RELIABILITY FOR SOME BIVARIATE GAMMA DISTRIBUTIONS

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In the area of stress-strength models, there has been a large amount of work as regards estimation of the reliability $R = \Pr(X < Y)$. The algebraic form for $R = \Pr(X < Y)$ has been worked out for the vast majority of the well-known distributions when X and Y are independent random variables belonging to the same univariate family. In this paper, we consider forms of R when (X, Y) follows a bivariate distribution with dependence between X and Y. In particular, we derive explicit expressions for R when the joint distribution is bivariate gamma. The calculations involve the use of special functions.

1. Introduction

Bivariate gamma distributions arise as tractable "lifetime" models in many areas, including life testing and telecommunications. In the context of reliability, the stress-strength model describes the life of a component which has a random strength Y and is subjected to a random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus, $R = \Pr(X < Y)$ is a measure of the component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the aging of concrete pressure vessels. Some examples are as follows.

(i) If X represents the maximum chamber pressure generated by ignition of a solid propellant and Y represents the strength of the rocket chamber, then R is the probability of successful firing of the engine.

(ii) If X represents the diameter of a shaft and Y represents the diameter of a bearing that is to be mounted on the shaft, then R is the probability that the bearing fits without interference.

(iii) Let Y and X be the remission times of two chemicals when they are administered to two kinds of mechanical systems. Inferences about R present a comparison of the effectiveness of the two chemicals.

(iv) If X and Y are future observations on the stability of an engineering design, then R would be the predictive probability that X is less than Y. Similarly, if X and

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Y represent lifetimes of two electronic devices, then R is the probability that one fails before the other.

(v) If Y represents the distance of a pyrotechnic igniter from its adjacent pellet and X represents its *ignition distance*, then R is the probability that the igniter succeeds to bridge the gap in the pyrotechnic chain.

(vi) A certain unit—be it a receptor in a human eye or ear or any other organ (including sexual)—operates only if it is stimulated by the source of random magnitude Y and the stimulus exceeds a lower threshold X specific for that unit. In this case, R is the probability that the unit functions.

(vii) In military warfare, R could be interpreted as the probability that a given round of ammunition will penetrate its target.

Because of these applications, the calculation and the estimation of $R = \Pr(X < Y)$ are important for the class of bivariate gamma distributions. The calculation of R has been extensively investigated in the literature when X and Y are independent random variables belonging to the same univariate family of distributions. The algebraic form for R has been worked out for the majority of the well-known distributions, including normal, uniform, exponential, gamma, beta, extreme value, Weibull, Laplace, logistic, and the Pareto distributions (Nadarajah [21, 22, 23, 24, 25]; Nadarajah and Kotz [27]). However, there is relative little work when X and Y are dependent random variables. A complete literature search shows that the cases of bivariate beta (Nadarajah [26]), bivariate exponential (Awad et al. [2], Klein and Basu [18], Jana [13], Jana and Roy [15], Jana [14], Jeevanand [17], Hanagal [10, 12]), bivariate normal (Mukherjee and Saran [20], Nandi and Aich [28, 29], Gupta and Subramanian [8]), and bivariate Pareto (Hanagal [11], Jeevanand [16]) have been considered. The cases of multivariate exponential (Cramer and Kamps [5]) and multivariate normal (Singh [33], R. D. Gupta and R. C. Gupta [9]) have also been considered. We have been able to trace no other work for dependent variables.

The aim of this paper is to calculate R when X and Y are dependent random variables from six flexible families of bivariate gamma distributions. We will assume throughout this paper that (X,Y) has a bivariate gamma distribution with joint probability density function (pdf) f and joint survivor function $\bar{F}$. One can write

$$R = \int_0^\infty \int_x^\infty f(x,y)\,dy\,dx. \tag{1.1}$$

Our calculations of R make use of a number of special functions. They are the complementary incomplete gamma function defined by

$$\Gamma(a,x) = \int_x^\infty t^{a-1}\exp(-t)\,dt, \tag{1.2}$$

the confluent hypergeometric function (${}_1F_1$) defined by

$${}_1F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!}, \tag{1.3}$$

the Gauss hypergeometric function (${}_2F_1$) defined by

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{x^k}{k!}, \tag{1.4} $$

the generalized hypergeometric function (${}_2F_2$) defined by

$$ {}_2F_2(a,b;c,d;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(d)_k} \frac{x^k}{k!}, \tag{1.5} $$

