REALIZATION PROBLEM FOR POSITIVE LINEAR SYSTEMS WITH TIME DELAY

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The realization problem for positive single-input single-output discrete-time systems with one time delay is formulated and solved. Necessary and sufficient conditions for the solvability of the realization problem are established. A procedure for computation of a minimal positive realization of a proper rational function is presented and illustrated by an example.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology, and medicine, and so forth.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [4, 5]. Recent developments in positive systems theory and some new results are given in [5]. Realizations problem of positive linear systems without time delays has been considered in many papers and books [1, 4, 5].

Explicit solution of equations describing the discrete-time systems with time delay has been given in [2].

Recently, the reachability, controllability, and minimum energy control of positive linear discrete-time systems with time delays have been considered in [3, 6].

In this paper, the realization problem for positive single-input single-output discrete-time systems with time delay will be formulated and solved. Necessary and sufficient conditions for the solvability of the realization problem will be established and a procedure for computation of a minimal positive realization of a proper rational function will be presented.

To the best knowledge of the author, the realization problem for positive linear systems with time delays has not been considered yet.

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2. Problem formulation

Consider the single-input single-output discrete-time linear system with one time delay

$$x_{i+1} = \mathbf{A}_0 x_i + \mathbf{A}_1 x_{i-1} + \mathbf{b} u_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \ldots\}, \tag{2.1a}$$

$$y_i = \mathbf{c} x_i + \mathbf{d} u_i, \tag{2.1b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $y_i \in \mathbb{R}$ are the state vector, input, and output, respectively, and $\mathbf{A}_k \in \mathbb{R}^{n\times n}$, $k = 0, 1$, $\mathbf{b} \in \mathbb{R}^n$, $\mathbf{c} \in \mathbb{R}^{1\times n}$, and $\mathbf{d} \in \mathbb{R}$.

Initial conditions for (2.1a) are given by

$$x_{-1}, x_0 \in \mathbb{R}^n. \tag{2.2}$$

Let $\mathbb{R}_+^{n\times m}$ be the set of $n \times m$ real matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

Definition 2.1 (see [3]). The system (2.1) is called (internally) positive if for every $x_{-1}, x_0 \in \mathbb{R}_+^n$ and all inputs $u_i \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$, $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+$ for $i \in \mathbb{Z}_+$.

THEOREM 2.2 (see [3]). *The system (2.1) is positive if and only if*

$$\mathbf{A}_0 \in \mathbb{R}_+^{n\times n}, \quad \mathbf{A}_1 \in \mathbb{R}_+^{n\times n}, \quad \mathbf{b} \in \mathbb{R}_+^n, \quad \mathbf{c} \in \mathbb{R}_+^{1\times n}, \quad d \in \mathbb{R}_+. \tag{2.3}$$

The transfer function of (2.1) is given by

$$T(z) = \mathbf{c}\left[\mathbf{I}_n z - \mathbf{A}_0 - \mathbf{A}_1 z^{-1}\right]^{-1}\mathbf{b} + d. \tag{2.4}$$

Definition 2.3. Matrices (2.3) are called positive realizations of a given proper rational function $T(z)$ if and only if they satisfy the equality (2.4). A realization (2.3) is called minimal if and only if the dimension n of $\mathbf{A}_0$ and $\mathbf{A}_1$ is minimal among all realizations of $T(z)$.

The positive realization problem can be stated as follows.

Given a proper rational function $T(z)$, find a positive realization (2.3) of the rational function $T(z)$.

Necessary and sufficient conditions for the solvability of the problem will be established and a procedure for computation of a positive realization will be presented.

3. Problem solution

The transfer function (2.4) can be rewritten in the form

$$\begin{aligned} T(z) &= \mathbf{c}\left[z^{-1}\left(\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1\right)\right]^{-1}\mathbf{b} + d \\ &= \frac{\mathbf{c}z\left[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1\right]_{ad}\mathbf{b}}{\det\left[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1\right]} + d = \frac{zl(z)}{d(z)} + d, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} l(z) &= \mathbf{c}\left[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1\right]_{ad}\mathbf{b} = l_{2(n-1)} z^{2(n-1)} + l_{2n-3} z^{2n-3} + \cdots + l_1 z + l_0, \\ d(z) &= \det\left[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1\right] = z^{2n} - a_{2n-1} z^{2n-1} - \cdots - a_1 z - a_0, \end{aligned} \tag{3.2}$$

and $[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1]_{ad}$ denotes the adjoint matrix for $[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1]$.

