Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2007, Article ID 94035, 8 pages
doi:10.1155/2007/94035

Research Article

Love and Rayleigh Correction Terms and Padé Approximants

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Received 29 September 2006; Accepted 16 October 2006

Recommended by Semyon M. Meerkov

Simplified theories governing behavior of beams and plates keeping the fundamental characteristics of the being modeled objects are proposed and discussed. By simplification, we mean decrease of order of partial differential equations (PDEs) with respect to spatial coordinates. Our approach is used for both discrete and continuous models. An advantage of Padé approximation is addressed. First part of this report deals with approximation of a beam equation by string-like one, and plate equation by membrane-like one. Second part is devoted to the construction of Love-type theory for rods vibrations and Rayleigh-type theory for beams vibrations.

1. Introduction

The classical equations governing longitudinal and bending vibrations of rods and bending vibrations of beams as well as vibrations of plates and shells are nothing but the approximations to the classical equations of the theory of elasticity. Note that classical theories of rods and beams are associated with a flat cross-section hypothesis, whereas those of plates (shells) are matched with the hypotheses of Kirchhoff (Kirchhoff-Love). On the other hand, in the theory of thin-walled structures, the so-called improved theories are applied. By the improved theories, one means these which include some additional terms in comparison to the classical theories that give an extension of the validity domain of the latter ones. In particular, Love [3] and Rayleigh [1, 2] proposed to include the inertia of normal motion in equation of longitudinal vibrations of rods and rotary inertia in equation of transverse vibrations of beams. (In fact, an analogous way for accounting of rotary inertia has been proposed by Bresse [4].) Then, Rayleigh theory was generalized for plates and shells [5, 6] and widely used in the analysis of vibrations of thin-walled structures. It is worth noting that these theories are not asymptotically accurate [7], that is, they cannot be derived from the equations of elasticity using a successive asymptotic approximation. In particular, the terms including mixed derivatives of the form w_{xxtt} appear. In this paper, we show that such terms can be obtained in a natural way with the help of Padé approximants (PA). In addition, we also study vibrations of both plates and a discrete system of masses. It is shown that the present approach can be generalized to nonlinear problems.

Our paper consists of two parts. In the first one, transitions from a beam dynamical equation to a string-like one as well as from a plate dynamical equation to a membrane-like one are discussed. Second part is focused on obtaining a Love-type theory of rods' vibrations and Rayleigh-type theory for beams vibrations using discrete governing models.

2. Reducing of continuous systems order

2.1. Beam and string-like models. In the theory related to suspended systems, usually the suspended construction members are substituted by simple models of beams or plates [8, 9]. However, the bending stiffness of the construction members is neglected, so a proper estimation of high frequencies and associated vibration modes is not provided. As an example of vibrations, a conveyor belt is considered, being treated as a 1D spatial variables object for which the belt speed is small with respect to the wave speed. In the simplest approximation, one gets a stretched beam equation

$$\rho F w_{tt} - T w_{xx} + EI w_{xxxx} = 0, \tag{2.1}$$

where ρ is the belt material density; T is the stretching force (see Figure 2.1); E is the Young modulus; F, I are the area and second moment of the transverse belt cross section, respectively; w is the normal displacement.

The following boundary conditions are applied:

$$w = w_{xx} = 0 \quad \text{for } x = 0, L. \tag{2.2}$$

Equations (2.1) and (2.2) can be transformed to the following nondimensional form:

$$w_{\tau\tau} - w_{\xi\xi} + \varepsilon w_{\xi\xi\xi\xi} = 0; \tag{2.3}$$

$$w = w_{\xi\xi} = 0 \quad \text{for } \xi = 0, 1, \tag{2.4}$$

where $\xi = x/L$, $\varepsilon = EI/(TL^2)$, and $\tau = (t/L)\sqrt{T/\rho F}$.

