ON THE OPTIMAL CONTROL OF AFFINE NONLINEAR SYSTEMS

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The minimization control problem of quadratic functionals for the class of affine nonlinear systems with the hypothesis of nilpotent associated Lie algebra is analyzed. The optimal control corresponding to the first-, second-, and third-order nilpotent operators is determined. In this paper, we have considered the minimum fuel problem for the multi-input nilpotent control and for a scalar input bilinear system for such systems. For the multi-input system, usually an analytic closed-form solution for the optimal control $u_i^\star(t)$ is not possible and it is necessary to use numerical integration for the set of m nonlinear coupled second-order differential equations. The optimal control of bilinear systems is obtained by considering the Lie algebra generated by the system matrices. It should be noted that we have obtained an open-loop control depending on the initial value of the state x_0.

1. Introduction

Optimal control theory offers modern methods regarding the control of systems, and plays a significant role in the analysis of the linear control characterizing quadratic linear regulators and also the Gaussian quadratic linear control [9, 11]. The use of optimal control in the class of linear systems permits a substantial reduction of the computations determining the laws of optimal control. Moreover, it is an efficient method for solving nonlinear optimal control problems [3]. The Lie brackets generated by the fields of vectors defining the nonlinear system represent a remarkable mathematical tool for the control of affine systems [7, 8, 9, 10, 11].

Optimal control of bilinear systems has been considered by Tzafestas et al. (1984) and by Banks and Yew (1985)—in the latter case the linear quadratic regulator problem is extended to the bilinear quadratic regulator problem.

Bourdache-Siguerdindjane [2] applied the method of Lie algebras to the study of the optimal control regulation of satellites. In [1], Banks and Yew studied the optimal control of energy consumption minimization for a class of bilinear systems and Liu et al. [6] generalized this result to the class of affine nonlinear systems.

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The objective of this paper is to obtain optimal controls for the general class of quadratic functionals with applications in minimum fuel control for affine nonlinear systems and bilinear systems.

2. The problem of optimal control

We consider the class of affine nonlinear dynamic systems

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i, \qquad x(t_0) = x_0, \\ &= f(x) + g(x)u, \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}^1$, $i = 1,\dots,m$, $f(x), g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$. The control problem is to find the optimal control functions $u_i^\star$, $i = 1,2,\dots,m$, which minimize the quadratic functionals

$$J = \frac{1}{2}\int_{t_0}^{t_f} (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)\,dt + \Phi(x(t_f)) \tag{2.2}$$

subject to differential restrictions represented by the dynamic systems (2.1), in which $Q = (q_{ij})$ and $R = (r_{ij})$ are constant symmetric positive definite $(n \times n)$ and $(m \times m)$ matrices, respectively, and the final time t_f is specified. The system vectors f, g, $\Phi(x(t_f))$ are all smooth.

We associate to the nonlinear systems (2.1) the Lie L algebra generated by the systems of the field of vectors

$$\{f, g_1, \dots, g_m\}. \tag{2.3}$$

We will use the notations

$$\begin{aligned} \mathrm{ad}_L^0 &:= L, \\ \mathrm{ad}_L L &= [L,L] = \{[X,Y];\ X \in L,\ Y \in L\}, \\ \mathrm{ad}_L^{k+1} L &= \mathrm{ad}_L \mathrm{ad}_L^k L, \end{aligned} \tag{2.4}$$

where $[X,Y]$ is the Lie bracket defined by

$$[X,Y] = \frac{\partial Y}{\partial x}X - \frac{\partial X}{\partial x}Y. \tag{2.5}$$

The Lie algebra is nilpotent if there exists a positive integer k such that

$$\mathrm{ad}_L^k L = 0. \tag{2.6}$$

The complex structure of the systems (2.1) creates difficulties in solving the optimal control problems and makes mandatory their approximation by systems with a simple structure. Hermes [5] and Bressan [4] show that, under certain conditions, the affine system (2.1) with, or without the passivity of the $f(x)$ term, may be approximated locally by a nilpotent system of the same form. The nonlinear system considered here is nilpotent if the associated Lie algebra L is nilpotent.

