_n \geq t\} = \Omega$ a.s., the result follows.

(b) Existence: we want to show existence on an arbitrary interval $[0,T]$ (T will be fixed below).

Let $\{H^n;\ n \in \mathbb{N}\}$ denote a progressively measurable localizing sequence for H in $(L^{1,2})^p$ on $[0,T]$. Since, from (H.3′), $\sup_{t\leq T}|H_t|$ is a.s. finite, we can and do assume, without loss of generality, that

$$|H_t^n(\omega)| \leq n, \quad \forall (t,\omega) \in [0,T]\times\Omega. \tag{5.5}$$

Note that $H_t(\omega) = H_t^n(\omega)$ a.s. on Ω_n^T, for all $t \in [0,T]$, where $\Omega_n^T \uparrow \Omega$ a.s. as $n \to \infty$. We, moreover, define $X_0^n = X_0 1_{\{|X_0|\leq n\}}$, $S_n = \inf\{t;\ \sup_{s<t}|D_s H_t^n| \vee V_t^n \geq n\}$. We consider the equation

$$X_t^n = X_0^n + \int_0^t \tilde{F}^n(t,s,X_s^n)ds + b\sum_{i=1}^k \int_0^t \frac{G_i^n(h;t,s,X_s^n)}{(t-s)^{\alpha/2q}} dW_s^i|_{h=H_t^n}, \tag{5.6}$$

where

$$\begin{gathered}\tilde{F}^n(t,s,x) = 1_{[0,S_n]}(s)\left[a\frac{F(t,s,x)}{(t-s)^{\alpha/q}} - b\sum_{i=1}^k \frac{G_i'(H_t^n;t,s,x)D_s^i H_t^n}{(t-s)^{\alpha/2q}}\right], \\ G_i^n(h;t,s,x) = 1_{[0,S_n]}(s)G_i(h;t,s,x).\end{gathered} \tag{5.7}$$

It is easy to see that theorem (4.15) applies to (5.6).

Define

$$\bar{S}_n(\omega) = \begin{cases} S_n(\omega) \wedge \inf\left\{t \leq T;\ \int_0^t |H_s(\omega) - H_s^n(\omega)|ds > 0\right\} & \text{if } |X_0(\omega)| < n, \\ 0 & \text{otherwise.}\end{cases} \tag{5.8}$$

$\bar{S}_n$ is a stopping time, and it follows from the uniqueness part of Theorem 4.4 that if $m > n$,

$$X_t^m = X_t^n \quad \text{on } [0, \bar{S}_n] \text{ a.s.} \tag{5.9}$$

Since, moreover, $\{\bar{S}_n = T\} \uparrow \Omega$ a.s., we can define the process $\{X_t\}$ on $[0,T]$ by $X_t = X_t^n$ on $[0, \bar{S}_n]$, for all $n \in \mathbb{N}$.

Clearly, $X \in L^q(0,T)$ a.s. and solves (3.1) on $[0,T]$. Since T is arbitrary, the existence is proved.

(c) Uniqueness: it suffices to prove uniqueness on an arbitrary interval $[0,T]$. Let $\{\bar{X}_t,\ t \in [0,T]\}$ be a progressively measurable process such that $\bar{X} \in L^q(0,T)$ a.s. and $\bar{X}$ solves (3.1). It suffices to show that $\bar{X}$ coincides with the solution we have just constructed.

Let

$$\begin{aligned} \tilde{S}_n(\omega) &= \bar{S}(\omega) \wedge \inf\left\{t \le T;\ \int_0^t \frac{|\bar{X}_s(\omega)|^q}{(t-s)^\alpha} ds > n\right\}, \\ \tilde{X}_t^n &= \bar{X}_{t \wedge \tilde{S}_n}; \end{aligned} \tag{5.10}$$

$\tilde{S}^n \in L^q(\Omega \times [0,T])$ and it solves (5.6) with S_n replaced by $\tilde{S}_n$.

Then

$$\tilde{X}_t^n(\omega) = X_t(\omega) \ell \times P \quad \text{a.e. on } [0, \tilde{S}_n], \tag{5.11}$$

where ℓ is the Lebesgue measure on the real line.

The result follows from the fact that $\{\tilde{S}_n \ge T\} \uparrow \Omega$ a.s.

Note that the above solution satisfies, in fact, $X \in \bigcap_{q>1} \bigcap_{t>0} L^q(0,t)$ a.s. □

6. Continuity of the solution

We want to give additional conditions under which the solution of (3.1) is an a.s. continuous process.

(H6) For all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $t \to F(t,s,x)/(t-s)^{\alpha/q}$ is a.s. continuous on $(s,+\infty)$.

(H7) $\{H_t;\ t \ge 0\}$ is a.s. continuous.