the Kummer function defined by

$$ \Psi(a,b,x) = \frac{1}{\Gamma(a)} \int_0^{\infty} t^{a-1}(1+t)^{b-a-1} \exp(-zt)dt, \tag{1.6} $$

and the modified Bessel function of the first kind defined by

$$ I_m(z) = \sum_{k=0}^{\infty} \frac{z^{2k+m}}{2^{2k+m}k!\Gamma(k+m+1)}, \tag{1.7} $$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The properties of these special functions being used can be found in Prudnikov et al. [30, 31, 32] and Gradshteyn and Ryzhik [7].

The paper is organized as follows. Sections 2 to 7 calculate expressions for R for the six distributions (Section 2 for McKay's bivariate gamma distribution, Section 3 for Dussauchoy and Berland's bivariate gamma distribution, Section 4 for Cherian's bivariate gamma distribution, Section 5 for Arnold and Strauss' bivariate gamma distribution, Section 6 for Becker and Roux's bivariate gamma distribution, and Section 7 for Smith and Adelfang's bivariate gamma distribution.) Conclusions and suggestions for future work are described in Section 8.

2. McKay's bivariate gamma distribution

McKay's [19] bivariate gamma distribution has the joint pdf given by

$$ f(x,y) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} x^{p-1}(y-x)^{q-1} \exp(-ay) \tag{2.1} $$

for $y > x > 0$, $a > 0$, $p > 0$, and $q > 0$. For this distribution, it is easily seen that R is given by

$$ \begin{aligned} R &= \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \int_0^{\infty} x^{p-1} \int_x^{\infty} (y-x)^{q-1} \exp(-ay)dy\,dx \\ &= \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \int_0^{\infty} x^{p-1}\Gamma(q)dx \\ &= a^{p+q}. \end{aligned} \tag{2.2} $$

3. Dussauchoy and Berland's bivariate gamma distribution

Dussauchoy and Berland's [6] bivariate gamma distribution has the joint pdf specified by

$$f(x,y)=\frac{\beta a_2^q}{\Gamma(p)\Gamma(q-p)}(\beta x)^{p-1}(y-\beta x)^{q-p-1}\exp(-a_2x)\exp\left\{-\frac{a_2}{\beta}(y-\beta x)\right\} \times {}_1F_1\left(p,q-p;\left(\frac{a_1}{\beta}-a_2\right)(y-\beta x)\right) \tag{3.1}$$

for $x>0$, $y>0$, $0\le\beta\le 1$, $0<a_2\le a_1/\beta$, and $0<p<q$. Using the definition of the confluent hypergeometric function given in Section 1, one can express the corresponding form of R as

$$\begin{aligned} R&=\frac{\beta a_2^q}{\Gamma(p)\Gamma(q-p)}\int_0^\infty(\beta x)^{p-1}\exp(-a_2x)\\ &\quad\times\int_x^\infty(y-\beta x)^{q-p-1}\exp\left\{-\frac{a_2}{\beta}(y-\beta x)\right\}\\ &\quad\times\sum_{k=0}^\infty\frac{(p)_k}{k!(q-p)_k}\left(\frac{a_1}{\beta}-a_2\right)^k(y-\beta x)^k\,dy\,dx\\ &=\frac{\beta a_2^q}{\Gamma(p)\Gamma(q-p)}\int_0^\infty(\beta x)^{p-1}\exp(-a_2x)\sum_{k=0}^\infty\frac{(p)_k}{k!(q-p)_k}\left(\frac{a_1}{\beta}-a_2\right)^k\\ &\quad\times\int_x^\infty(y-\beta x)^{k+q-p-1}\exp\left\{-\frac{a_2}{\beta}(y-\beta x)\right\}dy\,dx\\ &=\frac{\beta^q a_2^p}{\Gamma(p)\Gamma(q-p)}\int_0^\infty x^{p-1}\exp(-a_2x)\sum_{k=0}^\infty\frac{(p)_k}{k!(q-p)_k}\left(\frac{a_1}{a_2}-\beta\right)^k\\ &\quad\times\int_{a_2(1-\beta)x/\beta}^\infty w^{k+q-p-1}\exp(-w)\,dw\,dx\\ &=\frac{\beta^q a_2^p}{\Gamma(p)\Gamma(q-p)}\int_0^\infty x^{p-1}\exp(-a_2x)\sum_{k=0}^\infty\frac{(p)_k}{k!(q-p)_k}\left(\frac{a_1}{a_2}-\beta\right)^k\\ &\quad\times\Gamma\left(k+q-p,\frac{a_2(1-\beta)x}{\beta}\right)dx\\ &=\frac{\beta^q a_2^p}{\Gamma(p)\Gamma(q-p)}\sum_{k=0}^\infty\frac{(p)_k}{k!(q-p)_k}\left(\frac{a_1}{a_2}-\beta\right)^k I(k), \end{aligned} \tag{3.2}$$