The Hamiltonian associated to the optimal problem is

$$H = p^{\mathrm{T}}[f(x)+g(x)u] - \frac{1}{2}(x^{\mathrm{T}}Qx+u^{\mathrm{T}}Ru), \tag{2.7}$$

where $p \in \mathbb{R}^n$ is the adjoint $(n\times 1)$ vector.

The Hamiltonian system associated is

$$\begin{aligned} \dot{x} &= f(x)+g(x)u, \qquad x(t_0) = x_0, \\ \dot{p} &= -\frac{\partial}{\partial x}(f+gu)^{\mathrm{T}}p+Qx, \qquad p(t_f) = -\frac{\partial\Phi(x(t_f))}{\partial x}, \end{aligned} \tag{2.8}$$

with the $(m\times 1)$ vector added to (2.8):

$$y = \frac{\partial H}{\partial u} = \begin{pmatrix} p^{\mathrm{T}}g_1-(Ru)_1 \\ p^{\mathrm{T}}g_2-(Ru)_2 \\ \cdots \\ p^{\mathrm{T}}g_m-(Ru)_m \end{pmatrix}, \tag{2.9}$$

where $(Ru)_i$, for $i=1,\ldots,m$, represents the rows of the $(n\times 1)$ matrix (Ru).

The optimal control problem (2.1) and (2.2) is a nondegenerate problem because

$$\frac{\partial^2 H}{\partial u^2} = -R \tag{2.10}$$

is nonsingular for any (x,p,u).

The necessary conditions for the optimal control $u^\star$ are given by

$$y = \left.\frac{\partial H}{\partial u}\right|_{u^\star} = 0. \tag{2.11}$$

From (2.11), one obtains

$$y_i = p^{\mathrm{T}}g_i-(Ru)_i, \quad i=1,\ldots,m. \tag{2.12}$$

By derivation of (2.12), one has

$$\begin{aligned} \dot{y}_i &= \dot{p}^{\mathrm{T}}g_i+p^{\mathrm{T}}\dot{g}_i-(u^{\mathrm{T}}R)_i' \\ &= p^{\mathrm{T}}[f+gu,g_i]+x^{\mathrm{T}}Qg_i-(u^{\mathrm{T}}R)_i', \quad i=1,\ldots,m. \end{aligned} \tag{2.13}$$

Let

$$F := f+gu. \tag{2.14}$$

Equation (2.13) becomes

$$\frac{d}{dt}(Ru)_i = p^{\mathrm{T}}[F,g_i]+x^{\mathrm{T}}Qg_i-\dot{y}_i, \quad i=1,\ldots,m. \tag{2.15}$$

Since

$$[F,g_i] = \mathrm{ad}_F\, g_i, \tag{2.16}$$

(2.13) becomes

$$\frac{d}{dt}(Ru)_i = p^{\mathrm{T}}\,\mathrm{ad}_F\, g_i + x^{\mathrm{T}}Qg_i - \dot{y}_i, \quad i = 1,\dots,m. \tag{2.17}$$

The derivation will be made utilizing the following.

Lemma 2.1. *Let Y be a vector and let p be the adjoint optimal vector. Then,*

$$\frac{d}{dt}(p^{\mathrm{T}}Y) = p^{\mathrm{T}}\,\mathrm{ad}_F\, Y + (x^{\mathrm{T}}Q)\,Y. \tag{2.18}$$

The time derivative is calculated along the trajectory of the system.

Lemma 2.1, Proposition 2.2, and Corollary 3.1 have been proved by Popescu [9].

Substituting the optimal control $u^\star$ in (2.7), the optimal Hamiltonian is $H^\star(x,p) = H(x,p,u^\star)$.

Using the optimality condition $y^{(k)} = 0$, $k = 0,1,2,\dots$, we obtain the following result.

