SEVERAL DISSIPATIVITY AND PASSIVITY IMPLICATIONS IN THE LINEAR DISCRETE-TIME SETTING

EVA MARÍA NAVARRO-LÓPEZ

Some important features and implications of dissipativity and passivity properties in the discrete-time setting are collected in this paper. These properties are mainly referred to as the stability analysis (feedback stability systems and study of the zero dynamics), the relative degree, the feedback passivity property, and the preservation of passivity under feedback and parallel interconnections. Frequency-domain characteristics are exploited to show some of these properties. The main contribution is the proposal of necessary and sufficient conditions in order to render a multiple-input multiple-output linear discrete-time invariant system passive by means of a static-state feedback and using the properties of the relative degree and zero dynamics of the system. A discrete-time model for the DC-to-DC buck converter is used as an example to illustrate the passivation scheme proposed. In addition, dissipativity frequency-domain properties are related to some feedback stability criteria.

1. Introduction

Dissipativity and its particular case of passivity were born from the observation of physical systems behaviour. They are the formalization of physical energy processes. Passivity ideas emerged in the circuit theory field, from the phenomenon of dissipation of energy across resistors. The abstraction of the connections between input-output behaviour, internal system description, and properties of energy functions is the basis for dissipative systems. Precisely, due to the fact that dissipativity merges all these concepts, it acts as a powerful tool for analyzing systems behaviour.

The study of the behaviour of a system in terms of the energy it can store or dissipate has an extraordinary value, since it gives a rather physical and intuitive interpretation of problems, such as the system stability. The analysis of mechanical, electromechanical, or electrical systems, among others, by means of their associated energy is appropriate.

A dissipative system is a system which cannot store all the energy that has been given, that is, it dissipates energy in some way. Our definition of dissipative systems is based on the existence of a storage function (energy stored by the system) and a supply function (external received energy). There are different kinds of dissipative systems depending on the form of the supply function; passive systems are a special case of dissipative ones.

Dissipative and passive systems exhibit highly desirable properties, namely, the ones referring to stability and representation properties, which may simplify the system analysis and control design [30, 31].

Although the interest in discrete-time dynamics is important by itself, we can also bring up for consideration discrete-time dynamics obtained from continuous-time ones. Most discrete-time systems are sampled-data systems obtained from continuous-time ones by means of a sample-and-hold element. Computer-controlled systems are sampled-data systems. Sampling may transform the properties of the original system, including passivity- and dissipativity-related ones.

The motivation of studying dissipative discrete-time systems stems from the fact that dissipativity and passivity properties can simplify the system analysis. Furthermore, it is adequate to translate well-known properties in the continuous-time setting into the discrete-time framework, taking into account the increasing importance that is recently given to discrete control for continuous-time systems or to new hybrid representations used to describe more complex dynamical systems (see [29]) in which discrete-type subsystems are present.

Dissipativity and passivity implications in dynamical continuous-time systems have been broadly studied. Nevertheless, a lot of problems concerning dissipativity and passivity in the discrete-time setting remain unsolved, or they have not attracted as significant attention as in the continuous-time case. This is the case of the study of the frequency-domain implications of dissipativity in the discrete-time domain, the interconnection of passive discrete-time systems, the implications of dissipativity and passivity in the relative degree and the zero dynamics of discrete dynamics, the problem of rendering a system passive by means of a static-state feedback, or the study of absolute stability by means of the dissipativity approach in nonlinear discrete-time systems. This paper deals with these appealing problems and the further research on some of these topics is motivated.

Dissipativity and passivity are properties which reflect internal properties of systems. A comment on some of these properties for discrete-time systems is given in the following paragraphs.

One of the most important passivity results is that a negative feedback loop consisting of two passive systems is passive. In addition, under an additional detectability condition, this feedback system is also stable. This result is well known for continuous-time systems [24], but it has attracted less attention in the discrete-time setting. Passivity and dissipativity have been considered as a vehicle for producing stability criteria for interconnected systems. General input-output systems representations have been used in an operator-theoretic setting. This has allowed to treat discrete- and continuous-time systems within the same framework, see [6, 32]. In the seminal work [21], among other things, the concept of hyperstability is introduced, that is, a closed-loop system consisting of a linear system with a nonlinear block in the feedback path is hyperstable when the nonlinear block satisfies a passivity-like characteristic and the linear block is positive real. This result is given either for the discrete-time or the continuous-time case.

The study of the properties of the relative degree and zero dynamics of a passive system has played an important role in understanding problems such as feedback passivity or the stabilization of passive systems in the continuous-time setting, see [2]. For general

discrete-time systems, the implications of dissipativity and passivity in the relative degree and zero dynamics have not been established yet, these ones have only been studied for the losslessness case, see [3], and for the passivity case [18, 19].

The properties of the relative degree and zero dynamics of a passive system are closely connected to its feedback passivity property. The action of rendering a system passive by means of a static-state feedback is regarded as *feedback passivity* or *passivation*. Systems which can be rendered passive are referred to as *feedback passive systems*. For the linear case, the problem of feedback dissipativity for (Q, S, R)-dissipative systems (i.e., dissipative systems with a supply function of the form $s(y, u) = y^T Q y + 2 y^T S u + u^T R u$, with Q, S, R matrices with appropriate dimensions, and Q, R symmetric) has been solved in the framework of the positive real control [4, 5] and the (Q, S, R)-dissipative control problem [26, 27] in connection with H_∞ design.

