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*Research Article*

# Diffusive Synchronization of Hyperchaotic Lorenz Systems

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The synchronizing properties of two diffusively coupled hyperchaotic Lorenz 4D systems are investigated by calculating the transverse Lyapunov exponents and by observing the phase space trajectories near the synchronization hyperplane. The effect of parameter mismatch is also observed. A simple electrical circuit described by the Lorenz 4D equations is proposed. Some results from laboratory experiments with two coupled circuits are presented.



## 1. Introduction

Coupled oscillators are currently studied in physics, chemistry, biology, neural networks, and other fields. A large number of coupled oscillators form a complex system for which the investigation of coherence and synchronization is important and in the last decade the discovery that chaotic systems can synchronize added interest to this topic [1–4]. An approach for contributing to the investigation of large networks is studying the properties of a small number of coupled oscillators—this is the approach used in the present work. Chaos synchronization is also of interest in secure communication systems. For such application, hyperchaos has drawn attention for providing more complex waveforms than simply chaotic systems, thus improving the masking process. This is because hyperchaos is characterized by at least two positive Lyapunov exponents, while simple chaos shows a single one. Related to this subject is the question on how many variables are necessary to be coupled in order to obtain synchronization. Although chaotic systems can synchronize by a single variable coupling, it was for some time believed that in the case of hyperchaos the minimum number of coupling variables had to be equal to the number of positive Lyapunov exponents [5]. It was later demonstrated in [6] that it is not true, and some hyperchaotic systems can achieve synchronization by a single variable. Other systems, however, for example the Rössler equations for hyperchaos, are unable to synchronize by only one of its variables [6, 7]. Another problem refers to synchronization under parameter mismatch, since usually perfect synchronization is achieved only if the coupling systems are identical. In this work we are concerned with the question of whether a hyperchaotic Lorenz system can synchronize and with which variables. The effect of mismatch is also observed. We present some results from numerical and laboratory experiments on an eight-dimensional dynamical system obtained by diffusively coupling two hyperchaotic Lorenz systems.

## 2. Equations and Numerical Simulations

### 2.1. Diffusive Coupling

Consider the four-dimensional system $d\mathbf{x}/dt = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = [x, y, z, w]^t$. *Unidirectional* diffusive coupling involving two such systems is obtained by using the system $d\mathbf{x}_1/dt = \mathbf{f}_1(\mathbf{x}_1)$ to drive the response system $d\mathbf{x}_2/dt = \mathbf{f}_2(\mathbf{x}_2) + \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2)$. In this work we consider $\mathbf{K}$ a diagonal matrix, where each element $k_{ii}$ is the strength of the coupling related to the corresponding variable. We also consider $\mathbf{f}_1(\mathbf{x}) = \mathbf{f}_2(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, and each entry $k_{11}$, $k_{22}$, $k_{33}$, or $k_{44}$ is either zero or equal to a positive constant $k$. For example, the case $k_{11} = 0$, $k_{22} = k$, $k_{33} = 0$, and $k_{44} = k$ means simultaneous $y$-$y$ and $w$-$w$ coupling. The two coupled systems form a compound eight-dimensional system. If the systems synchronize, the motion must remain on the hyperplane $\mathbf{x}_1 = \mathbf{x}_2$. For small $(\mathbf{x}_1 - \mathbf{x}_2)$ we have $\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2) \approx (\partial \mathbf{f}(\mathbf{x}_1)/\partial \mathbf{x}_1) \cdot (\mathbf{x}_1 - \mathbf{x}_2)$, so we can write the variational equation

$$\frac{d\mathbf{x}_\perp}{dt} = \mathbf{A}(\mathbf{x}_1)\,\mathbf{x}_\perp, \tag{2.1}$$

where $\mathbf{A} = \partial\,\mathbf{f}(\mathbf{x}_1)/\partial \mathbf{x}_1 - \mathbf{K}$ and $\mathbf{x}_\perp = (\mathbf{x}_1 - \mathbf{x}_2)$. In the case of *bidirectional* diffusive coupling we have $d\mathbf{x}_1/dt = \mathbf{f}(\mathbf{x}_1) + \mathbf{K}(\mathbf{x}_2 - \mathbf{x}_1)$ and $d\mathbf{x}_2/dt = \mathbf{f}(\mathbf{x}_2) + \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2)$, therefore (2.1) also applies, now with $\mathbf{A} = \partial\,\mathbf{f}(\mathbf{x}_1)/\partial \mathbf{x}_1 - 2\mathbf{K}$. We use (2.1) as the locally linear dynamical system associated to a fiducial trajectory [8] of the coupled system to calculate the transverse Lyapunov exponents [2, 3, 7, 9], so called because the perturbation $\mathbf{x}_\perp$ is transverse to the synchronization hyperplane. The coupled system will remain stably synchronized if all transverse Lyapunov exponents (TLEs) are negative.