Proposition 2.2. *The necessary conditions of optimality for $u_i^\star$ are that along the optimal Hamiltonian $H^\star$*

$$\begin{aligned} \left[(Ru^\star)_i\right]^{(k)} = \Big\{ & p^{\mathrm{T}}\,\mathrm{ad}_F^k g_i + (x^{\mathrm{T}}Q)\,\mathrm{ad}_F^{k-1} g_i + \frac{d}{dt}\left[(x^{\mathrm{T}}Q)\,\mathrm{ad}_F^{k-2} g_i\right] \\ & + \frac{d^2}{dt^2}\left[(x^{\mathrm{T}}Q)\,\mathrm{ad}_F^{k-3} g_i\right] + \cdots + \frac{d^{k-2}}{dt^{k-2}}\left[(x^{\mathrm{T}}Q)\,\mathrm{ad}_F\, g_i\right]\Big\}_{u=u^\star}, \\ & k = 0,1,2,\dots,\ i = 1,2,\dots,m. \end{aligned} \tag{2.19}$$

Hence, the properties of the optimal control can be expressed as

$$\begin{aligned} (Ru^\star)_i &= p^{\mathrm{T}} g_i, \\ \frac{d}{dt}(Ru^\star)_i &= p^{\mathrm{T}}[f + gu^\star, g_i] + (x^{\mathrm{T}}Q)g_i. \end{aligned} \tag{2.20}$$

In the following we consider affine nonlinear systems with a nilpotent structure.

3. Optimal control for nilpotent operators

Corollary 3.1. *If L satisfies the nilpotent conditions,*

$$\mathrm{ad}_L^k = 0, \tag{3.1}$$

for some positive integer k, then it results that for any vector field $Y \in \mathrm{ad}_L^{k-1} L$,

$$\frac{d}{dt}[p^{\mathrm{T}}Y(x)] = (x^{\mathrm{T}}Q)\,Y. \tag{3.2}$$

The following three cases are important.

Case 1 (commutative, $k = 1$). In this case one has

$$\mathrm{ad}_L L = \{[X, Y],\ X, Y \in L\} = 0. \tag{3.3}$$

As the field of vectors is $\{f, g_1, \ldots, g_m\}$, by (3.3) we obtain

$$\begin{aligned} [f, g_i] &= 0, \quad i = 1, 2, \ldots, m, \\ [g_i, g_j] &= 0, \quad i, j = 1, 2, \ldots, m. \end{aligned} \tag{3.4}$$

The relations (3.4) express the commutativity of the operations defining the Lie algebras.

From relation (2.17), and by property $\mathrm{ad}_F g_i = 0$, one obtains

$$u_i^\star = \frac{\Delta_i}{\det(R)} + C_1^i, \quad i = 1, 2, \ldots, m, \tag{3.5}$$

where Δ_i are the determinants resulting from the substitution in $\det(R)$ of the column $\int x^{\mathrm{T}} Q g_i dt$, $i = 1, 2, \ldots, m$, for the column (i), and C_1^i are integrating constants.

Let α_i be the minors of the terms $\int x^{\mathrm{T}} Q g_i dt$ from Δ_i.

Using Corollary 3.1, the expression of the optimal control becomes

$$u_i^\star = \frac{1}{\det(R)} \sum_{k=1}^{m} \alpha_k (p^{\mathrm{T}} g_k) + C_1^i, \quad i = 1, 2, \ldots, m. \tag{3.6}$$

The constants for which the functional $J^\star$ is optimal result from the conditions

$$\frac{dJ^\star}{dC_1^i} = 0, \quad i = 1, 2, \ldots, m. \tag{3.7}$$

Case 2 ($k = 2$). In this case $\mathrm{ad}_F^2 g_i = 0$, then (2.19) becomes

$$\frac{d^2}{dt^2}(Ru^\star)_i = \left\{ (x^{\mathrm{T}} Q)\, \mathrm{ad}_F g_i + \frac{d}{dt}[(x^{\mathrm{T}} Q) g_i] \right\}_{u^\star}. \tag{3.8}$$

After some calculations, (3.8) becomes

$$R\ddot{u}^\star = (x^{\mathrm{T}} Q)[A(x) u^\star + B(x)] + \frac{d}{dt}(x^{\mathrm{T}} Q g), \tag{3.9}$$