This paper makes use of an important tool in order to illustrate some special properties that dissipative and passive systems exhibit: the frequency-domain interpretation of passivity and dissipativity by means of the positive realness property of a transfer function. Three main problems are treated. First, passivity preservation under block interconnection. Second, the relative degree and zero dynamics of passive linear discrete-time systems are analyzed and related to the feedback passivity property. Third, the frequency-domain characteristics of dissipativity are related to some frequency-based nonlinear feedback stability criteria in the discrete-time domain. A preliminary study of some of these properties was made in [17].

Some of the properties analyzed for the discrete-time case are compared with the continuous-time counterparts. This work can be considered as a results-collecting one, however, some contributions are given and some ideas which have not been completely exploited before are presented from another viewpoint. Two main results are highlighted, which are compared to previous related results as follows.

(1) Referring to the study of passivity preservation under block interconnection, here, the state-space system description is used and dissipativity and passivity are defined in terms of energy-like functions, unlike previous works, such as [6, 24, 32] which use an input-output description. In these works, passivity of operators, which describe the system, is defined. The state-space description facilitates the study of qualitative behaviour of system orbits and is computationally more feasible.

(2) Concerning the feedback passivity results for linear discrete-time invariant systems, the approach followed is different from the ones proposed previously in the literature [4, 5, 26, 27]. These works are oriented to achieve asymptotic stability of uncertain systems exploiting dissipativity properties. Here, the conditions under which a system can be transformed into a passive system are based on the properties of the relative degree and zero dynamics of the nonpassive system.

Although the results presented are inspired by their continuous-time counterparts [2, 24], they are different in essence due to the particular properties of discrete-time systems. For instance, passive continuous-time systems have relative degree one [2], while passive discrete-time ones have relative degree zero.

The paper is organized as follows. Section 2 recall the most commonly used definitions for dissipativity in the discrete-time setting in addition to its frequency-domain

characteristics, which will be used in the sequel. Section 3 is devoted to the study of the interconnection of passive discrete-time systems. Some notes on the preservation of some classes of dissipativity under feedback and parallel interconnections are also given. Section 4 presents the special properties that the relative degree and zero dynamics of passive discrete-time systems have. Section 5 is devoted to the feedback passivity problem. Section 6 relates the dissipativity frequency-domain properties of discrete-time systems to some of the most important nonlinear feedback stability criteria in the discrete-time domain, such as Popov's, Tsypkin's, and some cases of the circle criteria. Conclusions are given in the last section.

2. Definitions: dissipativity-related frequency-domain characteristics

Dissipativity can be formalized from two different points of view: considering the input-output description of the system via an operator on a function space or via the state-space or internal dynamical representation. The former endows the frequency-domain characterization of dissipativity; for the discrete-time case, see, for example, [8, 21, 32]. The latter interprets dissipativity by means of an energy balance equation; for the discrete-time case, see, for example, [3, 14, 16, 23].

The frequency-domain interpretation of passivity for linear systems is given by means of the positive realness property of a transfer function. Passivity is equivalent to positive realness, see for the discrete-time case [9]. (Q, S, R)-dissipativity is also equivalent to the positive realness of a transfer function [8]. The concept of positive real transfer functions is originated in the continuous-time setting in network theory as the frequency-domain formulation of the fact that the time integral of the energy input to a passive network must be positive, in other words, a linear time-invariant passive circuit, having positive resistance, inductance, and capacitance values, has a positive real impedance function. This property can be easily identified via the Nyquist diagram of the associated transfer function of the system, which is confined in the right-hand side half of the Nyquist plane. In addition, positive real transfer functions do not have poles with modulus greater than one, and their poles lying on $|z| = 1$ are simple with positive real residues. These features will be used in the sequel.

The interest in studying positive realness properties arises from their implications in systems stability. Positive real systems have played a major role in stability theory. An interesting linear control problem which has attracted broad attention is the *positive real control problem* which consists in designing a controller which renders the closed-loop transfer function positive real, see [10, 22, 25] and references therein. The study of the positive real control problem is motivated because robust stability can be guaranteed provided that an appropriate closed-loop system is strictly positive real [1, 12]; in these last two mentioned works, the positive real control problem is considered as a passivation problem. The robust quadratic dissipative control problem, that is, the problem of rendering an uncertain linear system (Q, S, R)-dissipative and asymptotic stable is treated in [33]. All these works are in the framework of continuous-time systems. The discrete-time counterpart of these results is given in [4, 5] for the positive control problem and in [26, 27] for the quadratic dissipative control problem and the robust quadratic dissipative control problem, respectively.

In the present work, the state-space formalization of dissipativity and passivity will be followed and use is made of storage and supply functions.

Let the system

$$x(k+1) = f(x(k), u(k)), \quad x \in \mathcal{X},\ u \in \mathcal{U},$$
$$y(k) = h(x(k), u(k)), \quad y \in \mathcal{Y}, \tag{2.1}$$

where $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $h : \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$ are smooth maps, with $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{U}, \mathcal{Y} \subset \mathbb{R}^m$, $k \in \mathcal{Z}_+ := \{0, 1, 2, \ldots\}$.

Definition 2.1. A positive definite $\mathcal{C}^2$ function $V : \mathcal{X} \to \mathbb{R}$ such that $V(0) = 0$ is addressed as storage function. A $\mathcal{C}^2$ function denoted by $s(y, u)$ with $s : \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$ is addressed as supply function.