### 2.2. Complete Replacement

If the drive system 1 transmits the scalar component $x_1$ and the corresponding variable $x_2$ of the response system 2 is replaced by the transmitted one $x_1$, this is called complete replacement [4]. As explained in [4], unidirectional diffusive coupling and complete replacements are related, since at very high values of $k$ the variable $x_1$ slaves $x_2$. Therefore, in our numerical and experimental investigations, complete replacement of one or more variables corresponds to $k \to \infty$.

### 2.3. Coupled Hyperchaotic Lorenz Systems

The preceding coupling scheme will now be applied to systems described by the following Lorenz equations linearly extended to four dimensions:

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x),\\ \frac{dy}{dt} &= x(r - z) - y + w,\\ \frac{dz}{dt} &= xy - bz,\\ \frac{dw}{dt} &= -\gamma x,\end{aligned} \tag{2.2}$$

which shows hyperchaos and was theoretically analyzed in [10]. In the case of unidirectional coupling the eight-dimensional system is given by

$$\begin{aligned}\frac{dx_1}{dt} &= \sigma(y_1 - x_1), & \frac{dx_2}{dt} &= \sigma(y_2 - x_2) + k_{11}(x_1 - x_2),\\ \frac{dy_1}{dt} &= x_1(r - z_1) - y_1 + w_1, & \frac{dy_2}{dt} &= x_2(r - z_2) - y_2 + w_2 + k_{22}(y_1 - y_2),\\ \frac{dz_1}{dt} &= x_1y_1 - bz_1, & \frac{dz_2}{dt} &= x_2y_2 - bz_2 + k_{33}(z_1 - z_2),\\ \frac{dw_1}{dt} &= -\gamma x_1, & \frac{dw_2}{dt} &= -\gamma x_2 + k_{44}(w_1 - w_2).\end{aligned} \tag{2.3}$$

For this system the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} -\sigma - k_{11} & \sigma & 0 & 0\\ z_1 - r & -1 - k_{22} & -x_1 & 1\\ y_1 & x_1 & -b - k_{33} & 0\\ -\gamma & 0 & 0 & -k_{44}\end{bmatrix} \tag{2.4}$$

for unidirectional coupling. In the case of bidirectional coupling, the only modification is replacing $k_{ii}$ by $2k_{ii}$ along the diagonal. The synchronizing properties of system (2.2) will now be numerically investigated by calculating the TLE of (2.3) as a function of the coupling strength $k$. Parameter mismatch is also examined. In the following, only unidirectional coupling is considered.

#### 2.3.1. Parameters $\sigma = 10$, $b = 8/3$, $r = 30$, $\gamma = 10$

We first illustrate the synchronizing properties of (2.2) for the above parameter values. The values $\sigma = 10$ and $b = 8/3$ are the classical, or most popular, used in studies of the original