where

$$
\begin{aligned}
R &= (r_{ij}) = (r_{ji}), \quad i \neq j, \text{ for } i,j = 1,2,\dots,m,\\
a_{ij}(x) &= [g_i,g_j] = -[g_j,g_i], \quad i > j,\ i,j = 1,2,\dots,m,\\
b_k(x) &= [f,g_k], \quad k = 1,2,\dots,m,\\
A(x) &= \begin{pmatrix}
0 & a_{21} & a_{31} & a_{41} & \cdots & a_{m1}\\
-a_{21} & 0 & a_{32} & a_{42} & \cdots & a_{m2}\\
-a_{31} & -a_{32} & 0 & a_{43} & \cdots & a_{m3}\\
-a_{41} & -a_{42} & -a_{43} & 0 & \cdots & a_{m4}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-a_{m1} & -a_{m2} & -a_{m3} & \cdots & -a_{m,m-1} & 0
\end{pmatrix},\\
B(x) &= (b_1,b_2,\dots,b_m)^{\mathrm{T}},\\
u^\star &= (u_1^\star,u_2^\star,\dots,u_m^\star)^{\mathrm{T}}.
\end{aligned}
\tag{3.10}
$$

Using Corollary 3.1, we get the following result:

$$
R\ddot{u}^\star = \frac{d}{dt}\left(p^{\mathrm{T}}A(x)\right)u^\star + \frac{d}{dt}\left(p^{\mathrm{T}}B(x)\right) + \frac{d}{dt}\left(x^{\mathrm{T}}Qg\right). \tag{3.11}
$$

The optimal control $u_j^\star$ $(j = 1,2,\dots,m)$ is represented by the solution of the differential system (3.11).

Case 3 ($k = 3$). The optimality conditions (2.19) becomes

$$
\frac{d^3}{dt^3}(Ru^\star)_i = \left\{(x^{\mathrm{T}}Q)\,\mathrm{ad}_F^2 g_i + \frac{d}{dt}\left[(x^{\mathrm{T}}Q)\,\mathrm{ad}_F g_i\right] + \frac{d^2}{dt^2}\left[(x^{\mathrm{T}}Q)g_i\right]\right\}_{u^\star}. \tag{3.12}
$$

We have

$$
\mathrm{ad}_F^2 g_i = [F,[F,g_i]] = \left[F,[f,g_i] + \sum_{k=1}^{m} u_k[g_k,g_i]\right]. \tag{3.13}
$$

Therefore

$$
\begin{aligned}
\mathrm{ad}_F^2 g_i = {}& [f,[f,g_i]] + \sum_{k=1}^{m}[g_k[f,g_i]]u_k + \sum_{j=1}[f,[g_j,g_i]]u_j\\
&+ \sum_{j=1}^{m}\sum_{k=1}^{m}[g_k,[g_j,g_i]]u_ju_k.
\end{aligned}
\tag{3.14}
$$

From Corollary 3.1, we can write

$$
\frac{d^3}{dt^3}(Ru^\star)_i = \left\{\frac{d}{dt}\left(p^{\mathrm{T}}\,\mathrm{ad}_F^2 g_i\right) + \frac{d}{dt}\left[(x^{\mathrm{T}}Q)\,\mathrm{ad}_F g_i\right] + \frac{d^2}{dt^2}\left[(x^{\mathrm{T}}Q)g_i\right]\right\}_{u^\star}. \tag{3.15}
$$

The optimal control $u_i^\star$ $(i = 1,2,\dots,m)$ can be calculated by numerical integration of the nonlinear differential system

$$\sum_{j=1}^{m} r_{ij}\ddot{u}_j = p^{\mathrm{T}}\Big\{[f,[f,g_i]] + \sum_{k=1}^{m}[g_k,[f,g_i]]u_k^\star + \sum_{j=1}^{m}[f,[g_j,g_i]]u_j^\star + \sum_{j=1}^{m}\sum_{k=1}^{m}[g_k,[g_j,g_i]]u_j^\star u_k^\star\Big\} + (x^{\mathrm{T}}Q)\Big\{[f,g_i] + \sum_{k=1}^{m}[g_k,g_i]u_k^\star\Big\} + \frac{d}{dt}[(x^{\mathrm{T}}Qg_i)]. \tag{3.16}$$

The third, fourth, and sixth terms from the right-side of (3.16) characterize the nilpotent structure of the nonlinear systems considered.