where the transformation $w=a_2(y-\beta x)/\beta$ has been applied and $I(k)$ denotes the integral

$$I(k)=\int_0^\infty x^{p-1}\exp(-a_2x)\Gamma\left(k+q-p,\frac{a_2(1-\beta)x}{\beta}\right)dx. \tag{3.3}$$

Application of Lemma A.1 in the appendix shows that $I(k)$ can be expressed in terms of the Gauss hypergeometric function as

$$I(k) = \frac{\beta^p (1-\beta)^{k+q}}{p a_2^p} \Gamma(k+q) \, {}_2F_1(1, k+q; p+1; \beta). \tag{3.4}$$

If $q-p$ is an integer, then by using the property

$$\Gamma(n,z) = (n-1)! \exp(-z) \sum_{m=0}^{n-1} \frac{z^k}{k!}, \tag{3.5}$$

one can obtain the elementary expression

$$\begin{aligned}
I(k) &= \int_0^\infty x^{p-1} \exp\left(-a_2 x\right)(k+q-p-1)! \sum_{m=0}^{k+q-p-1} \frac{1}{m!} \left\{\frac{a_2(1-\beta)x}{\beta}\right\}^m dx \\
&= (k+q-p-1)! \sum_{m=0}^{k+q-p-1} \frac{1}{m!} \left\{\frac{a_2(1-\beta)}{\beta}\right\}^m \int_0^\infty x^{m+p-1} \exp\left(-a_2 x\right) dx \\
&= (k+q-p-1)! \left(\frac{a_2}{\beta}\right)^p \sum_{m=0}^{k+q-p-1} \frac{\Gamma(p+m)(1-\beta)^m}{m!}.
\end{aligned} \tag{3.6}$$

Expressions for R can be obtained by substituting (3.4) and (3.6) into (3.2).

4. Cherian's bivariate gamma distribution

Cherian's [4] bivariate gamma distribution has the joint pdf specified by

$$f(x,y) = K \exp(-x) \exp(-y) \int_0^{\min(x,y)} (x-z)^{\theta_1 - 1} (y-z)^{\theta_2 - 1} z^{\theta_3 - 1} \exp(z) dz \tag{4.1}$$

for $x > 0$, $y > 0$, $\theta_1 > 0$, $\theta_2 > 0$, and $\theta_3 > 0$, where K denotes the normalizing constant given by

$$\frac{1}{K} = \Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3). \tag{4.2}$$

For this distribution, the form of R can be expressed as

$$\begin{aligned}
R &= K \int_0^\infty \int_x^\infty \int_0^{\min(x,y)} (x-z)^{\theta_1-1} (y-z)^{\theta_2-1} z^{\theta_3-1} \exp(z-x-y) dz\, dy\, dx \\
&= K \int_0^\infty \int_0^x (x-z)^{\theta_1-1} \exp(-x) z^{\theta_3-1} \exp(z) \left\{ \int_x^\infty (y-z)^{\theta_2-1} \exp(-y) dy \right\} dz\, dx \\
&= K \int_0^\infty \int_0^x (x-z)^{\theta_1-1} \exp(-x) z^{\theta_3-1} \exp(z) \left\{ \int_{x-z}^\infty w^{\theta_2-1} \exp(-w) dw \right\} dz\, dx
\end{aligned}$$