In the above, ε is a small parameter. A string-like model is obtained from (2.1) for $\varepsilon = 0$ [8, 9],

$$w_{\tau\tau} - w_{\xi\xi} = 0; \tag{2.5}$$

$$w = 0 \quad \text{for } \xi = 0, 1. \tag{2.6}$$

One may observe that the solution to (2.5) satisfies both of the boundary conditions (2.4), and hence BVP (2.3), (2.4) is regular perturbated. However, if instead of the conditions

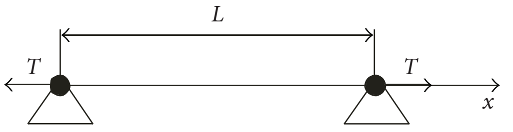

Figure 2.1. Conveyor belt model.

of a simple support (2.4) clamping is taken, then in the vicinity of rod ends boundary layers appear.

Note that now an extremely simplified PDE is obtained. Namely, one has a PDE of second order which essentially simplifies our considerations. It is possible to keep the second order of the approximating equation and to increase the approximation accuracy by Padé approximants [10, 11].

Let us briefly describe the PA using as an example the following series:

$$\varphi(p) = \sum_{i=0}^{\infty} c_i p^i. \tag{2.7}$$

The PA is defined via the following rational function:

$$\varphi_{[m/n]} = \frac{\sum_{i=0}^{m} a_i p^i}{\sum_{j=0}^{n} b_j p^j}, \tag{2.8}$$

where the coefficients a_i and b_i are determined from the following conditions. The first $(m+n)$ components of the expansion of the rational function $\varphi_{[m/n]}$ in a Maclaurin series coincide with the first $(m+n+1)$ components of the series $\varphi(p)$.

Namely, in (2.3) instead of the differential operator

$$-\frac{\partial^2}{\partial\xi^2} + \frac{\varepsilon\partial^4}{\partial\xi^4}, \tag{2.9}$$

one can use the following PA:

$$\frac{-\partial^2/\partial\xi^2}{(1+\varepsilon\partial^2/\partial\xi^2)}. \tag{2.10}$$

It gives the following approximation to (2.3):

$$\left(1+\varepsilon\frac{\partial^2}{\partial\xi^2}\right)w_{\tau\tau} - w_{\xi\xi} = 0. \tag{2.11}$$

The associated boundary conditions have the form (2.6). Solution to (2.11) satisfies both boundary conditions of (2.4). If one takes clamping instead of the boundary conditions (2.4), then boundary layer occurs in neighborhood of the rod faces.

Observe that BVP (2.5), (2.6) approximates eigenvalues of the governing BVP up to the order of ε, but BVP (2.11), (2.6) includes the second-order approximation of ε^2 preserving the equation order.

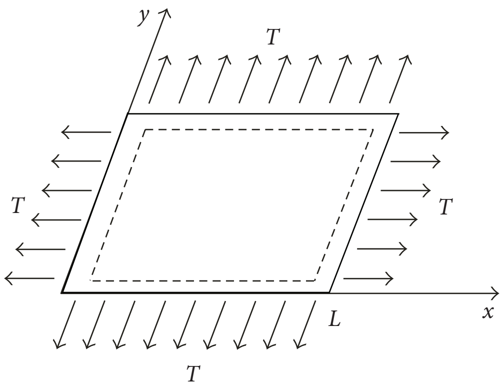

Figure 2.2. Stretched square plate.

The proposed approach can be also applied to nonlinear problems. In what follows, we consider a stretched rod on a nonlinear elastic foundation governed by the following equation:

$$w_{\tau\tau} - w_{\xi\xi} + dw^3 + \varepsilon w_{\xi\xi\xi\xi} = 0, \tag{2.12}$$

where d is a constant coefficient.

The boundary conditions have the form (2.4), whereas the reduced nonlinear equation has the form

$$w_{\tau\tau} - w_{\xi\xi} + dw^3 = 0 \tag{2.13}$$

with the boundary conditions (2.6).

On the other hand, the modified nonlinear equation has the following form:

$$\left(1 + \varepsilon\frac{\partial^2}{\partial\xi^2}\right)\left(w_{\tau\tau} + dw^3\right) - w_{\xi\xi} = 0 \tag{2.14}$$

also with the boundary conditions (2.6).