For $m = 1$, the optimal control $u^\star$ is the solution of the equation

$$R\ddot{u}^\star = p^{\mathrm{T}}[g,[f,g]]u^\star + p^{\mathrm{T}}[f,[f,g]] + (x^{\mathrm{T}}Q)[f,g] + \frac{d}{dt}[(x^{\mathrm{T}}Q)g]. \tag{3.17}$$

The results regarding the minimization of the quadratic functionals are used in solving some problems of optimum representing the minimum energy criterion in the regulator design. These cases correspond to the $L^2[t_0,t_f]$ norm (resp., norm for the product space $U \times X$).

4. On minimum energy control of affine nonlinear systems with a nilpotent structure

Next we consider the following performance index:

$$J(x_0,u) = \frac{1}{2}\int_0^{t_1} u^T u\, dt + u^T(t_1)\Theta_0 x(t) + x^T(t_1)\varphi_0, \tag{4.1}$$

where Θ_0 and φ_0 are constant $(n\times n)$ matrix and $(n\times 1)$ vector, respectively.

The performance index has to fulfill the following restrictions:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \qquad x(0) = x_0. \tag{4.2}$$

We analyze the cases corresponding to the first, second, and third degree nilpotent operators.

Case 1 (commutative, $k = 1$). In this case

$$p^T Y(x) = \text{const} \quad \text{where } Y \in \operatorname{ad}_L^{k-1} L, \tag{4.3}$$

therefore

$$\dot{u}_i^* = p^T \operatorname{ad}_F g_i = 0. \tag{4.4}$$

Consequently, the minimum fuel control for the nilpotent system, with $\mathrm{ad}_F = 0$ is the constant vector

$$u^* = C_1. \tag{4.5}$$

In this case, the minimum performance index is given by

$$\begin{aligned} J^*(x_0) = J(x_0, u^*) &= \frac{1}{2}\int_0^{t_1} u^{*T}u^*\,dt + x^T(t_1)\Theta_0 x(t_1) + x^T(t_1)\varphi_0 \\ &= \frac{t_1}{2}\sum_{i=1}^{m}(C_1^i)^2 + x^T(t_1)\Theta_0 x(t_1) + x^T(t_1)\varphi_0, \end{aligned} \tag{4.6}$$

and the associated dynamic system becomes

$$\dot{x} = f(x) + g(x)C_1, \qquad x(0) = x_0. \tag{4.7}$$

The system can be solved for $x(t_1)$ and thus J^* is a function of C_1^i. For optimality of C_1^i, we require that

$$\frac{dJ^*}{dC_1^i} = TC_1^i + \Theta_0^T x(t_1) + \varphi_0 = 0, \quad i = 1, 2, \dots, m. \tag{4.8}$$

These constitute a set of m algebraic equations.

Case 2 ($k = 2$ or $\mathrm{ad}_F^2 = 0$). In this case the optimal control is given by

$$\ddot{u}_i^*(t) = p^T \mathrm{ad}_F^2 g_i = 0. \tag{4.9}$$

This system admits the solution

$$u_i^*(t) = C_2^1 + C_2^2 t, \tag{4.10}$$

where C_2^k ($k = 1, 2$) are constants.

Case 3 ($k = 3$ or $\mathrm{ad}_F^3 = 0$). Here $\mathrm{ad}_F^3 g_i = 0$ and we obtain

$$u_i^{*(3)} = 0, \quad i = 1, 2, \dots, m. \tag{4.11}$$

Using Corollary 3.1, we have

$$\ddot{u}_i^* = a^i + \sum_{j=1}^{m} b_j^i u_j^* + \sum_{k=1}^{m} c_k^i u_k^* + \sum_{j=1}^{m}\sum_{k=1}^{m} d_{jk}^i u_j^* u_k^*, \tag{4.12}$$

where a^i, b_j^i, c_k^i, d_{jk}^i are constants defined by

$$\begin{aligned} a^i &= p^T[f, [f, g]], \\ b_j^i &= p^T[f, [g_j, g_i]], \\ c_k^i &= p^T[g_k, [f, g_i]], \\ d_{jk}^i &= p^T[g_j, [g_k, g_i]]. \end{aligned} \tag{4.13}$$

This set of m nonlinear coupled second-order differential equations may be solved for u_i^* using numerical integration techniques.