$$
\begin{aligned}
&= K\int_0^\infty \int_0^x (x-z)^{\theta_1-1}\exp(-x)z^{\theta_3-1}\Gamma(\theta_2,x-z)\,dz\,dx\\
&= K\int_0^\infty \int_0^x (x-z)^{\theta_1-1}\exp(-x)z^{\theta_3-1}\Gamma(\theta_2,x-z)\,dz\,dx\\
&= K\int_0^\infty x^{\theta_1+\theta_3-1}\exp(-x)\int_0^1 t^{\theta_1-1}(1-t)^{\theta_3-1}\Gamma(\theta_2,xt)\,dt\,dx,
\end{aligned}
\tag{4.3}
$$

where the transformations $w = y - z$ and $t = (x - z)/x$ have been applied. Application of Lemma A.2 shows that the inner integral in (4.3) can be calculated as

$$
\begin{aligned}
&\int_0^1 t^{\theta_1-1}(1-t)^{\theta_3-1}\Gamma(\theta_2,xt)\,dt\\
&\qquad= \Gamma(\theta_2)B(\theta_1,\theta_3) + \theta_2^{-1}B(\theta_3,\theta_1+\theta_2)x^{\theta_2}\,{}_2F_2(\theta_2,\theta_1+\theta_2;\theta_2+1;\theta_1+\theta_2+\theta_3;-x),
\end{aligned}
\tag{4.4}
$$

and thus R can be rewritten as

$$
R = \frac{1}{\Gamma(\theta_1)\Gamma(\theta_3)} + \theta_2^{-1}B(\theta_3,\theta_1+\theta_2)I, \tag{4.5}
$$

where I denotes the integral

$$
I = \int_0^\infty x^{\theta_1+\theta_2+\theta_3-1}\exp(-x)\,{}_2F_2(\theta_2,\theta_1+\theta_2;\theta_2+1;\theta_1+\theta_2+\theta_3;-x)\,dx. \tag{4.6}
$$

This integral can be calculated by an application of Lemma A.3 to yield

$$
I = \Gamma(\theta_1+\theta_2+\theta_3)\,{}_2F_1(\theta_2,\theta_1+\theta_2;\theta_2+1;-1), \tag{4.7}
$$

and hence it follows from (4.5) that the form of R for Cherian's bivariate gamma distribution is given by

$$
R = 1 + \theta_2^{-1}\Gamma(\theta_1+\theta_2)\Gamma(\theta_3)\,{}_2F_1(\theta_2,\theta_1+\theta_2;\theta_2+1;-1). \tag{4.8}
$$

5. Arnold and Strauss' bivariate gamma distribution

Arnold and Strauss's [1] bivariate gamma distribution has the joint pdf specified by

$$
f(x,y) = Kx^{\alpha-1}y^{\beta-1}\exp\{-(ax+by+cxy)\} \tag{5.1}
$$

for $x > 0$, $y > 0$, $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$, and $c > 0$, where $K = K(a,b,c,\alpha,\beta)$ denotes the normalizing constant given by

$$
\frac{1}{K} = b^{\alpha-\beta}c^{-\alpha}\Gamma(\alpha)\Gamma(\beta)\Psi\left(\alpha,\alpha-\beta+1,\frac{ab}{c}\right). \tag{5.2}
$$

For this distribution, the form of R can be derived as follows:

$$\begin{aligned} R &= K\int_0^\infty x^{\alpha-1}\exp(-ax)\int_x^\infty y^{\beta-1}\exp\{-(b+cx)y\}\,dy\,dx \\ &= K\int_0^\infty \frac{x^{\alpha-1}\exp(-ax)}{(b+cx)^\beta}\int_{(b+cx)x}^\infty z^{\beta-1}\exp(-z)dz\,dx \\ &= K\int_0^\infty \frac{x^{\alpha-1}\exp(-ax)\Gamma(\beta,(b+cx)x)}{(b+cx)^\beta}dx, \end{aligned} \tag{5.3}$$