Definition 2.2 [3]. System (2.1) with supply function s is said to be dissipative if there exists a storage function V such that

$$V(x(k+1)) - V(x(k)) \le s(y(k), u(k)), \quad \forall (x(k), u(k)) \in \mathcal{X} \times \mathcal{U},\ \forall k. \tag{2.2}$$

Definition 2.3. System (2.1) is said to be passive if it is dissipative with respect to the supply function $s(y(k), u(k)) = y^T(k)u(k)$.

In the linear case, the relation between the input-output and the state-space representations of passivity properties is given by the Kalman-Yakubovich-Popov (KYP) lemma, which is proposed for the discrete-time setting in [9]. The generalized version of the KYP lemma, also called discrete positive real lemma, for the dissipativity discrete-time case is given in [8] for supply functions of the form

$$s(y, u) = y^T Q y + 2 y^T S u + u^T R u, \tag{2.3}$$

where Q, S, R are appropriately dimensioned matrices, with Q and R symmetric.

LEMMA 2.4 [8]. *Let $G(z)$ be a transfer function description, and $M(z) = R + G^H(z)S + S^T G(z) + G^H(z)QG(z)$, with $G^H(z)$ denoting the Hermitian transpose of $G(z)$. Let*

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k) \tag{2.4}$$

be a minimal realization of $G(z)$. Then for all z such that $|z| \ge 1$, $M(z) \ge 0$ if and only if there exist a real symmetric positive definite matrix P and real matrices L and W such that

$$A^T P A - P = C^T Q C - L^T L,$$
$$A^T P B = C^T Q D + C^T S - L W, \tag{2.5}$$
$$B^T P B = R + D^T S + S^T D + D^T Q D - W^T W.$$

Conditions (2.5) can be considered as the characterization of dissipativity, for storage functions of the form $V = x^T(k)Px(k)$, with P a positive definite symmetric matrix, and
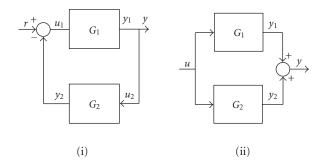
Figure 3.1. (i) Feedback interconnection and (ii) parallel interconnection.

supply functions as in (2.3). Special cases of dissipativity can be derived choosing different values for Q, S, and R [8]:

(1) passivity: $Q = R = 0$, $S = (1/2)I$;
(2) input strict passivity (ISP): $Q = 0$, $S = (1/2)I$, $R = -\epsilon I$;
(3) output strict passivity (OSP): $Q = -\delta I$, $S = (1/2)I$, $R = 0$;
(4) very strict passivity (VSP): $Q = -\delta I$, $S = (1/2)I$, $R = -\epsilon I$;
(5) finite-gain stable (FGS): $Q = -I$, $S = 0$, $R = k^2 I$,

with ϵ and δ small positive scalars, I the identity matrix, and k an arbitrary constant.

3. Implications of passivity in interconnected systems

The purpose of this section is to show an alternative to input-output approaches for studying whether the feedback and the parallel interconnections of two discrete-time passive systems given in Figure 3.1 result in a passive system. It is inspired by the continuous results given in [24].

THEOREM 3.1. *Consider systems G_1 and G_2 (linear or nonlinear) to be passive. Then, the systems resulting from the feedback and the parallel interconnections of systems G_1 and G_2 given in Figure 3.1 are passive.*

Proof. Let x_1 denote states of G_1, and x_2 denote states of G_2. Taking into account the dissipativity definition in (2.2), and particularizing it for the passivity case, that is, $s(y, u) = y^T u$, it is concluded that if G_1 and G_2 are passive, then there exist two storage functions $V_1(x_1)$ and $V_2(x_2)$ such that

$$V_1(x_1(k+1)) - V_1(x_1(k)) \le y_1^T u_1,$$
$$V_2(x_2(k+1)) - V_2(x_2(k)) \le y_2^T u_2. \tag{3.1}$$

A new state vector is defined as $x := (x_1, x_2)$, which will be the new state vector for the interconnected system, and a new positive definite storage function V is also considered,

$$V(x) := V_1(x_1) + V_2(x_2). \tag{3.2}$$

For the feedback interconnection (i) of **Figure 3.1**, one has

$$V(x(k+1)) - V(x(k)) \le y_1^T u_1 + y_2^T u_2. \tag{3.3}$$

Taking into account that $u_2 = y_1$, $u_1 = r - y_2$, it follows that $y_1^T(r - y_2) + y_2^T y_1 = y_1^T r$. Consequently,

$$V(x(k+1)) - V(x(k)) \le y_1^T r, \tag{3.4}$$

that is, the feedback interconnected system is passive.

For the parallel interconnection, the output of the system is $y_1 + y_2 = y$. If G_1 and G_2 are passive,

$$\begin{aligned}
V_1(x_1(k+1)) - V_1(x_1(k)) &\le y_1^T u, \\
V_2(x_2(k+1)) - V_2(x_2(k)) &\le y_2^T u.
\end{aligned} \tag{3.5}$$

Adding (3.5), we obtain

$$V(x(k+1)) - V(x(k)) \le (y_1 + y_2)^T u = y^T u, \tag{3.6}$$

that is, the system corresponding to the parallel interconnection is passive.   □

Remark 3.2. Following the same procedure, it can be seen that the property of OSP for supply functions of the form (2.3) is preserved under feedback block interconnection. Besides, ISP is preserved under parallel interconnection.