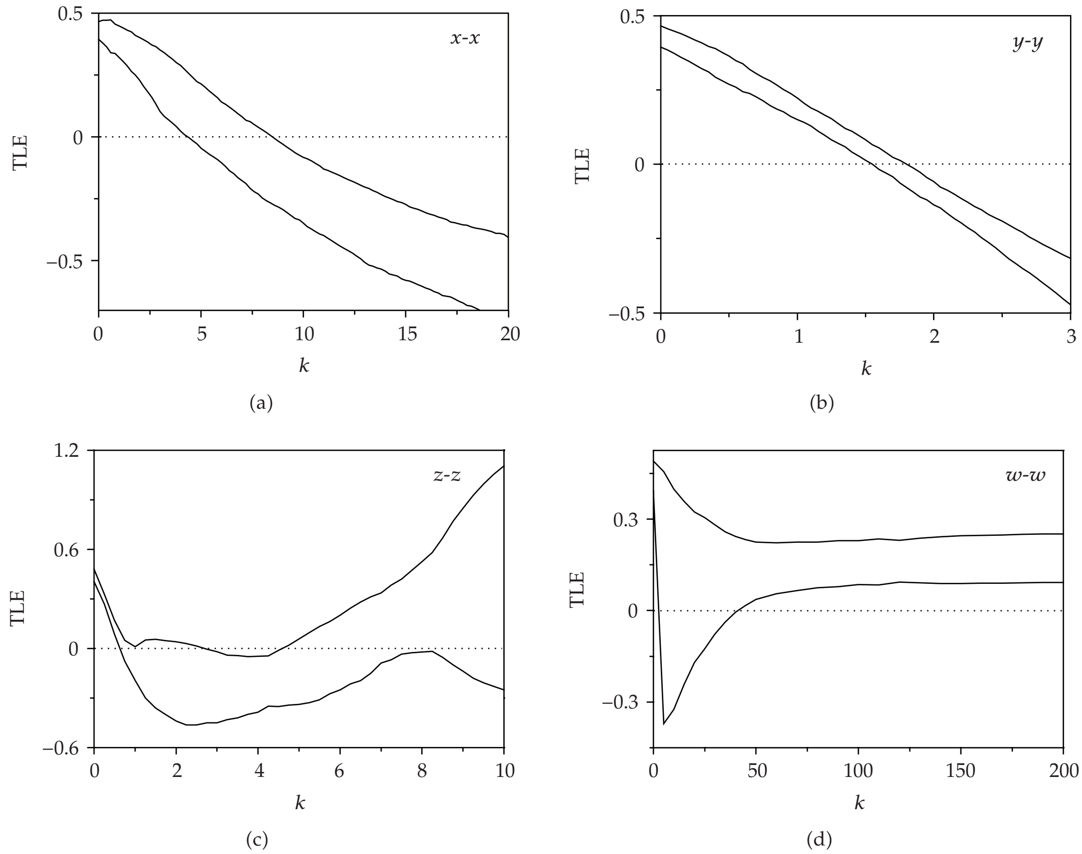



**Figure 1:** Transversal Lyapunov exponents (TLEs) for $\sigma = 10$, $b = 8/3$, $r = 30$, $\gamma = 10$.

Lorenz 3D system. For the hyperchaotic Lorenz 4D system the extra parameter $\gamma = 10$ is included. In Figure 1, the two largest TLE for single-variable coupling are plotted as a function of $k$, for $r = 30$. The synchronization thresholds are $k = 8.1$ and $k = 1.8$ for $x$-$x$ and $y$-$y$ couplings, respectively. The systems will never get synchronized if $w$ alone is the coupling variable. On the other hand, $z$-$z$ coupling provides a small window of stable synchronization.

### 2.3.2. *Parameters* $\sigma = 4$, $b = 0.3$, $r = 30$, $\gamma = 1.6$

The above values of the parameters are of interest in this work because small values of $\sigma$ and $b$ are easier to realize with practical component values in the circuit model presented in Section 3. (At this point it is worth remembering the observation by Sparrow [11] that small $b$ leads to very complex behavior of the Lorenz equations.) Therefore, the system properties for such small parameter values will be examined with more detail in the following. In Figure 2, the Lyapunov spectrum for these parameter values is plotted as a function of $r$, showing a broad range of hyperchaotic behavior. Also shown is the one-dimensional bifurcation diagram along the same $r$ range.

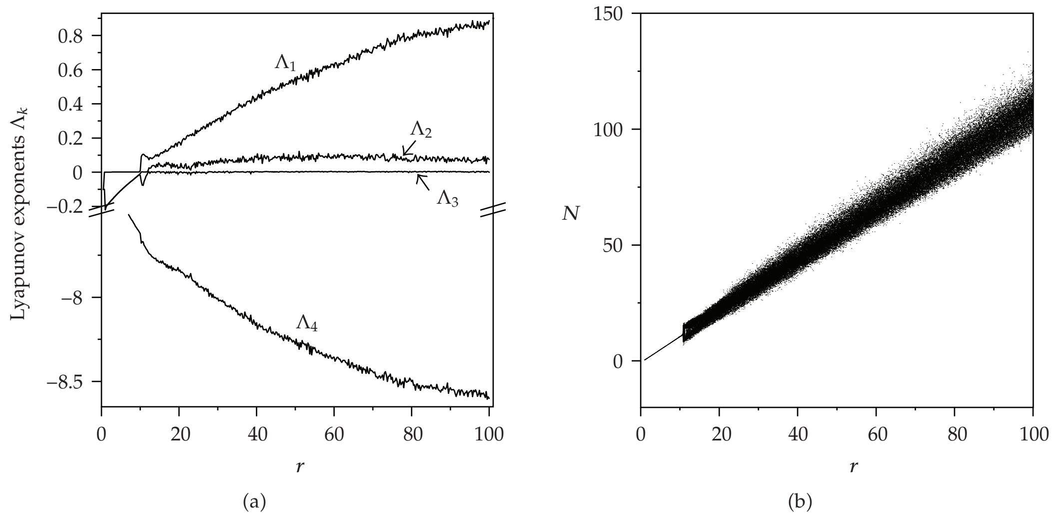



**Figure 2:** (a) Lyapunov spectrum $\Lambda_k(r)$; (b) bifurcation diagram obtained from the crossings through the Poincaré surface $dz/dt = 0$, corresponding to maxima of $z(t)$. Parameter values: $\sigma = 4$, $b = 0.3, r = 30$, $\gamma = 1.6$.

The two largest TLEs for single-variable coupling, plotted as a function of $k$, are shown in Figures 3(a)–3(c). The synchronization thresholds are $k = 3.2$ and $k = 1.3$ for $x$-$x$ and $y$-$y$ couplings, respectively. For $z$-$z$ coupling, the system shows a window of stable synchronization ranging from $k = 0.7$ to $k = 9.0$. The system does not synchronize in the case of $w$-$w$ coupling. However, the nonsynchronizing variables $z$ and $w$, when working together in the $z$-$z$ plus $w$-$w$ double-coupling scheme, provide stable synchronization above $k = 0.34$. In the case of all-variable coupling, stable synchronization is achieved above $k = 0.21$.