From (3.1), we have

$$d = \lim_{z\to\infty} T(z) \tag{3.3}$$

since $\lim_{z\to\infty}[z^{-1}(\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1)]^{-1} = 0$.

The strictly proper part of $T(z)$ is given by

$$T_{sp}(z) = T(z) - d = \frac{zl(z)}{d(z)}. \tag{3.4}$$

Therefore, the positive realization problem has been reduced to finding matrices

$$\mathbf{A}_0 \in \mathbb{R}_+^{n\times n}, \quad \mathbf{A}_1 \in \mathbb{R}_+^{n\times n}, \quad \mathbf{b} \in \mathbb{R}_+^{n}, \quad \mathbf{c} \in \mathbb{R}_+^{1\times n} \tag{3.5}$$

for a given strictly proper rational matrix (3.4).

Lemma 3.1. *The strictly proper transfer function (3.4) has the form*

$$T'_{sp}(z) = \frac{l(z)}{d'(z)} \tag{3.6}$$

if and only if $\det \mathbf{A}_1 = 0$, *where*

$$d'(z) = z^{2n-1} - a_{2n-1}z^{2n-2} - \cdots - a_2 z - a_1. \tag{3.7}$$

Proof. From the definition of (3.2) of $d(z)$ for $z = 0$, it follows that $a_0 = \det \mathbf{A}_1$. Note that $d(z) = zd'(z)$ if and only if $a_0 = 0$ and (3.4) can be reduced to (3.6). □

Lemma 3.2. *If the matrices* $\mathbf{A}_0$ *and* $\mathbf{A}_1$ *have the forms*

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & \cdots & 0 & 0 \\ a_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-7} & 0 & \cdots & 0 & 0 \\ a_{2n-5} & 0 & \cdots & 0 & a_{2n-3} \\ 0 & 0 & \cdots & 0 & a_{2n-1} \end{bmatrix} \in \mathbb{R}^{n\times n},$$
$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{2(n-4)} & 0 & \cdots & 0 & 0 & 0 \\ a_{2(n-3)} & 0 & \cdots & 1 & 0 & a_{2(n-2)} \\ 0 & 0 & \cdots & 0 & 1 & a_{2(n-1)} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{3.8}$$

then

$$\det[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1] = z^{2n} - a_{2n-1}z^{2n-1} - a_{2(n-1)}z^{2(n-1)} - \cdots - a_1 z - a_0. \tag{3.9}$$

Proof. Expansion of the determinant with respect to the first row yields

$$\begin{aligned}
&\det[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1] \\
&= \begin{vmatrix}
z^2 & 0 & \cdots & 0 & 0 & -1 \\
-a_1 z - a_0 & z^2 & \cdots & 0 & 0 & 0 \\
-a_3 z - a_2 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{2n-7}z - a_{2(n-4)} & 0 & \cdots & z^2 & 0 & 0 \\
-a_{2n-5}z - a_{2(n-3)} & 0 & \cdots & -1 & z^2 & -a_{2n-3}z - a_{2(n-2)} \\
0 & 0 & \cdots & 0 & -1 & z^2 - a_{2n-1}z - a_{2(n-1)}
\end{vmatrix} \\
&= z^{2(n-2)}\left(z^4 - a_{2n-1}z^3 - a_{2(n-1)}z^2 - a_{2n-3}z - a_{2(n-2)}\right) + (-1)^n \\
&\quad \times \begin{vmatrix}
-a_1 z - a_0 & z^2 & 0 & \cdots & 0 & 0 \\
-a_3 z - a_2 & -1 & z^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-a_{2n-7}z - a_{2(n-4)} & 0 & 0 & \cdots & z^2 & 0 \\
-a_{2n-5}z - a_{2(n-3)} & 0 & 0 & \cdots & -1 & z^2 \\
0 & 0 & 0 & \cdots & 0 & -1
\end{vmatrix} = \cdots \\
&= z^{2n} - a_{2n-1}z^{2n-1} - a_{2(n-1)}z^{2(n-1)} - \cdots - a_1 z - a_0.
\end{aligned} \tag{3.10}$$

□

Matrices $\mathbf{A}_0$ and $\mathbf{A}_1$ having the forms (3.8) will be called the matrices in canonical forms.

The following two remarks are in order.