2.2. Plate and membrane-like models. Now vibrations of a stretched square plate with side lengths L (see Figure 2.2) are under consideration. The governing equations read

$$\begin{gathered} D\nabla_1^4 w - T\nabla_1^2 w + \rho_1 h_1 w_{tt} = 0, \\ w = \nabla_1^2 w = 0 \quad \text{for } x, y = L, \end{gathered} \tag{2.15}$$

where $D = Eh_1^3/(12(1-\nu^2))$; $\nabla_1^4 = \nabla_1^2\nabla_1^2$; $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$; ν is the Poisson coefficient; h_1 is the plate thickness; and ρ_1 is the plate material density.

In nondimensional form, one has

$$\varepsilon_1 \nabla^4 w - \nabla^2 w + w_{\tau_1\tau_1} = 0, \tag{2.16}$$

$$w = \nabla^2 w = 0 \quad \text{for } \xi, \eta = 0, 1, \tag{2.17}$$

where $\nabla^4 = \nabla^2\nabla^2$; $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$; $\xi = x/L$; $\eta = y/L$; $\varepsilon_1 = D/(TL^2)$; $\tau_1 = (t/L)\sqrt{T/\rho_1 h_1}$; and T is stretched stress.

For small values of ε, the membrane model is obtained from (2.16) for $\varepsilon = 0$,

$$w_{\tau_1\tau_1} - \nabla^2 w = 0, \tag{2.18}$$

$$w = 0 \quad \text{for } \xi, \eta = 0, 1. \tag{2.19}$$

One may observe that a solution to (2.18) satisfies both boundary conditions (2.16), and BVP (2.16), (2.17) is a regular perturbated problem. If one takes boundary conditions of clamping instead of (2.17), then in vicinity the plate edges boundary layers appear.

Owing to the PA

$$-\nabla^2 + \varepsilon\nabla^4 \approx \frac{-\nabla^2}{1 + \varepsilon\nabla^2}, \tag{2.20}$$

the following improved membrane model is obtained from (2.16):

$$\left(1 + \varepsilon_1 \nabla^2\right) w_{\tau_1\tau_1} - \nabla^2 w = 0. \tag{2.21}$$

Associated boundary conditions have the form (2.19). Solution to (2.21) satisfies both boundary conditions (2.16). If one takes clamping instead of the boundary condition (2.17), then in the neighborhood of plate edges, boundary layers appear.

3. Continualization of discrete models

3.1. Love model. Consider now a discrete model. Observe that usually, in order to take into account rotary inertia, rather artificial physical assumptions are applied. Now, we show how the Padé approximants can be applied. The key steps and methodology of our approach are illustrated owing to the analysis of vibrations of a mass chain shown in Figure 3.1.

$$m\frac{d^2 y_i}{dt^2} + c\left(y_{i+1} - 2y_i - y_{i-1}\right) = 0, \quad i = 1, 2, 3, \ldots, n; \tag{3.1}$$

$$y_0 = y_{n+1} = 0, \tag{3.2}$$

where m is the mass of the chain particle; c is the spring stiffness; and h is the distance between particles.

Figure 3.1. Mass chain.

The system of difference-differential equations (3.1) can be given in the following form:

$$m\frac{d^2 y_i}{dt^2} + cB(y_i) = 0, \tag{3.3}$$

where B is the difference operator.

A classical continuous BVP yields

$$m\frac{\partial^2 y(x,t)}{\partial t^2} + ch^2\frac{\partial^2 y(x,t)}{\partial x^2} = 0, \quad y = 0 \text{ for } x = 0, L, \tag{3.4}$$

where $L = (n+1)h$.