### *2.3.3. Parameter Mismatch*

A qualitative method of investigating the hyperchaos synchronization phenomena is by observing the projections of the eight-dimensional attractor onto the planes $(x_1, x_2)$, $(y_1, y_2)$, $(z_1, z_2)$, and $(w_1, w_2)$. In these planes the straight lines $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$, and $w_1 = w_2$ correspond to the synchronization hyperplane. In the following we show only the $(z_1, z_2)$ plane, since the components $z_1$ and $z_2$ seem to be the most difficult to synchronize. For all plots we used $x$-$x$ coupling. Figure 4 refers to identical parameters (i.e., without mismatch), illustrating the inability of the systems to synchronize if $k$ is less than the threshold value obtained from Figure 3(a), while perfect synchronization is achieved above the threshold: the same alignment along the diagonal is observed in all the four projection planes.

In Figures 5 and 6, we observe the effect of mismatch on synchronization (for $x$-$x$ coupling). In these examples we applied the same mismatch to all parameters, that is, $\Delta\sigma/\sigma = \Delta b/b = \Delta r/r = \Delta\gamma/\gamma$. In Figure 5, where $k = 4.0$, we see that for $k$ values just above the threshold, some good degree of synchronization is obtained for 1% mismatch; however, for 5% large deviations from the diagonal are observed. Figure 6 shows the effect of mismatch for 1% and 5% in the case of $k = 10$.

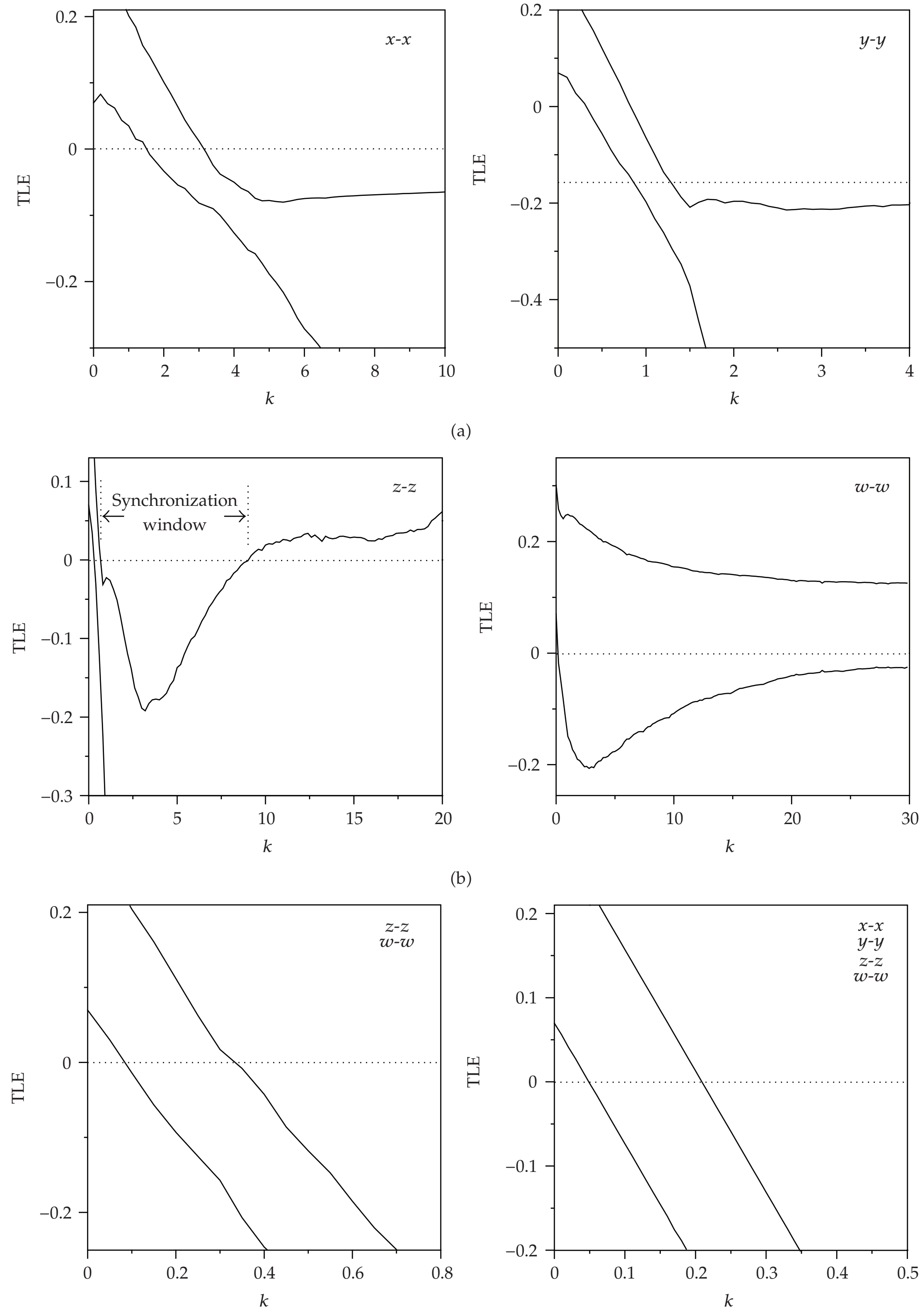



**Figure 3:** Transversal Lyapunov exponents (TLEs) for $\sigma = 4$, $b = 0.3$, $r = 30$, $\gamma = 1.6$. (a) and (b) One-variable coupling; (c) two- and four-variable couplings.