3.1. Interconnection of passive linear discrete-time systems: an example. For single-input single-output linear time-invariant dynamics, a way of illustrating that the feedback and parallel interconnections of two passive systems result in a passive system is by means of the positive realness property of the transfer function of the interconnected resulting systems.

An example is considered. A normalized model of the buck converter [11], proved to be passive with respect to the current output in the continuous-time setting, will be discretized and connected to itself by means of a negative feedback and a parallel interconnection. Due to the fact that usually the implementation of control systems results in sampled-data systems, it is adequate to study systems to control in the discrete-time setting. This is the case of the example given in this section.

The DC-to-DC buck converter is a well-known system employed in power electronics. The simplified scheme of this converter is depicted in **Figure 3.2**. This circuit is used to produce a constant (DC) voltage in the load R lower than the power supply voltage (V_{in}). This goal is achieved by means of the adequate commutation of the switch. The circuit operates as follows: when the switch position is 1, the inductor and the capacitor store energy, whereas if the switch position is 0, there is an exchange of energy between the inductor, the capacitor, and the load. This circuit can be modelled by the following system
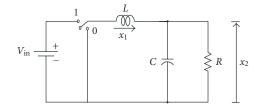
Figure 3.2. The DC-to-DC buck converter.

of ordinary differential equations [11]:

$$\frac{d}{d\tau}\begin{pmatrix} i \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{RC} \end{pmatrix}\begin{pmatrix} i \\ v \end{pmatrix} + u\begin{pmatrix} \dfrac{V_{\text{in}}}{L} \\ 0 \end{pmatrix}, \tag{3.7}$$

where i is the current flowing through the inductor, v the voltage across the capacitor, u the switch position, $u \in \{0,1\}$, and V_{in} the power supply. In order to have a normalized model, the following coordinate transformation is considered:

$$x_1 = \frac{i}{V_{\text{in}}}\sqrt{\frac{L}{C}}, \qquad x_2 = \frac{v}{V_{\text{in}}}, \qquad t = \frac{\tau}{\sqrt{LC}}, \qquad \gamma = \frac{\sqrt{L}}{R\sqrt{C}}, \tag{3.8}$$

which yields to

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -\gamma \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u \in \{0,1\}, \tag{3.9}$$

with x_1 the normalized current and x_2 the normalized voltage. The paremeter γ plays the role of the normalized load. Perturbation theory may be used to obtain the continuous-time averaged model [13]

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -\gamma \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \hat{u}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{u} \in [0,1]. \tag{3.10}$$

Model (3.10) will be discretized by means of the trapezoidal or bilinear transformation, shown to preserve passivity under sampling in linear systems, see [28]. The continuous-transfer function having the current through the inductor as the output, with a normalized load of $\gamma = 0.3536$, takes the form

$$G_c(s) = \frac{s + 0.3536}{s^2 + 0.3536s + 1}. \tag{3.11}$$

Applying the trapezoidal transformation on system (3.11) and choosing the sampling period time $T = 0.35355$, the following positive real transfer function in z is obtained:

$$G_c(z) = \frac{0.17173(z+1)(z-0.8823)}{z^2 - 1.771z + 0.8857}. \tag{3.12}$$
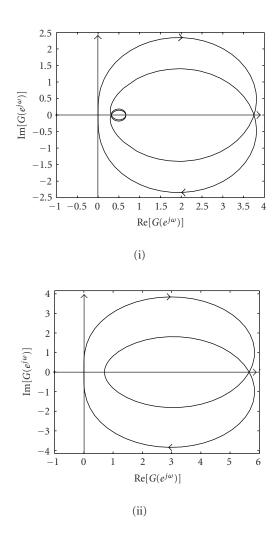
(i)

(ii)

Figure 3.3. Nyquist plots for the feedback and parallel interconnections of the discretized normalized model of the buck converter obtained by means of the trapezoid-rule transformation: (i) Nyquist plot for the feedback interconnection and (ii) Nyquist plot for the parallel interconnection.

Nyquist diagrams for the feedback and parallel interconnections of (3.12) are presented in Figure 3.3. They both correspond to positive real transfer functions or to passive systems.

4. Implications of dissipativity and passivity in the relative degree and the zero dynamics of a system

The characteristics of the relative degree and zero dynamics of passive linear discrete-time systems of the form (2.4) will be analyzed. These properties give a valuable information of the relation between the input and the output of a system.

The basis of the analysis will be dissipativity conditions (2.5) considered for the passivity case.

PROPOSITION 4.1 [9]. *Let a storage function of the form $V = x^T P x$, with P a real positive definite and symmetric matrix. A system of the form (2.4) is passive with V, if and only if, there exist P and real matrices L, K such that*

$$A^T P A - P = -L^T L, \qquad (4.1a)$$

$$A^T P B = \frac{1}{2} C^T - LK, \qquad (4.1b)$$

$$B^T P B - \frac{1}{2}(D^T + D) = -K^T K. \qquad (4.1c)$$

Remark 4.2. Conditions (4.1) are often presented for storage functions V of the form $V = (1/2)x^T P x$. In this case, conditions (4.1b) and (4.1c) would take the following forms, respectively:

$$A^T P B = C^T - LK,$$
$$B^T P B - (D^T + D) = -K^T K. \qquad (4.2)$$

PROPOSITION 4.3. *If system (2.4) is passive, then it has relative degree zero.*

Proof. Having relative degree zero is equivalent to D being nonsingular, that is, the output depends directly on the input. From condition (4.1c), with P a positive definite matrix, one concludes that $D^T + D$ must be a positive definite matrix, and therefore D is nonsingular. □

Remark 4.4. This result has been briefly pointed out in [3, 15].