For $m=1$ we have $b_j^i = d_{jk}^k = 0$, then the optimal control is given by

$$u^\star = C_1 \exp(C_3 t) + C_2 \exp(-C_3 t) + C_4, \tag{4.14}$$

with C_i $(i = 1,\dots,4)$ constants.

5. Application to bilinear systems

We consider the bilinear system

$$\dot{x} = Ax + uBx, \qquad x(t_0) = x_0, \tag{5.1}$$

where $x \in R^n$ and u is a scalar control, A, B are $(n \times n)$ constant matrices. The performance index is given by

$$J = \frac{1}{2}\int_0^{t_1} u^2 dt + x^T(t_1)\Theta_0 x(t_1) + x^T(t_1)\varphi_0, \tag{5.2}$$

where final time t_1 is specified.

By considering the Lie algebra $M(A,B)$ generated by A and B, when $M(A,B)$ is a nilpotent, we can obtain a simple method to determine the optimal control.

Case 1. If $[A,B] = AB - BA = 0$ (i.e., if A and B commute), then the optimal control $u^\star$ is constant. By Corollary 3.1 we have

$$\dot{u}^\star = 0. \tag{5.3}$$

Then

$$J(u^\star) = \frac{1}{2}u^{\star 2} t_1 + x^T(t_1)\Theta_0 x(t_1) + x^T(t_1)\varphi_0. \tag{5.4}$$

By

$$\begin{gathered} \dot{x} = (A + u^\star B)x, \\ x(t_1) = \exp((A + u^\star B)t_1)x_0, \end{gathered} \tag{5.5}$$

we have

$$\begin{aligned} J(u^\star) = \frac{1}{2}u^{\star 2} t_1 &+ x_0^T \exp((A^T + u^\star B^T)t_1)\Theta_0 \exp((A + u^\star B)t_1)x_0 \\ &+ x_0^T \exp((A^T + u^\star B^T)t_1)\varphi_0. \end{aligned} \tag{5.6}$$

From here, by condition

$$\frac{dJ(u^\star)}{du^\star} = 0, \tag{5.7}$$

one can determine the optimal control $u^\star$.

Case 2. In this case if $M(A,B)$ is nilpotent with $(\operatorname{ad} M(A,B))^2 = 0$ (i.e., $[[A,B],A] = [[A,B],B] = 0$), then the optimal control of the problem takes the form (see (4.1))

$$u^\star = c_1 + c_2 t, \qquad c_i = \text{const} \quad (i = 1,2). \tag{5.8}$$

For the optimality of c_i we require

$$\frac{dJ^\star}{dc_i} = 0, \quad i = 1,2. \tag{5.9}$$

Case 3. If $(\operatorname{ad} M(A,B))^3 = 0$ (i.e., $[[[A,B],A],A] = [[[A,B],B],A] = [[[A,B],A],B] = [[[A,B],B],B] = 0$), the optimal control $u^\star$ is the solution of the equation

$$\ddot{u}^\star = c_1 + c_2 u^\star, \tag{5.10}$$

where

$$c_1 = p^T[A,[A,B]], \qquad c_2 = p^T[B,[A,B]]. \tag{5.11}$$

The general solution of (5.10) is

$$u^\star(t) = \begin{cases} k_1\left(e^{k_2 t} - e^{-k_2 t} - \dfrac{c_1}{c_2}\right) & \text{if } c_2 \neq 0, \\ \dfrac{1}{2}c_1 t^2 + c_3 t + c_4 & \text{if } c_2 = 0, \end{cases} \tag{5.12}$$

where k_2 is the solution of the characteristic equation $k_2^2 = c_2$.

6. Conclusions

We have considered the optimal control problem for the class of affine nonlinear systems under investigation such that the Lie algebra generated by the system vector fields is nilpotent. The key for optimal control $u^\star$ are (2.19) representing a hierarchy for the necessary conditions of $u^\star$. These equations play an important role in obtaining the open-loop optimal control $u^\star(t)$ at least for $k = 1,2,3$ which were studied. The optimal control determination of nonlinear system with a nilpotent structure minimizing the quadratic functionals generalizes the results of Liu et al. [6] and Banks and Yew [1], respectively, regarding the energy minimization of the affine nonlinear and bilinear systems.

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