Remark 3.3. The matrices (3.8) have nonnegative entries if and only if the coefficients a_k, $k = 0, 1, \ldots, 2n-1$, of the polynomial (3.9) are nonnegative.

Remark 3.4. The dimension $n \times n$ of matrices (3.8) is the smallest possible one for (3.4).

Definition 3.5. The pair of matrices $(\mathbf{A}_0, \mathbf{A}_1)$ is called cyclic if and only if the characteristic polynomial $\varphi(z) = \det[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1]$ is equal to the minimal polynomial $\psi(z)$ of the matrix $[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1]$, $\varphi(z) = \psi(z)$.

Lemma 3.6. *If the matrices* $\mathbf{A}_0$ *and* $\mathbf{A}_1$ *have the canonical forms (3.8), then the pair* $(\mathbf{A}_0, \mathbf{A}_1)$ *is cyclic.*

Proof. It is well known that $\varphi(z) = \psi(z)$ if and only if the greatest common divisor of all $n-1$ degree minors of the polynomial matrix $[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1]$ is equal to 1. Using (3.8), we obtain

$$
\begin{aligned}
&[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1] = \\
&\begin{bmatrix}
z^2 & 0 & \cdots & 0 & 0 & -1 \\
-a_1 z - a_0 & z^2 & \cdots & 0 & 0 & 0 \\
-a_3 z - a_2 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{2n-7} z - a_{2(n-4)} & 0 & \cdots & z^2 & 0 & 0 \\
-a_{2n-5} z - a_{2(n-3)} & 0 & \cdots & -1 & z^2 & -a_{2n-3} z - a_{2(n-2)} \\
0 & 0 & \cdots & 0 & -1 & z^2 - a_{2n-1} z - a_{2(n-1)}
\end{bmatrix}.
\end{aligned} \tag{3.11}
$$

Note that the $n-1$ degree minor corresponding to the entry $-a_1 z - a_0$ of the matrix (3.11) is equal to 1. Therefore, we have $\varphi(z) = \psi(z)$ and by Definition 3.5, the pair $(\mathbf{A}_0, \mathbf{A}_1)$ is cyclic. □

Let

$$
[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1]_{ad} = \begin{bmatrix} a_{11}(z) & \cdots & a_{1n}(z) \\ \vdots & \ddots & \vdots \\ a_{n1}(z) & \cdots & a_{nn}(z) \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}, \tag{3.12a}
$$

where

$$
a_{ij}(z) = \sum_{k=0}^{2(n-1)} a_{ij}^k z^k, \quad i, j = 1, \ldots, n. \tag{3.12b}
$$

Using (3.2) and (3.12), we obtain

$$
\begin{aligned}
zl(z) &= z\mathbf{c}[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1]_{ad}\mathbf{b} = \sum_{i=1}^{n}\sum_{j=1}^{n} z a_{ij}(z) c_i b_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=0}^{2(n-1)} a_{ij}^k c_i b_j z^{k+1} \\
&= l_{2(n-1)} z^{2n-1} + l_{2n-3} z^{2(n-1)} + \cdots + l_1 z^2 + l_0 z.
\end{aligned} \tag{3.13}
$$

Comparison of the coefficients at like powers of z in (3.13) yields

$$
\mathbf{A}\mathbf{x} = \mathbf{l}, \tag{3.14}
$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11}^{0} & a_{12}^{0} & \cdots & a_{1n}^{0} & a_{21}^{0} & a_{22}^{0} & \cdots & a_{n,n-1}^{0} & a_{n,n}^{0} \\ a_{11}^{1} & a_{12}^{1} & \cdots & a_{1n}^{1} & a_{21}^{1} & a_{22}^{1} & \cdots & a_{n,n-1}^{1} & a_{n,n}^{1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{11}^{2n-3} & a_{12}^{2n-3} & \cdots & a_{1n}^{2n-3} & a_{21}^{2n-3} & a_{22}^{2n-3} & \cdots & a_{n,n-1}^{2n-3} & a_{n,n}^{2n-3} \\ a_{11}^{2(n-1)} & a_{12}^{2(n-1)} & \cdots & a_{1n}^{2(n-1)} & a_{21}^{2(n-1)} & a_{22}^{2(n-1)} & \cdots & a_{n,n-1}^{2(n-1)} & a_{n,n}^{2(n-1)} \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} c_1 b_1 \\ c_1 b_2 \\ \vdots \\ c_1 b_n \\ c_2 b_1 \\ c_2 b_2 \\ \vdots \\ c_n b_{n-1} \\ c_n b_n \end{bmatrix}; \qquad \mathbf{l} = \begin{bmatrix} l_0 \\ l_1 \\ \vdots \\ l_{2n-3} \\ l_{2(n-1)} \end{bmatrix}. \tag{3.15}$$

By Kronecker-Capelli theorem, the matrix equation (3.14) has a solution $\mathbf{x}$ if and only if

$$\operatorname{rank}[\mathbf{A},\mathbf{l}] = \operatorname{rank}\mathbf{A}, \tag{3.16}$$

therefore, we have the following theorem.