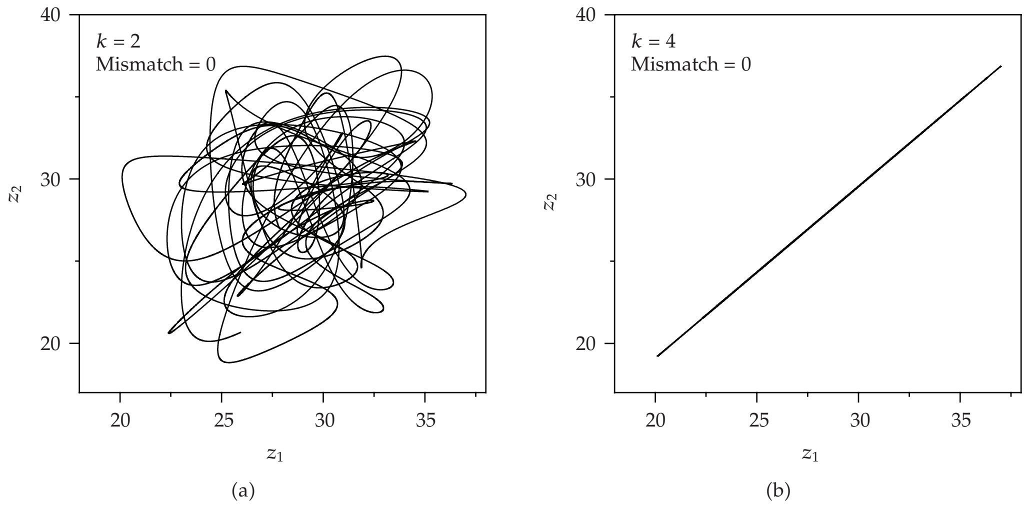



**Figure 4:** Synchronization for identical parameters: $k$ below and above the threshold $k = 3.2$ ($\sigma = 4$, $b = 0.3$, $r = 30$, $\gamma = 1.6$).

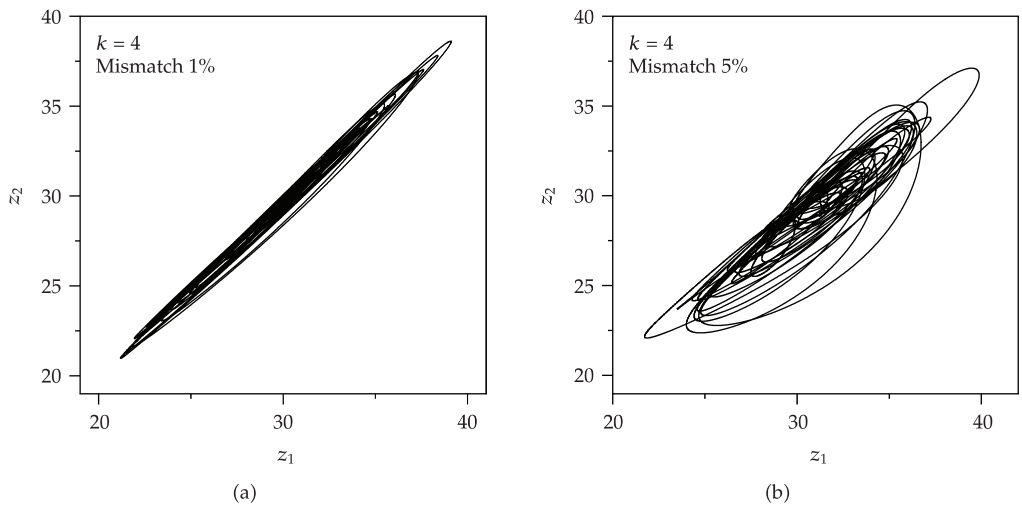



**Figure 5:** Effect of mismatch on synchronization for $k = 4$ ($\sigma = 4$, $b = 0.3$, $r = 30$, $\gamma = 1.6$).

## 3. Experiments with a Simple Electrical Circuit

### *3.1. Circuit Description and Equations*

The Lorenz system, being one of the most important paradigms of chaos, has inspired many attempts to make a physical system representing its equations, mainly in the form of an electrical circuit. Some authors have proposed replacing the cross-products of variables by discontinuities (switching circuits) as in [12, 13], or by continuous piecewise linear resistors [14], thus resulting in very simple and practical circuits, although not truly described by the Lorenz equations. More accurate realization, though more complex, is by the analog computer approach using smooth cross-product functions, as in [15], which employs 10