where the transformation $z=(b+cx)y$ has been applied. The integral in (5.3) cannot be simplified further into its general form. However, if β is an integer, then using (3.5) one can rewrite (5.3) as

$$\begin{aligned} R &= K(\beta-1)!\int_0^\infty \frac{x^{\alpha-1}\exp(-ax)}{(b+cx)^\beta}\exp\{-(b+cx)x\}\sum_{l=0}^{\beta-1}\frac{(b+cx)^l x^l}{l!}dx \\ &= K(\beta-1)!\sum_{l=0}^{\beta-1}\frac{I(l)}{l!}, \end{aligned} \tag{5.4}$$

where $I(l)$ denotes the integral

$$I(l) = \int_0^\infty (b+cx)^{l-\beta}x^{l+\alpha-1}\exp\{-(ax+bx+cx^2)\}dx. \tag{5.5}$$

Again, this integral cannot be simplified further unless α is also an integer. If this is assumed then—by transforming $y=b+cx$—one can express

$$\begin{aligned} I(l) &= c^{-(l+\alpha)}\exp\left\{\frac{(a+b)^2}{4c}\right\}\int_b^\infty y^{l-\beta}(y-b)^{l+\alpha-1}\exp\left\{-\frac{1}{c}\left(y+\frac{a-b}{2}\right)^2\right\}dy \\ &= c^{-(l+\alpha)}\exp\left\{\frac{(a+b)^2}{4c}\right\}\int_b^\infty y^{l-\beta}\left[\sum_{m=0}^{l+\alpha-1}\binom{l+\alpha-1}{m}(-b)^{l+\alpha-1-m}y^m\right] \\ &\qquad\qquad \times\exp\left\{-\frac{1}{c}\left(y+\frac{a-b}{2}\right)^2\right\}dy \\ &= c^{-(l+\alpha)}\exp\left\{\frac{(a+b)^2}{4c}\right\}\sum_{m=0}^{l+\alpha-1}\binom{l+\alpha-1}{m}(-b)^{l+\alpha-1-m}J(m,l), \end{aligned} \tag{5.6}$$

where

$$J(m,l) = \int_b^\infty y^{m-\beta+l}\exp\left\{-\frac{1}{c}\left(y+\frac{a-b}{2}\right)^2\right\}dy. \tag{5.7}$$

The integral $J(m,l)$ can be easily calculated. Three cases need to be considered.

(1) If $m=\beta-l$, then

$$J(m,l)=\sqrt{\pi c}\left\{1-\Phi\left(\frac{a+b}{\sqrt{2c}}\right)\right\}, \tag{5.8}$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution.

(2) If $m>\beta-l$, then—by transforming $z=\{y+(a-b)/2\}^2$—one can write

$$\begin{aligned}
J(m,l)&=\frac{1}{2}\int_{(a+b)^2/4}^{\infty}\frac{1}{\sqrt{z}}\left(\sqrt{z}+\frac{b-a}{2}\right)^{m-\beta+l}\exp\left(-\frac{z}{c}\right)dz\\
&=\frac{1}{2}\int_{(a+b)^2/4}^{\infty}\frac{1}{\sqrt{z}}\left\{\sum_{n=0}^{m-\beta+l}\binom{m-\beta+l}{n}\left(\frac{b-a}{2}\right)^{m-\beta+l-n}z^{n/2}\right\}\exp\left(-\frac{z}{c}\right)dz\\
&=\frac{1}{2}\sum_{n=0}^{m-\beta+l}\binom{m-\beta+l}{n}\left(\frac{b-a}{2}\right)^{m-\beta+l-n}\int_{(a+b)^2/4}^{\infty}z^{(n-1)/2}\exp\left(-\frac{z}{c}\right)dz\\
&=\frac{1}{2}\sum_{n=0}^{m-\beta+l}\binom{m-\beta+l}{n}\left(\frac{b-a}{2}\right)^{m-\beta+l-n}c^{(n+1)/2}\Gamma\left(\frac{n+1}{2},\frac{(a+b)^2}{4c}\right),
\end{aligned} \tag{5.9}$$

a finite sum of incomplete gamma functions.