Theorem 3.7. *The positive realization problem has a solution only if the condition (3.16) is satisfied.*

Note that the matrix $\mathbf{A}$ of the dimension $(2n-1)\times n^2$ has more columns than rows. If the condition (3.16) is satisfied then without loss of generality, we may assume that the matrix $\mathbf{A}$ has full row rank equal to $2n-1$ (otherwise, we may eliminate the linearly dependent equations from (3.14)).

Choosing $n^2 - 2n + 1 = (n-1)^2$ nonnegative components of the vector $\mathbf{x}$ and solving the corresponding matrix equation with nonsingular $(2n-1)\times(2n-1)$ coefficient matrix, we may compute the desired entries of $\mathbf{b}$ and $\mathbf{c}$ (that should be nonnegative). Therefore, we have established the following necessary and sufficient conditions for the existence of the solution to the positive realization problem.

Theorem 3.8. *The positive realization problem has a solution if and only if the following conditions are satisfied.*

(1) $T(\infty) = \lim_{z\to\infty} T(z) \in \mathbb{R}_+$.
(2) *The coefficients* a_k, $k = 0,1,\ldots,2n-1$, *of the polynomial* $d(z)$ *are nonnegative.*
(3) *The matrix equation (3.14) has a nonnegative solution,* $\mathbf{x} \in \mathbb{R}_+^{n^2}$.

If the conditions of Theorem 3.7 are satisfied, then the desired positive realization (2.3) of $\mathbf{T}(z)$ can be found by the use of the following procedure.

Procedure 3.9.

Step 1 . Using (3.3) and (3.4), find d and the strictly proper rational function $T_{sp}(z)$.

Step 2 . Knowing the coefficients a_k, $k = 0,1,\ldots,2n-1$, of $d(z)$, find the matrices (3.8).

Step 3 . Find the coefficients a_{ij}^k, $i,j = 1,\ldots,n$, $k = 0,1,\ldots,2(n-1)$, of the adjoint matrix (3.12a) and the matrix equation (3.14).

Step 4 . Find the nonnegative solution $\mathbf{x} \in \mathbb{R}_+^{n^2}$ of (3.14) and the matrices $\mathbf{b}$ and $\mathbf{c}$.

Remark 3.10. A positive realization computed by the use of Procedure 3.9 is a minimal one.

4. Example

Find a positive realization (3.5) of the strictly proper function

$$T_{sp}(z) = \frac{z(l_2z^2 + l_1z + l_0)}{z^4 - a_3z^3 - a_2z^2 - a_1z - a_0}, \quad (l_i \geq 0,\ i = 0,1,2;\ a_k \geq 0,\ k = 0,1,2,3). \tag{4.1}$$

Using Procedure 3.9, we obtain successively the following steps.

Step 1. $d = 0$ since $T(\infty) = \lim_{z\to\infty} T(z) = 0$.

Step 2. Using (3.8) and (4.1), we obtain

$$\mathbf{A}_0 = \begin{bmatrix} 0 & a_1 \\ 0 & a_3 \end{bmatrix}, \qquad \mathbf{A}_1 = \begin{bmatrix} 0 & a_0 \\ 1 & a_2 \end{bmatrix}. \tag{4.2}$$

Step 3. Taking into account that

$$[\mathbf{I}_n z^2 - \mathbf{A}_0 z - \mathbf{A}_1]_{ad} = \begin{bmatrix} z^2 & -a_1z - a_0 \\ -1 & z^2 - a_3z - a_2 \end{bmatrix}_{ad} = \begin{bmatrix} z^2 - a_3z - a_2 & a_1z + a_0 \\ 1 & z^2 \end{bmatrix}, \tag{4.3}$$

we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -a_3 & a_1 & 0 & 0 \\ -a_2 & a_0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1b_1 \\ c_1b_2 \\ c_2b_1 \\ c_2b_2 \end{bmatrix} = \begin{bmatrix} l_2 \\ l_1 \\ l_0 \end{bmatrix}. \tag{4.4}$$