This approximation can be improved [12, 13]. Note that the system of ODEs (3.1) can be reduced to one pseudodifferential equation using the following pseudodifferential operator [14]:

$$B = \sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right). \tag{3.5}$$

Hence, with the help of this operator, the system of (3.1) is transformed into the following pseudodifferential equation:

$$m\frac{\partial^2 y}{\partial t^2} + 4c\sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right)y = 0. \tag{3.6}$$

The pseudodifferential operator B can be developed into the Maclaurin series of the form

$$\sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right) = -\left(\frac{h^2}{4}\frac{\partial^2}{\partial x^2} + \frac{h^4}{48}\frac{\partial^4}{\partial x^4} + \frac{h^6}{1440}\frac{\partial^6}{\partial x^6} + \cdots\right). \tag{3.7}$$

A transformation of the first two terms of series (3.7) by PA gives the following result:

$$-0.25h^2\frac{\partial^2}{\partial x^2}\left(1 + \frac{h^2}{6}\frac{\partial^2}{\partial x^2}\right) \sim -0.25h^2\frac{\partial^2}{\partial x^2}\frac{1}{1 - (h^2/12)(\partial^2/\partial x^2)}. \tag{3.8}$$

Hence, a continuous approximation is

$$ch^2\frac{\partial^2 y}{\partial x^2} + m\left(1 - \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 y}{\partial t^2} = 0. \tag{3.9}$$

Following [15], let us introduce Young modulus $E = c/h$ and the density $\rho = m/h^3$. Then one obtains the Love-type equation of the form

$$E\frac{\partial^2 y}{\partial x^2} + \rho\left(1 - \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 y}{\partial t^2} = 0, \tag{3.10}$$

which takes into account a microstructure of the governing material.

3.2. Rayleigh model. Let us study transversal vibrations. The governing equations follow

$$m\frac{d^2 y_i}{dt^2} + c\left(6y_i - 4y_{i+1} - 4y_{i-1} + y_{i+2} + y_{i-2}\right) = 0, \quad i = 1,2,3,\dots,n; \tag{3.11}$$

$$y_0 = y_{n+1} = 0; \qquad y_{-1} = -y_1; \qquad y_{n+2} = -y_n. \tag{3.12}$$

The system of difference-differential equations (3.11) can be cast into the following form:

$$m\frac{\partial^2 y}{\partial t^2} + 16c\sin^4\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right)y = 0. \tag{3.13}$$

The classical continuous BVP yields

$$m\frac{\partial^2 y(x,t)}{\partial t^2} + ch^4\frac{\partial^4 y(x,t)}{\partial x^4} = 0, \tag{3.14}$$

$$y = \frac{\partial^2 y}{\partial x^2} = 0 \quad \text{for } x = 0, L. \tag{3.15}$$

The pseudodifferential operator can be developed into the Maclaurin series of the form

$$16\sin^4\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right) = \frac{h^4\partial^4}{\partial x^4} + \frac{h^6}{6}\frac{\partial^6}{\partial x^6} + \frac{h^8}{80}\frac{\partial^8}{\partial x^8} + \cdots. \tag{3.16}$$

A transformation of the first two terms of series (3.16) by PA gives the following result:

$$h^4\frac{\partial^4}{\partial x^4}\left(1 + \frac{h^2}{6}\frac{\partial^2}{\partial x^2}\right) \sim h^4\frac{\partial^4}{\partial x^4}\frac{1}{1 - (h^2/6)\left(\partial^2/\partial x^2\right)}. \tag{3.17}$$

Hence, a continuous approximation is

$$ch^4\frac{\partial^4 y}{\partial x^4} + m\left(1 - \frac{h^2}{6}\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 y}{\partial t^2} = 0, \tag{3.18}$$

and the boundary conditions have the form of (3.15).

Note that the derived equation (3.18) belongs to the class of hyperbolic PDEs, but it is not purely hyperbolic [5].

Finally, following [16], one may obtain the following Rayleigh equation:

$$EI\frac{\partial^4 y}{\partial x^4} + \rho F\left(1 - \frac{h^2}{6}\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 y}{\partial t^2} = 0. \tag{3.19}$$

4. Concluding remarks

The procedure described illustrates how the PA can serve as an effective tool for the construction of Love- and Rayleigh-type theories. It is shown that Love and Rayleigh models can be obtained as hyperbolic approximations of the governing discrete models. Furthermore, using the mentioned approach, two improved string (2.11) and membrane (2.21) models have been also derived.

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