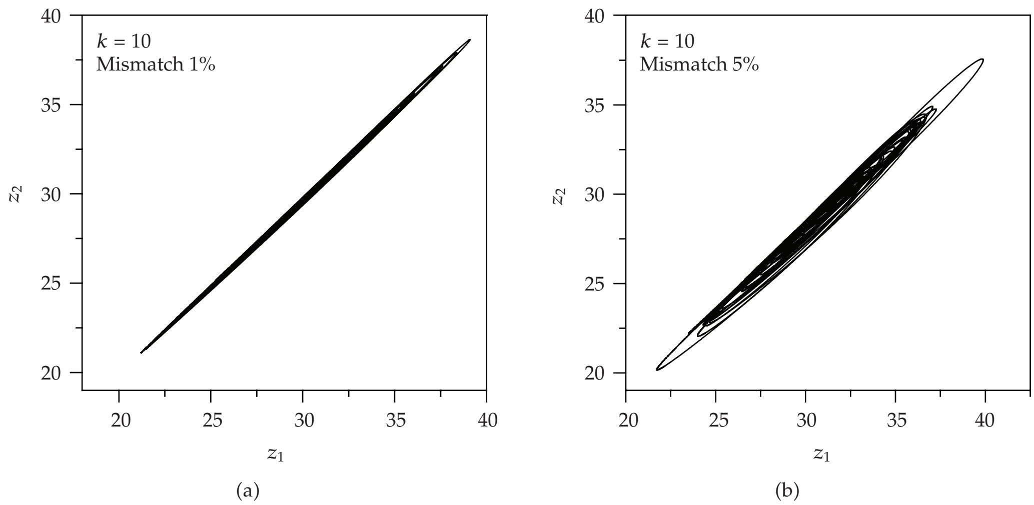
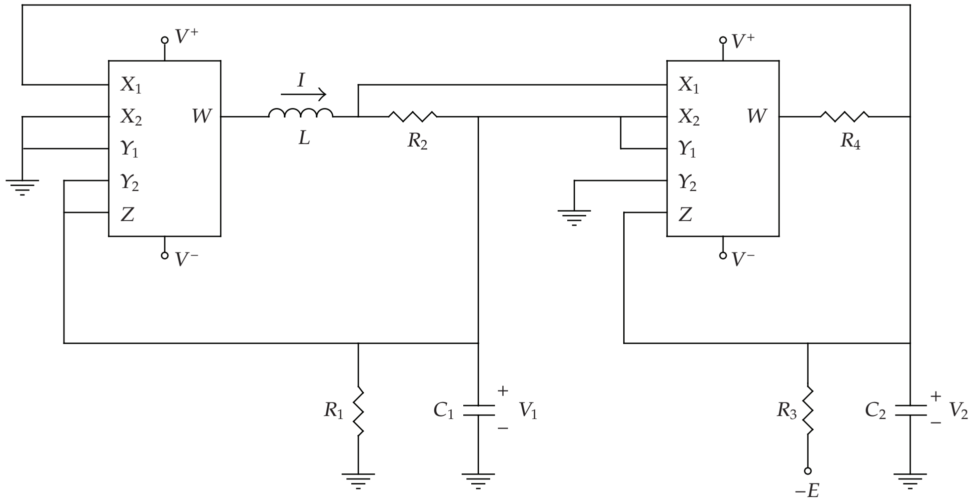



**Figure 6:** Effect of mismatch on synchronization for $k = 10$ ($\sigma = 4$, $b = 0.3$, $r = 30$, $\gamma = 1.6$).

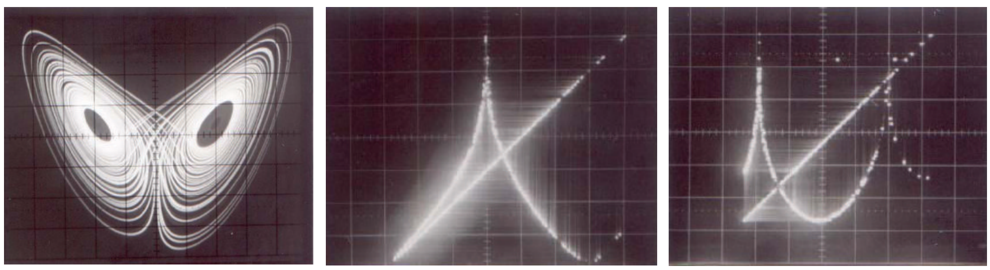
integrated circuits (2 multipliers and 8 op amps) and 23 passive components, therefore a total of 33 circuit components for the Lorenz 3D circuit. In the present work we are proposing a simpler easy-to-build circuit with smooth functions, aiming at encouraging more experimental approaches on hyperchaos investigation, even by those researchers not trained in electronics. As stated in [3], to facilitate experiments with coupled chaotic oscillators the circuit is required to exhibit chaos in a large range of parameters in order that the coupling will not destroy the attractor, and it is needed to be simple enough so that several practically identical oscillators can be easily constructed. The simplest possible smooth Lorenz 3D circuit appeared in [16], using only 2 integrated circuits (2 multipliers) and 7 passive components (a total of 9 components, thus about 70% smaller than the circuit by Cuomo et al. in [15]), as shown in Figure 7(a). The good performance of this circuit is illustrated in Figure 7(b), which shows the experimental attractor and examples of single-cusp and double-cusp Lorenz maps (displayed in-line by the circuit, via a Poincaré-section circuitry). In the present work we extended to 4D that simplest circuit by adding 2 op amps and 6 passive components, obtaining the hyperchaotic Lorenz circuit shown in Figure 8(a), redrawn in Figure 8(b) using circuit theoretic symbols. The following equations describe the circuit:

$$\begin{aligned} C_1\frac{dv_1}{dt} &= i - \frac{v_1}{R_1}, \\ L\frac{di}{dt} &= -\frac{v_1 v_2}{10} - R_2 i + v_3, \\ C_2\frac{dv_2}{dt} &= \frac{R_2 i v_1}{10R_4} - \frac{v_2 + E}{R_3}, \\ C_3\frac{dv_3}{dt} &= -\frac{v_1}{R_1}, \end{aligned} \tag{3.1}$$

where we used the multipliers transfer function $W = 0.1(X_1 - X_2)(Y_1 - Y_2) + Z$, where $Z = v_1 + v_3$ on the input of the first multiplier (on the left side in Figure 8(a)), and $Z = v_2$ on




(a)



(b)

**Figure 7:** (a) Simplest circuit for the 3D standard Lorenz equations. (b) Experimental results from the circuit of Figure 7(a): butterfly attractor on the plane $v_1 \times v_2$; Lorenz maps showing single and double cusps $v_2(t_n) \times v_2(t_{n+1})$. The straight line in each picture is given by $v_2(t_{n+1}) = v_2(t_n)$. $L = 10\,\text{mH}$, $C_1 = 22\,\text{nF}$, $C_2 = 1\,\text{nF}$, $R_1 = 1.5\,\text{k}\Omega$, $R_2 = 100\,\Omega$, $R_3 = 160\,\text{k}\Omega$, $R_4 = 1\,\text{k}\Omega$.

the input of the second one. In deriving (3.1) we assumed $R_8 = 2R_9$ and $R_5 = R_6 = R_7 \gg R_1$. Now, by defining the new variables and parameters

$$x = \frac{v_1}{10}\sqrt{\frac{L}{R_1R_4C_2}}, \qquad y = \frac{R_1 i}{10}\sqrt{\frac{L}{R_1R_4C_2}}, \qquad z = \frac{v_2}{10}\frac{R_1}{R_2} + r,$$
$$w = \frac{v_3}{10}\frac{R_1}{R_2}\sqrt{\frac{L}{R_1R_4C_2}}, \qquad \sigma = \frac{L}{R_1R_2C_1}, \qquad b = \frac{L}{R_2R_3C_2}, \tag{3.2}$$
$$r = \frac{R_1E}{10R_2}, \qquad \gamma = \frac{L}{R_2^2C_3},$$

we obtain (2.2). Therefore, the proposed circuit realizes, exactly, the Lorenz hyperchaotic system given by (2.2), obviously with some usual practical restrictions imposed by parasitic effects and finite bandwidth, slew-rate, excursion range, and so forth.