(3) If $m<\beta-l$, then the same transformation as above yields

$$\begin{aligned}
J(m,l)&=\frac{1}{2}\int_{(a+b)^2/4}^{\infty}z^{(m-\beta+l-1)/2}\left(1+\frac{b-a}{2\sqrt{z}}\right)^{m-\beta+l}\exp\left(-\frac{z}{c}\right)dz\\
&=\frac{1}{2}\int_{(a+b)^2/4}^{\infty}z^{(m-\beta+l-1)/2}\left\{\sum_{n=0}^{\infty}\binom{m-\beta+l}{n}\left(\frac{b-a}{2\sqrt{z}}\right)^{n}\right\}\exp\left(-\frac{z}{c}\right)dz\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\binom{m-\beta+l}{n}\left(\frac{b-a}{2}\right)^{n}\int_{(a+b)^2/4}^{\infty}z^{(m-\beta+l-n-1)/2}\exp\left(-\frac{z}{c}\right)dz\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\binom{m-\beta+l}{n}\left(\frac{b-a}{2}\right)^{n}c^{(m-\beta+l-n+1)/2}\times\Gamma\left(\frac{m-\beta+l-n+1}{2},\frac{(a+b)^2}{4c}\right),
\end{aligned} \tag{5.10}$$

an infinite sum of incomplete gamma functions.

Combining (5.4)–(5.10), the reliability R can be expressed as an infinite sum of incomplete gamma functions for integer values of α and β and general a, b, and c. However, if $a=b$, then one can express R as a finite sum of incomplete gamma functions. This follows because if $a=b$, then $J(m,l)$ reduces to

$$\begin{aligned}
J(m,l)&=2^{-1}c^{1+(m-\beta)/2}\int_{a^2/c}^{\infty}z^{(m-\beta)/2}\exp(-z)dz\\
&=2^{-1}c^{1+(m-\beta)/2}\Gamma\left(1+\frac{m-\beta}{2},\frac{a^2}{c}\right)
\end{aligned} \tag{5.11}$$

(after substituting $z = y^2/c$), and combining (5.11) with (5.4)–(5.6), one obtains

$$R = \frac{K(\beta-1)!\exp\left(a^2/c\right)}{2(-a)^{1-\alpha}c^{\alpha+\beta/2-1}}\sum_{l=0}^{\beta-1}\sum_{m=0}^{l+\alpha-1}\frac{(-a)^{l-m}}{l!c^{l-m/2}}\binom{l+\alpha-1}{m}\Gamma\left(1+\frac{m-\beta}{2},\frac{a^2}{c}\right), \tag{5.12}$$

a finite sum of incomplete gamma functions. If also $\alpha = \beta$, then it is easy to see that $R = 1/2$.

6. Becker and Roux's bivariate gamma distribution

Becker and Roux's [3] bivariate gamma distribution has the joint pdf given by

$$f(x,y) = \begin{cases} \dfrac{\lambda_2}{\Gamma(h)\Gamma(l)\alpha_1^h\alpha_2^l}x^{h-1}\{\lambda_2(y-x)+x\}^{l-1}\exp\left\{-\left(\dfrac{1}{\alpha_1}+\dfrac{1-\lambda_2}{\alpha_2}\right)x-\dfrac{\lambda_2}{\alpha_2}y\right\} & \text{if } x<y, \\ \dfrac{\lambda_1}{\Gamma(h)\Gamma(l)\alpha_1^h\alpha_2^l}y^{l-1}\{\lambda_1(x-y)+y\}^{h-1}\exp\left\{-\left(\dfrac{1}{\alpha_1}+\dfrac{1-\lambda_1}{\alpha_1}\right)y-\dfrac{\lambda_1}{\alpha_1}x\right\} & \text{if } x>y, \end{cases} \tag{6.1}$$

for $x>0$, $y>0$, $h>0$, $l>0$, $\lambda_1>0$, $\lambda_2>0$, $\alpha_1>0$, and $\alpha_2>0$. For this distribution, the form of R can be expressed as