(H8) For all $i \in \{1,\dots,k\}$, $s \in \mathbb{R}_+$, $t \to D_s^i H_t$ is a.s. continuous on $(s,+\infty)$.

(H9) For all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $i \in \{1,\dots,k\}$, $(t,h) \to G_i'(h;t,s,x)/(t-s)^{\alpha/2q}$ is a.s. continuous on $(s,+\infty) \times \mathbb{R}^P$.

We also suppose that there exist $\delta > 0$, $\ell > 0$ such that for all $N > 0$, $|h| \le N$, $0 \le s < t \wedge r$, $x \in \mathbb{R}^d$, $|t-r| \le 1$.

(H10) There exists an increasing process $\{V_t^N;\ t \ge 0\}$ such that

$$|G_i(h;t,s,x) - G_i(h;r,s,x)| \le V_t^N |t-r|^\delta (1 + |x|^\ell). \tag{6.1}$$

Theorem 6.1. *Under conditions (H1′), (H2′), (H3′), (H4′), (H5′) and (H6), (H7), (H8), (H9), (H10), the unique solution of (3.1) (which is progressively measurable and belongs a.s. to* $\bigcap_{q>1} \bigcap_{t>0} L^q(0,t)$*) has a.s. continuous modification.*

Proof. We need to show only that whenever $X \in \bigcap_{q>1}\bigcap_{t>0} L^q(0,t)$ a.s., $\{J_t(X);\ t>0\}$ has an a.s. continuous modification.

(a) We first show that $t \to \int_0^t \tilde{F}(t,s,X_s)ds$ is a.s. continuous.

Note that (H.6), (H.7), (H.8), and (H.9) imply that for all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $t \to \tilde{F}(t,s,x)$ is a.s. continuous on $(s,+\infty)$.

Moreover, from (H.2′), (H.3′), (H.4′), and the fact that $X \in \bigcap_{q>1}\bigcap_{t>0} L^q(0,t)$ a.s. for any $T>0$, there exists a process $\{Z_s^T;\ s \in [0,T]\}$ such that

$$
\begin{gathered}
|F(t,s,X_s)| \le Z_s^T, \quad |G_i'(H_t;t,s,X_s)D_s^i H_t| \le Z_s^T, \quad 0 \le s < t \le T \quad \text{a.s.}, \\
\int_0^t Z_s^T \left\{ \frac{1}{(t-s)^{\alpha/2q}} + \frac{1}{(t-s)^{\alpha/q}} \right\} ds < \infty \quad \text{a.s.}
\end{gathered} \tag{6.2}
$$

Let first $\{t_n;\ n \in \mathbb{N}\}$ be a sequence such that $t_n < t$ for any n and $t_n \to t$ as $n \to \infty$; then

$$
\begin{gathered}
\int_0^t \tilde{F}(t,s,X_s)ds - \int_0^{t_n} \tilde{F}(t_n,s,X_s)ds = \int_{t_n}^t \tilde{F}(t,s,X_s)ds + \int_0^{t_n} [\tilde{F}(t,s,X_s) - \tilde{F}(t_n,s,X_s)]ds, \\
\left| \int_{t_n}^t \tilde{F}(t,s,X_s)ds \right| \le \int_{t_n}^t Z_s^T \left\{ \frac{1}{(t-s)^{\alpha/2q}} + \frac{1}{(t-s)^{\alpha/q}} \right\} ds,
\end{gathered} \tag{6.3}
$$

and the latter tends a.s. to 0 as $n \to \infty$.

$$
\left| \int_0^{t_n} [\tilde{F}(t,s,X_s) - \tilde{F}(t_n,s,X_s)]ds \right| \le \int_0^t |\tilde{F}(t,s,X_s) - \tilde{F}(t_n,s,X_s)| ds \tag{6.4}
$$

which tends to 0 as $n \to \infty$. A similar argument gives the same result when $t_n > t$, $t_n \to t$.

(b) We next show that $t \to I_t(X,H_t)$ possesses an a.s. continuous modification. This follows from (H.7) and

$$
(t,h) \longrightarrow I_t(X,h) \tag{6.5}
$$

has an a.s. continuous modification.

By localization, it suffices to prove (6.5) under assumptions (H.2), (H.3), (H.4), (H.5) and (H.6), (H.7), (H.8), (H.9), (H.10), with $V_t^N(\omega)$ in (H.10) replaced by a constant K, and in case $X_0 \in \bigcap_{q>1} L^q(\Omega;\mathbb{R}^d)$.