Remark 4.5. In [3], it is stated that it does not make sense to study passivity and losslessness of discrete-time systems having outputs independent of u. This is the case for $s(y, u) = y^T u$. Indeed, dissipative systems can have relative degree greater than zero, that is, D can be singular. For example, considering dissipative systems with supply functions of the form (2.3), it can be concluded that ISP, VSP, and FGS systems may have relative degree greater than zero.

If system (2.4) has relative degree zero, its zero dynamics takes the following form:

$$f^*(x(k)) = (A - BD^{-1}C)x(k). \qquad (4.3)$$

Definition 4.6. A system of the form (2.4) has passive zero dynamics if there exists a storage function V such that

$$V(f^*(x)) \leq V(x), \quad \forall x \in \mathcal{X}, \qquad (4.4)$$

with $f^*(x)$ as given in (4.3).

Remark 4.7. Systems with passive zero dynamics are also referred to as *weakly minimum phase* systems (see Byrnes et al. [2]).

PROPOSITION 4.8. *Let a system of the form (2.4) be passive. Then, the zero dynamics of the system exists and is passive.*

Proof. Since system (2.4) is assumed to be passive, there exists P a positive definite and symmetric matrix satisfying (4.1). Consider $V = x^T P x$. As the system is passive, it has relative degree zero and the zero dynamics is given by (4.3), then $V(f^*(x)) - V(x) = x^T M x$, where

$$M = (A - BD^{-1}C)^T P (A - BD^{-1}C) - P. \tag{4.5}$$

Thus, it is needed to be proved that M is negative semidefinite.

Considering condition (4.1b), M can be written as follows:

$$\begin{aligned} M = (A^T P A - P) &- \frac{1}{2} C^T \big[D^{-1} + (D^{-1})^T \big] C + (LKD^{-1}C) \\ &+ (LKD^{-1}C)^T + C^T (D^{-1})^T B^T P B D^{-1} C. \end{aligned} \tag{4.6}$$

Taking into account that $(1/2)C^T(D^{-1})^T(D^T + D)D^{-1}C = (1/2)C^T[D^{-1}+(D^{-1})^T]C$, and using (4.1c), one yields to

$$\begin{aligned} M &= (A^T P A - P) + (LKD^{-1}C) + (LKD^{-1}C)^T - (D^{-1}C)^T K^T K (D^{-1}C) \\ &= (A^T P A - P) - [L - (D^{-1}C)^T K^T][L - (D^{-1}C)^T K^T]^T + LL^T. \end{aligned} \tag{4.7}$$

Using condition (4.1a) in (4.7), M is concluded to be negative semidefinite. □

Remark 4.9. The properties of the relative degree zero and passive zero dynamics presented for linear discrete-time passive systems are accomplished by the passive or positive real transfer function (3.12) and its feedback and parallel interconnections shown to be passive in Section 3. It is interesting to notice that continuous-time passive systems present relative degree one, see (3.11), while discrete-time passive systems have relative degree zero, see (3.12).

5. The feedback passivity problem

In this section, a passifying scheme for systems of the form (2.4) is proposed. Conclusions given in Section 4 are used. The feedback passivity methodology can be considered as an adaptation for the linear case of the feedback passivity scheme proposed for nonlinear discrete-time systems which are affine in the control input given in [18]. A discretized model for the DC-to-DC buck converter shown in Section 3.1 will be used in order to illustrate the passivation method.

5.1. Passivation of a linear discrete-time system.
A linear static-state feedback control law is denoted by

$$u = Rx + Sv, \tag{5.1}$$

with R and S constant matrices of appropriate dimensions.

Definition 5.1. A feedback control law of the form (5.1) is regular if S is nonsingular.

THEOREM 5.2. *Consider a system of the form (2.4). Suppose that there exists a storage function V such that $V = x^T P x$, with P a real positive definite symmetric matrix. Then, system (2.4) is feedback equivalent to a passive system with V as storage function by means of a regular feedback control law of the form (5.1) if and only if the system has relative degree zero and passive zero dynamics.*

Proof. Necessity. If system (2.4) is feedback equivalent to a passive system by means of a static-state feedback control law of the form (5.1), with v the new input to the system, the feedback transformed system

$$\begin{aligned} x(k+1) &= \overline{A}x(k) + \overline{B}v(k), \\ y(k) &= \overline{C}x(k) + \overline{D}v(k) \end{aligned} \qquad (5.2)$$

is passive for some V, with $\overline{A} = A + BR$, $\overline{B} = BS$, $\overline{C} = C + DR$, $\overline{D} = DS$. On the one hand, making use of Proposition 4.3, system (5.2) has relative degree zero and $\overline{D}$ is nonsingular, consequently, D is nonsingular and system (2.4) has relative degree zero. On the other hand, by Proposition 4.8, system (5.2) has passive zero dynamics. It can be seen that the zero dynamics of system (2.4) is the same as that of system (5.2).

  Sufficiency. It will be shown that if system (2.4) has relative degree zero and passive zero dynamics, it is feedback equivalent to a passive system with $V = x^T P x$, and P a real positive definite matrix, that is, there exists a control $u(k) = Rx(k) + Sv(k)$ such that the feedback system (5.2) is passive.