$$\begin{aligned} R &= \frac{\lambda_2}{\Gamma(h)\Gamma(l)\alpha_1^h\alpha_2^l}\int_0^\infty x^{h-1}\exp\left\{-\left(\frac{1}{\alpha_1}+\frac{1-\lambda_2}{\alpha_2}\right)x\right\} \\ &\qquad\times\int_x^\infty\{\lambda_2(y-x)+x\}^{l-1}\exp\left\{-\frac{\lambda_2}{\alpha_2}y\right\}dy\,dx \\ &= \frac{1}{\Gamma(h)\Gamma(l)\alpha_1^h\alpha_2^l}\int_0^\infty x^{h-1}\exp\left(-\frac{x}{\alpha_1}\right)\int_x^\infty z^{l-1}\exp\left(-\frac{z}{\alpha_2}\right)dz\,dx \\ &= \frac{1}{\Gamma(h)\Gamma(l)\alpha_1^h}\int_0^\infty x^{h-1}\exp\left(-\frac{x}{\alpha_1}\right)\Gamma\left(l,\frac{x}{\alpha_2}\right)dx, \end{aligned} \tag{6.2}$$

where the transformation $z=\lambda_2(y-x)+x$ has been applied. By application of Lemma A.1 in the appendix, the integral in (6.2) can be calculated as

$$\int_0^\infty x^{h-1}\exp\left(-\frac{x}{\alpha_1}\right)\Gamma\left(l,\frac{x}{\alpha_2}\right)dx = \frac{\alpha_1^{h+l}\alpha_2^h}{h\left(\alpha_1+\alpha_2\right)^{h+l}}\,{}_2F_1\left(1,h+l;h+1;\frac{\alpha_2}{\alpha_1+\alpha_2}\right). \tag{6.3}$$

Thus, the form of R can be expressed in terms of the Gauss hypergeometric function as

$$R = \frac{\alpha_1^l\alpha_2^h}{hB(h,l)\left(\alpha_1+\alpha_2\right)^{h+l}}\,{}_2F_1\left(1,h+l;h+1;\frac{\alpha_2}{\alpha_1+\alpha_2}\right). \tag{6.4}$$

If l is an integer, then the hypergeometric term reduces to a finite sum, yielding

$$R = \frac{\alpha_2^h}{hB(h,l)\left(\alpha_1+\alpha_2\right)^h}\sum_{k=0}^{l-1}\frac{(h)_k(1-l)_k}{(h+1)_k k!}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^k. \tag{6.5}$$

7. Smith and Adelfang's bivariate gamma distribution

Smith and Adelfang's [34] bivariate gamma distribution has the joint pdf specified by

$$f(x,y) = Kx^{\gamma_1-1}y^{\gamma_2-1}\exp\left(-\frac{x+y}{1-\eta}\right)\sum_{k=0}^{\infty}a_k I_{\gamma_2+k-1}\left(\frac{2\sqrt{\eta xy}}{1-\eta}\right) \tag{7.1}$$

for $x>0$, $y>0$, $\gamma_1>0$, $\gamma_2>0$, and $0<\eta<1$, where K denotes the normalizing constant given by

$$\frac{1}{K} = (1-\eta)^{\gamma_1}\Gamma(\gamma_1)\Gamma(\gamma_2-\gamma_1). \tag{7.2}$$

Using the definition of $I_m(\cdot)$ given in Section 1, the corresponding form of R can be expressed as

$$\begin{aligned}R &= K\sum_{k=0}^{\infty}a_k\sum_{l=0}^{\infty}\frac{1}{l!\Gamma(\gamma_2+l+k)}\left(\frac{\sqrt{\eta}}{1-\eta}\right)^{k+2l+\gamma_2-1}\\ &\qquad\times\int_0^{\infty}x^{\gamma_1+l+(\gamma_2+k)/2-1}\int_x^{\infty}y^{\gamma_2+l+(\gamma_2+k-1)/2-1}\exp\left(-\frac{x+y}{1-\eta}\right)dy\,dx\\ &= K\sum_{k=0}^{\infty}a_k\sum_{l=0}^{\infty}\frac{\eta^{(k+2l+\gamma_2-1)/2}(1-\eta)^{(\gamma_2-k-2l+1)/2}}{l!\Gamma(\gamma_2+l+k)}J(k,l),\end{aligned} \tag{7.3}$$

where $J(k,l)$ denotes the integral

$$J(k,l) = \int_0^{\infty}x^{\gamma_1+(k+2l+\gamma_2-1)/2-1}\exp\left(-\frac{x}{1-\eta}\right)\Gamma\left(\frac{k+2l+3\gamma_2-1}{2},\frac{x}{1-\eta}\right)dx. \tag{7.4}$$