It then suffices to show that under the above hypotheses, there exists $C,q>0$ such that for any $h,k \in \mathbb{R}^P$, the numbers t,r will be positive. Suppose, to fix the ideas, that $0 \le r < t$; then

$$
\begin{aligned}
I_t(X,h) - I_r(X,k) = {} & \int_r^t \frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}} dW_s^i + \int_0^r \left[\frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}} - \frac{G_i(h;r,s,X_s)}{(r-s)^{\alpha/2q}} \right] dW_s^i \\
& + \int_0^r \left[\frac{G_i(h;t,s,X_s)}{(r-s)^{\alpha/2q}} - \frac{G_i(k;r,s,X_s)}{(r-s)^{\alpha/2q}} \right] dW_s^i.
\end{aligned} \tag{6.6}
$$

It follows from the Burkholder-Gundy inequality that

$$\begin{aligned} E\left(\left|\int_r^t \frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}}dW_s^i\right|^q\right) &\le c_q\sum_{i=1}^k E\left[\left(\int_r^t \frac{|G_i(h;t,s,X_s)|^2}{(t-s)^{\alpha/q}}ds\right)^{q/2}\right] \\ &\le c_q(t-r)^{(q-2)/2}\sum_{i=1}^k E\int_r^t \frac{|G_i(h;t,s,X_s)|^q}{(t-s)^{\alpha/2}}ds \\ &\le c_q(t-r)^{(q-2)/2}\int_r^t \frac{E(1+|X_s|^q)}{(t-s)^{\alpha/2}}ds \\ &\le C_q(t-r)^{(q-2)/2}(t-r)^{1-\alpha/2}. \end{aligned} \tag{6.7}$$

From (H4), for G_i, we deduce, as in Lemma 4.1, that

$$\begin{aligned} &E\left(\left|\int_0^r \left[\frac{G_i(h;r,s,X_s)-G_i(k;r,s,X_s)}{(r-s)^{\alpha/2q}}\right]dW_s^i\right|^q\right) \\ &\quad\le c_q(h-k)^q\int_0^r \frac{E(1+|X_s|^q)}{(r-s)^{\alpha/2}}ds \le C_q(h-k)^q r^{1-\alpha/2}. \end{aligned} \tag{6.8}$$

From (H.10) and the fact that

$$\begin{aligned} &X\in\bigcap_{q>1}\bigcap_{t>0}L^q(\Omega\times(0,t)), \\ &E\left(\left|\int_0^r\left[\frac{G_i(h;t,s,X_s)}{(t-s)^{\alpha/2q}}-\frac{G_i(h;r,s,X_s)}{(r-s)^{\alpha/2q}}\right]dW_s^i\right|^q\right) \\ &\quad= E\left(\left|\int_0^r\left(\frac{1}{(r-s)^{\alpha/2q}}[G_i(h;t,s,X_s)-G_i(h;r,s,X_s)]\right.\right.\right. \\ &\qquad\qquad \left.\left.\left.+G_i(h;t,s,X_s)\left[\frac{1}{(t-s)^{\alpha/2q}}-\frac{1}{(r-s)^{\alpha/2q}}\right]\right)dW_s^i\right|^q\right) \\ &\quad\le E\left(c_q\int_0^r\left(\frac{|G_i(h;t,s,X_s)-G_i(h;r,s,X_s)|^q}{(r-s)^\alpha}\right.\right. \\ &\qquad\qquad \left.\left.+|G_i(h;t,s,X_s)|^q\left((r-s)^{-\alpha/2q}-(t-s)^{-\alpha/2q}\right)^q\right)ds\right) \\ &\quad\le c_qK^q\left(\int_0^r\left(\frac{|t-r|^{\delta q}}{(r-s)^\alpha}E\left(1+|X_s|^{q\ell}\right)\right)ds\right. \\ &\qquad\qquad \left.+\int_0^r\left(E(1+|X_s|^q)\right)\left((r-s)^{-\alpha/2}-(t-s)^{-\alpha/2}\right)ds\right) \\ &\quad\le \tilde c_q\left[|t-r|^{\delta q}\int_0^r (r-s)^{-\alpha/2}ds+\int_0^r\left((r-s)^{-\alpha/2}-(t-s)^{-\alpha/2}\right)ds\right] \\ &\quad\le C_q\left(|t-r|^{\delta q}r^{1-\alpha/2}+\left(r^{1-\alpha/2}+(t-r)^{1-\alpha/2}-t^{1-\alpha/2}\right)\right) \end{aligned} \tag{6.9}$$

which from the above estimate yields

$$E\big(\,|\,I_t(X,h) - I_r(X,k)\,|^{\,q}\big) \leq C_q\big[\,|t-r|^{1-\alpha/2} + |t-r|^{(q-\alpha)/2} + r^{1-\alpha/2}\big(|t-r|^{\delta q} + |h-k|^{q}\big) + \big(r^{1-\alpha/2} - t^{1-\alpha/2}\big)\big]; \tag{6.10}$$

this completes the proof. □

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