  Since the system relative degree is zero, the matrix D^{-1} is well defined. It is chosen as

$$u(k) = D^{-1}(\overline{C} - C)x(k) + D^{-1}\overline{D}v(k). \qquad (5.3)$$

System (2.4) with (5.3) yields to (5.2) with

$$\begin{aligned} \overline{A} &= [I - 2B(B^T PB)^{-1}B^T P](A - BD^{-1}C), \\ \overline{B} &= B(B^T PB)^{-1}D^T, \\ \overline{C} &= -2D(B^T PB)^{-1}B^T P(A - BD^{-1}C), \\ \overline{D} &= D(B^T PB)^{-1}D^T. \end{aligned} \qquad (5.4)$$

It can be seen that system (5.2) with (5.4) fullfils conditions (4.1). In order to check condition (4.1a), the fact that the zero dynamics of system (2.4) is passive is used. Conditions (4.1b) and (4.1c) are satisfied considering $K = 0$. In conclusion, system (2.4) is feedback passive with the passifying control law (5.3).   □

5.2. A linear example: frequency-domain interpretation.   In this section, the passifying control law (5.3) will be applied to a discrete-time model for the DC-to-DC buck converter (3.10) which has been proposed in [20]. A discrete-time model of a normalized

averaged DC-to-DC buck converter takes the form given in (2.4) with

$$A = \begin{pmatrix} a & -b \\ b & c \end{pmatrix}, \qquad B = \begin{pmatrix} (-a+1)\gamma + b \\ -b\gamma - c + 1 \end{pmatrix},$$
$$C = (0,1), \qquad D = 1,$$

(5.5)

with $u = \hat{u}$, $u \in [0,1]$, $x = (x_1, x_2)^T$, and a, b, c, constants related to physical parameters. The storage function associated with the system is considered as the sum of the stored energy in the inductor and in the capacitor, and takes the following form for the normalized model:

$$V = \frac{\eta}{2}(x_1^2 + x_2^2) = x^T \begin{pmatrix} \frac{\eta}{2} & 0 \\ 0 & \frac{\eta}{2} \end{pmatrix} x = x^T P x,$$

(5.6)

with $\eta = V_{in}^2 C$. The following parameters, obtained from a real physical system, are considered in the model:

$$a = 0.9406416964, \qquad b = 0.3254699438, \qquad c = 0.8255706942,$$
$$\gamma = 0.3535533906, \qquad \eta = 13.25192,$$

(5.7)

and a sampling period of $T = 0.3535533906$. The corresponding transfer function of the buck converter state-space representation is not a positive real transfer function (i.e., the sytem is not passive) as the Nyquist diagram shows, see Figure 5.1. Although the original system is not passive, it can be rendered passive by means of a static-state feedback law as defined in (5.3), due to the fact that the system has relative degree zero and passive zero dynamics. The system is passified by means of control (5.3), and the feedback transformed system has an associated positive real transfer function. This fact is illustrated by the Nyquist diagrams depicted in Figure 5.1.

6. Notes on implications of dissipativity and passivity in feedback systems stability

The study of stability of nonlinear systems using frequency criteria instead of Lyapunov's direct method has been proposed for linear systems with a nonlinearity in the feedback path. These methods, mainly, Popov's, Tsypkin's, and the circle criteria establish stability criteria based upon the frequency response of the linear part. Reference [21] proposes that if the transfer function corresponding to the linear block is positive real or passive and the nonlinearity satisfies a Popov-like inequality, that is, it is a sector bounded nonlinear function, then the resulting closed-loop system is said to be absolutely stable (the zero solution of the system is globally asymptotically stable).

This section presents the importance that dissipativity and passivity concepts have in the stability analysis of nonlinear interconnected systems. The most interesting and remarkable property of passivity is that in linear systems (either discrete or continuous), the positive realness characteristic is equivalent to the passivity property, and in addition,
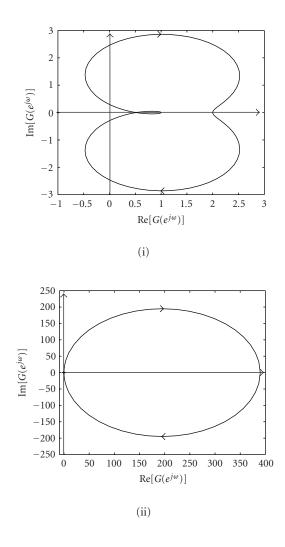
(i)

(ii)

Figure 5.1. (i) Nyquist plot of the discrete nonpassive buck converter and (ii) Nyquist plot for the passified discrete-time buck converter.

it presents highly interesting stability properties in the frequency domain. The fact of having a Nyquist plot on the right-half plane, means that an infinite-gain proportional control can be introduced without destabilizing the system.

Since the geometric interpretation of stability criteria such as Popov's, the circle, and Tsypkin's ones are based on the positive realness of a transfer function, and a particular emplacement of the Nyquist plot, dissipativity formalism can be considered to have interesting relations with these stability criteria. Indeed, a passive nonlinear function has the property of falling in sector $[0, \infty)$ [7], consequently, the passivity property increases

the validity of Popov's, the circle, and Tsypkin's criteria. If a sector-bounded nonlinearity is passive, its sector boundaries are augmented in comparison to the boundaries proposed in the mentioned stability criteria.