Application of Lemma A.1 in the appendix shows that $J(k,l)$ can be expressed in terms of the Gauss hypergeometric function as

$$\begin{aligned}J(k,l) = {}&\frac{(1-\eta)^{\gamma_1+l+(\gamma_2+k-1)/2}\Gamma(\gamma_1+2\gamma_2+k+2l-1)}{2^{\gamma_1+2\gamma_2+k+2l-2}(2\gamma_1+\gamma_2+k+2l-1)}\\ &\times{}_2F_1\left(1,\gamma_1+2\gamma_2+k+2l-1;\gamma_1+l+\frac{\gamma_2+k+1}{2};\frac{1}{2}\right).\end{aligned} \tag{7.5}$$

Substituting this into (7.3), one obtains the expression

$$\begin{aligned}R &= K(1-\eta)^{\gamma_1+\gamma_2}2^{2-\gamma_1-2\gamma_2}\\ &\quad\times\sum_{k=0}^{\infty}a_k\sum_{l=0}^{\infty}\frac{\eta^{(k+2l+\gamma_2-1)/2}\Gamma(\gamma_1+2\gamma_2+k+2l-1)}{l!2^{k+2l}(2\gamma_1+\gamma_2+k+2l-1)\Gamma(\gamma_2+l+k)}\\ &\qquad\times{}_2F_1\left(1,\gamma_1+2\gamma_2+k+2l-1;\gamma_1+l+\frac{\gamma_2+k+1}{2};\frac{1}{2}\right),\end{aligned} \tag{7.6}$$

a double infinite sum of terms involving the Gauss hypergeometric function.

8. Conclusions

We have calculated the forms of $R = \Pr(X < Y)$ for six flexible families of bivariate gamma distributions. It would be of interest to emulate this work for other continuous bivariate distributions, including bivariate Laplace distributions, bivariate logistic distributions, and bivariate extreme value distributions. It would also be of interest to extend this work for continuous multivariate distributions. We hope to address some of these issues in a future paper.

Appendix

Some technical lemmas required for the calculations above are noted below.

Lemma A.1 (Gradshteyn and Ryzhik [7, equation (6.455.1)]). *For $c>0$, $p>0$, and $\alpha+\nu>0$,*

$$\int_0^\infty x^{\alpha-1}\exp(-px)\Gamma(\nu,cx)dx = -\frac{c^\nu\Gamma(\alpha+\nu)}{\nu p^{\alpha+\nu}}\,{}_2F_1\left(\nu,\alpha+\nu;\nu+1;-\frac{c}{p}\right)+\frac{\Gamma(\nu)\Gamma(\alpha)}{p^\alpha}. \quad \text{(A.1)}$$

Lemma A.2 (Prudnikov et al. [31, Volume 2, equation (2.10.2.2)]). *For $\alpha > 0$ and $\beta > 0$,*

$$\begin{aligned}&\int_0^a x^{\alpha-1}(a-x)^{\beta-1}\Gamma(\nu,cx)dx\\ &\quad = a^{\alpha+\beta-1}\Gamma(\nu)B(\alpha,\beta)-\nu^{-1}a^{\alpha+\beta+\nu-1}c^\nu B(\beta,\alpha+\nu)\,{}_2F_2(\nu,\alpha+\nu;\nu+1,\alpha+\beta+\nu;-ac).\end{aligned} \quad \text{(A.2)}$$

Lemma A.3 (Prudnikov et al. [32, Volume 3, equation (2.22.3.5)]). *For $c > 0$ and $\sigma > w$,*

$$\int_0^\infty x^{c-1}\exp(-\sigma x)\,{}_2F_2(a,b;d,c;wx)dx = \sigma^{-c}\Gamma(c)\,{}_2F_1\left(a,b;d;\frac{w}{\sigma}\right). \quad \text{(A.3)}$$

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