Choosing $c_1 > 0$ and $b_1 > 0$ so that $l_2 - c_1b_1 \geq 0$ from (4.4), we obtain

$$\begin{bmatrix} 0 & 0 & 1 \\ a_1 & 0 & 0 \\ a_0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1b_2 \\ c_2b_1 \\ c_2b_2 \end{bmatrix} = \begin{bmatrix} l_2 - c_1b_1 \\ l_1 + a_3c_1b_1 \\ l_0 + a_2c_1b_1 \end{bmatrix}, \tag{4.5}$$

and for $a_1 > 0$,

$$\begin{bmatrix} c_1b_2 \\ c_2b_1 \\ c_2b_2 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{a_1} & 0 \\ 0 & -\dfrac{a_0}{a_1} & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_2 - c_1b_1 \\ l_1 + a_3c_1b_1 \\ l_0 + a_2c_1b_1 \end{bmatrix} = \begin{bmatrix} \dfrac{l_1 + a_3c_1b_1}{a_1} \\ l_0 + a_2c_1b_1 - \dfrac{a_0(l_1 + a_3c_1b_1)}{a_1} \\ l_2 - c_1b_1 \end{bmatrix}. \tag{4.6}$$

Therefore, (4.4) has a positive solution

$$b_1 > 0, \quad c_1 > 0, \quad b_2 = \frac{l_1 + a_3 c_1 b_1}{a_1 c_1}, \quad c_2 = \frac{a_1(l_0 + a_2 c_1 b_1) - a_0(l_1 + a_3 c_1 b_1)}{a_1 b_1} \tag{4.7}$$

if

$$l_2 - c_1 b_1 \geq 0, \qquad a_1(l_0 + a_2 c_1 b_1) > a_0(l_1 + a_3 c_1 b_1). \tag{4.8}$$

From $c_1 b_1 + c_2 b_2 = l_2$, it follows that there exists a positive realization if b_1 and c_1 are chosen so that $(l_1 + a_3 c_1 b_1)[a_1(l_0 + a_2 c_1 b_1) - a_0(l_1 + a_3 c_1 b_1)] + (a_1 c_1 b_1) = a_1^2 c_1 b_1 l_2$.

If $a_0 = 0$, then

$$\mathbf{T}'_{sp}(z) = \frac{l_2 z^2 + l_1 z + l_0}{z^3 - a_3 z^2 - a_2 z - a_1} \tag{4.9}$$

and

$$\mathbf{A}_0 = \begin{bmatrix} 0 & a_1 \\ 0 & a_3 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ 1 & a_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}, \tag{4.10a}$$

where

$$b_2 = \frac{l_1 + a_3 c_1 b_1}{a_1 c_1}, \qquad c_2 = \frac{l_0 + a_2 c_1 b_1}{b_1} \tag{4.10b}$$

for any $b_1 > 0$ and $c_1 > 0$ such that $l_2 - c_1 b_1 \geq 0$ and $c_2 b_2 = l_2 - c_1 b_1$.

From (4.10), it follows that for (4.9) with $l_2 > 0$, $l_1 \geq 0$, $l_0 \geq 0$, and $a_1 > 0$, $a_2 \geq 0$, $a_3 \geq 0$, there exists a positive realization of the form (4.10) if b_1 and c_1 are chosen so that $(l_1 + a_3 c_1 b_1)(l_0 + a_2 c_1 b_1) + a_1(b_1 c_1)^2 = a_1 b_1 c_1 l_2$.

5. Concluding remarks

The realization problem for positive single-input single-output discrete-time systems with one time delay has been formulated and solved. Canonical forms (3.8) of the system matrices $\mathbf{A}_0$ and $\mathbf{A}_1$ have been introduced. It has been shown that the pair (3.8) is cyclic. Necessary and sufficient conditions for the existence of positive minimal realization (2.3) of a proper rational function $T(z)$ have been established. A procedure for computation of a minimal positive realization of proper rational function has been presented and illustrated by an example. The considerations can be extended for the following:

(1) single-input single-output discrete-time linear systems with many time delays;
(2) multi-input multi-output discrete-time linear systems with one and many time delays.

An extension of the considerations for continuous-time linear systems with time delays is also possible.

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