The generalized KYP or discrete positive real lemma is proposed for dissipative discrete-time linear systems with supply function (3) in [8], see Lemma 2.4. In addition, the characteristics of the Nyquist plot of $G(e^{j\omega})$ for single-input single-output systems are presented depending on the form of the supply function. Two cases are analyzed: Q being negative definite and $Q = 0$. On the one hand, if $Q < 0$, the Nyquist plot of $G(e^{j\omega})$ lies inside the circle with center $S/|Q|$ and radius $(1/|Q|)\sqrt{S^2 + R|Q|}$. On the other hand, if $Q = 0$, the Nyquist plot of $G(e^{j\omega})$ lies to the right (if $S > 0$) or to the left (if $S < 0$) of the vertical line $\mathrm{Re}\, z = -R/2S$.

From the characteristics of the Nyquist plot of $G(e^{j\omega})$, dissipativity frequency-domain properties could be considered as the generalization of the stability conditions of the mentioned criteria for the discrete-time setting.

Tsypkin's criterion for nonlinear sampled-data systems establishes that the closed-loop system consisting of a linear transfer function with a nonlinear function in the feedback path is absolutely stable if the nonlinear function falls in a sector bounded by two straight lines with slopes 0 and b, and the Nyquist plot of the discrete transfer function lies to the right of the vertical line $\mathrm{Re}\, z = -1/b$ [28]. Considering dissipative systems with supply functions of the form (2.3), it can be seen that the geometric interpretation of Tsypkin's criterion in the framework of the frequency domain is a special case of dissipativity with $Q = 0$, $S = I/2$, $R = I/b$.

The circle criterion gives a sufficient condition for the absolute stability of a linear system with a nonlinear function gain in the feedback path which falls in a sector bounded by two straight lines with slopes a and b. This class of system will be absolutely stable if the Nyquist plot of the transfer function associated to the linear block does not intersect a region C defined by the points $(-1/a + 0j)$ and $(-1/b + j0)$. In case $a, b \neq 0$, the region C will be a circle. On the other hand, if $a = 0$, $b \neq 0$, or $b = 0$, $a \neq 0$, the critical disk is converted into a critical line which the Nyquist plot must not cross.

The discrete-time version of the circle criterion is obtained from the continuous-time result via the bilinear transformation, and using $z = e^{j\omega T}$, with T the sampling period, see [7]. Considering the frequency-domain characteristics of dissipativity, the conditions that the linear block of the nonlinear system under consideration must accomplish can be seen as different classes of dissipativity. For example, the case of having $a = 0$, $b \neq 0$ corresponds to the dissipativity case considering the supply function (2.3) with $Q = 0$, $S = I/2$, $R = I/b$, where the Nyquist plot of the transfer function corresponding to the linear part lies to the right of the vertical line $\mathrm{Re}\, z = -1/b$. The case of having $b = 0$, $a \neq 0$ corresponds to the dissipativity case considering the supply function (2.3) with $Q = 0$, $S = -I/2$, $R = -I/a$, where the Nyquist plot of the transfer function corresponding to the linear part lies to the left of the vertical line $\mathrm{Re}\, z = -1/a$. When the critical region corresponds to the interior or the outside of the circle determined by the points $(-1/a + 0j)$ and $(-1/b + j0)$, the stability conditions proposed by the circle criterion may also be obtained from the dissipativity frequency-domain properties, considering supply functions of the form (2.3) with Q negative definite.

Dissipativity characterization in the frequency domain can also be used in order to extend Popov's stability criterion to the discrete-time setting, however, a different analysis than the one made for the Tsypkin's and the circle criteria is required; probably, another kind of supply functions different from (2.3) are suspected to be proposed.

7. Conclusions

Some implications of dissipativity and passivity properties for the discrete-time case have been presented in the framework of state-space systems representation, mainly, the preservation of passivity under feedback and parallel interconnections, the study of the relative degree and zero dynamics of linear discrete-time passive systems, and how they can be used in order to solve the feedback passivity problem.

Dissipativity characterization in the frequency domain has been used to illustrate by means of an example the preservation of passivity under feedback and parallel interconnections and the feedback passivity scheme proposed. The frequency-domain characteristics of dissipative systems have also been used to present dissipativity as an interesting tool for the study of nonlinear systems stability in the discrete-time setting, and it can be considered as a way for obtaining frequency-based stability criteria types, such as Tsypkin's, the circle, and Popov's criteria, for discrete-time systems.

This work has shown some of the interesting properties that dissipative and passive systems exhibit; they motivate the transformation of a system which is not dissipative or passive into a dissipative or passive one.

Acknowledgments

References

[1]   J. Bao, F. Wang, P. L. Lee, and W. Zhou, *New frequency-domain phase-related properties of MIMO LTI passive systems and robust controller synthesis*, 13th IFAC World Congress (California, 1996), 1996, pp. 405–410.

[2]   C. I. Byrnes, A. Isidori, and J. C. Willems, *Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems*, IEEE Trans. Automat. Control **36** (1991), no. 11, 1217, 1228–1240.

[3]   C. I. Byrnes and W. Lin, *Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems*, IEEE Trans. Automat. Control **39** (1994), no. 1, 83–98.

[4]   C. E. de Souza, M. Fu, and L. Xie, H_∞ *analysis and synthesis of discrete-time systems with time-varying uncertainty*, IEEE Trans. Automat. Control **38** (1993), no. 3, 459–462.

[5]   C. E. de Souza and L. Xie, *On the discrete-time bounded real lemma with application in the characterization of static state feedback H_∞ controllers*, Systems Control Lett. **18** (1992), no. 1, 61–71.

[6]   C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Electrical Science, New York; Academic Press, London, 1975.

[7]   G. F. Franklin, J. D. Powell, and M. L. Workman, *Digital Control of Dynamic Systems*, 2nd ed., Addison-Wesley, Massachusetts, 1990.

[8]   G. C. Goodwin and K. S. Sin, *Adaptive Filtering Prediction and Control*, Prentice-Hall, New Jersey, 1984.

[9]   L. Hitz and B. D. O. Anderson, *Discrete positive-real functions and their application to system stability*, Proc. Inst. Elec. Engrs. **116** (1969), 153–155.

[10] S. M. Joshi and S. Gupta, *On a class of marginally stable positive-real systems*, IEEE Trans. Automat. Control **41** (1996), no. 1, 152–155.

[11] J. G. Kassakian, M. F. Schlecht, and G. C. Verghese, *Principles of Power Electronics*, Addison-Wesley, Massachusetts, 1991.

[12] A. G. Kelkar and S. M. Joshi, *Robust passification and control of non-passive systems*, American Control Conference (Pennsylvania, 1998), 1998, pp. 3133–3137.

[13] H. K. Khalil, *Nonlinear Systems*, Prentice-Hall, New Jersey, 1996.

[14] W. Lin, *Synthesis of discrete-time nonlinear systems*, Ph.D. thesis, Washington University in St. Louis, Missouri, 1993.

[15] S. Monaco and D. Normand-Cyrot, *Nonlinear representations and passivity conditions in discrete time*, Robustness in Identification and Control (Siena, 1998), Lecture Notes in Control and Inform. Sci., vol. 245, Springer, London, 1999, pp. 422–433.

[16] E. M. Navarro-López, *Dissipativity and passivity-related properties in nonlinear discrete-time systems*, Ph.D. thesis, Universitat Politècnica de Catalunya, Barcelona, 2002, ISBN: 84-688-1941-7. In http://www.tdx.cesca.es/TDCat-0328103-104942/.

[17] E. M. Navarro-López, D. Cortés, and E. Fossas-Colet, *Implications of dissipativity and passivity in the discrete-time setting*, 15th IFAC World Congress (Barcelona, 2002), 2002.

[18] E. M. Navarro-López and E. Fossas-Colet, *Dissipativity, passivity and feedback passivity in the nonlinear discrete-time setting*, 15th IFAC World Congress (Barcelona, 2002), 2002.

[19] ———, *Feedback passivity of nonlinear discrete-time systems with direct input-output link*, Automatica J. IFAC **40** (2004), no. 8, 1423–1428.

[20] E. M. Navarro-López, H. Sira-Ramírez, and E. Fossas-Colet, *Dissipativity and feedback dissipativity properties of general nonlinear discrete-time systems*, Eur. J. Control **8** (2002), no. 3, 265–274, Special issue on dissipativity of dynamical systems: application in control.

[21] V.-M. Popov, *Hyperstability of Control Systems*, Die Grundlehren der mathematischen Wissenschaften, vol. 204, Editura Academiei, Bucharest; Springer, Berlin, 1973.

[22] M. G. Safonov, E. A. Jonckheere, M. Verma, and D. J. N. Limebeer, *Synthesis of positive real multivariable feedback systems*, Internat. J. Control **45** (1987), no. 3, 817–842.

[23] N. S. Sengör, *Energy related concepts in nonlinear systems*, Ph.D. thesis, Institute of Science and Technology, Istanbul Technical University, Istanbul, 1995.

[24] R. Sepulchre, M. Janković, and P. V. Kokotović, *Constructive Nonlinear Control*, Communications and Control Engineering Series, Springer, Berlin, 1997.

[25] W. Sun, P. P. Khargonekar, and D. Shim, *Solution to the positive real control problem for linear time-invariant systems*, IEEE Trans. Automat. Control **39** (1994), no. 10, 2034–2046.

[26] Z. Tan, Y. C. Soh, and L. Xie, *Dissipative control for linear discrete-time systems*, Automatica J. IFAC **35** (1999), no. 9, 1557–1564.

[27] ———, *Dissipative control of linear discrete-time systems with dissipative uncertainty*, Internat. J. Control **73** (2000), no. 4, 317–328.

[28] J.-C. Tsai, *Toward guaranteed stability in the haptic display of virtual environments*, Ph.D. dissertation, Northwestern University, Illinois, 1996.

[29] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*, Lecture Notes in Control and Information Sciences, vol. 251, Springer, London, 2000.

[30] J. C. Willems, *Dissipative dynamical systems. I. General theory*, Arch. Rational Mech. Anal. **45** (1972), no. 5, 321–351.

[31]   —————, *Dissipative dynamical systems. II. Linear systems with quadratic supply rates*, Arch. Rational Mech. Anal. **45** (1972), 352–393.

[32]   M. Y. Wu and C. A. Desoer, *Input-output properties of multiple-input, multiple-output discrete systems. II*, J. Franklin Inst. **290** (1970), no. 2, 85–101.

[33]   S. Xie, L. Xie, and C. E. de Souza, *Robust dissipative control for linear systems with dissipative uncertainty*, Internat. J. Control **70** (1998), no. 2, 169–191.

Eva María Navarro-López: Programa de Investigación en Matemáticas Aplicadas y Computación, Instituto Mexicano del Petróleo, Eje Central Lázaro Cárdenas 152 AP. 14-805, Colonia San Bartolo Atepehuacan, México, DF 07730, Mexico

E-mail address: navarro.lopez@gmail.com