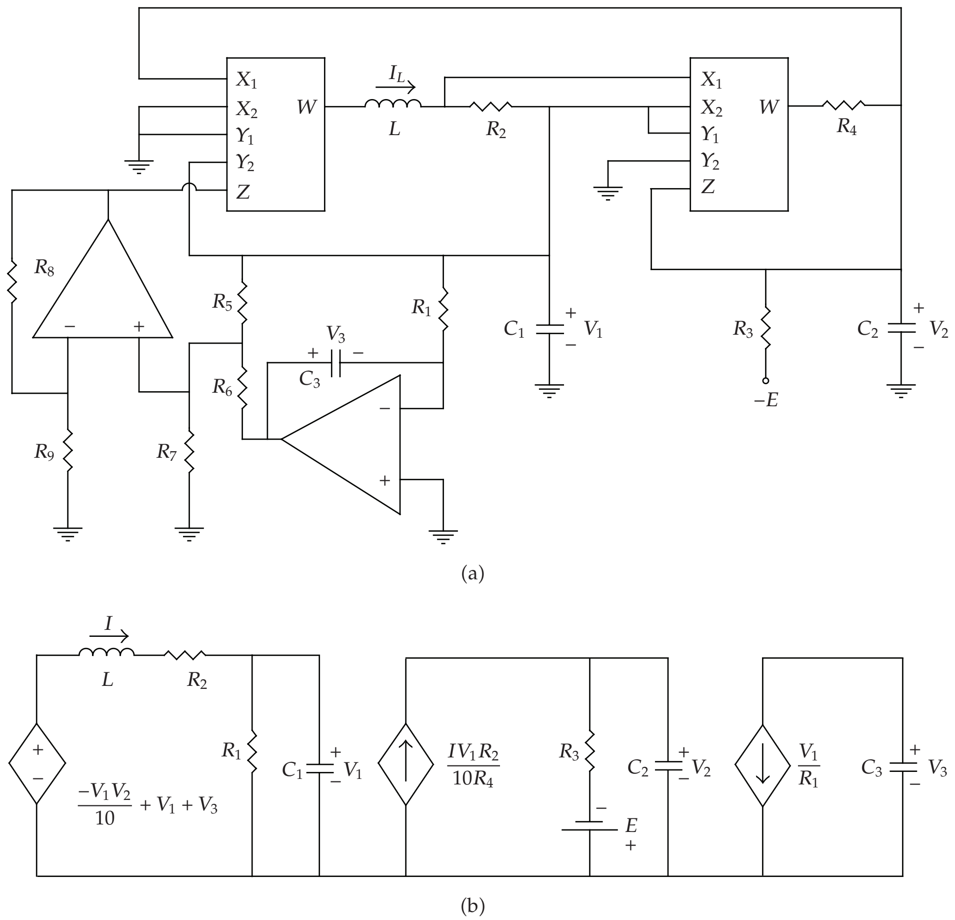
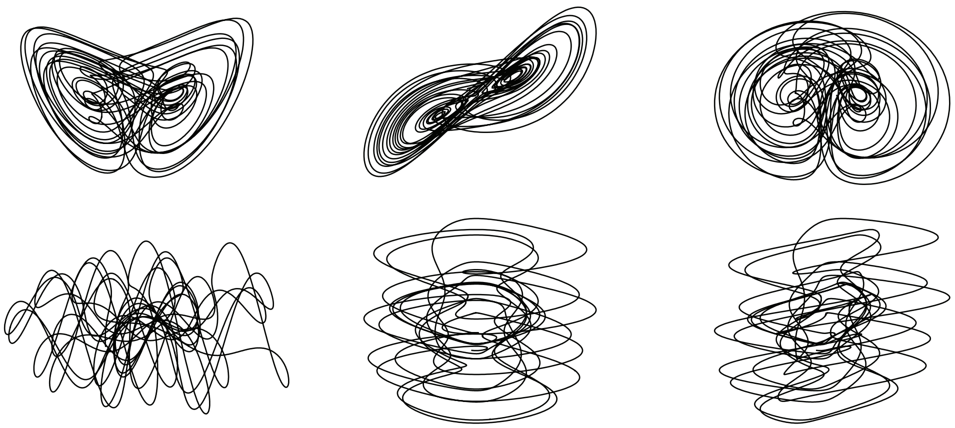


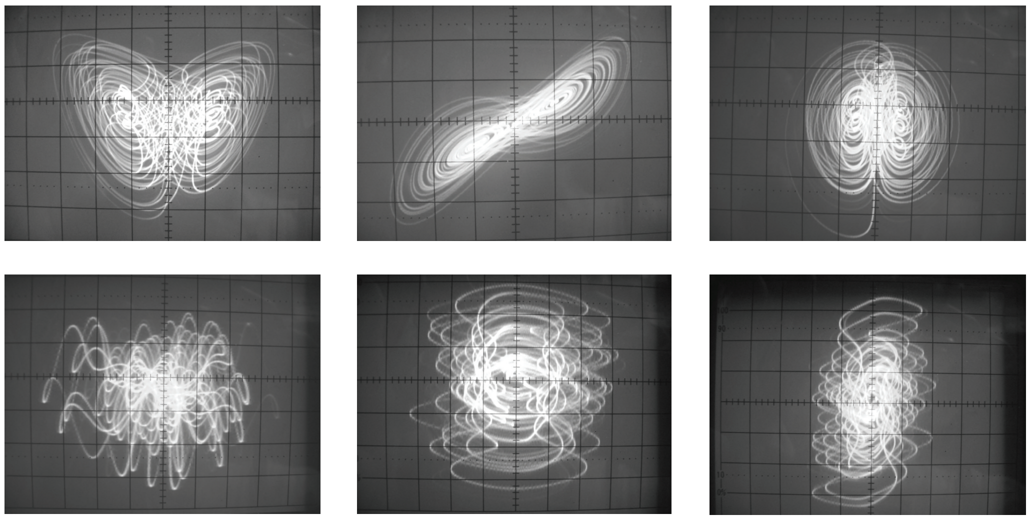

**Figure 8:** (a) Simplest circuit for the 4D hyperchaotic Lorenz system of (2.2). (b) Equivalent Lorenz hyperchaotic circuit.

### *3.2. Some Experimental Results*

For our experiments, two circuits following the schematic diagram of Figure 8(a) were constructed, each one using two AD633 multipliers and two LM351 op amps, all powered with ±15 V, and the following passive components: $L = 11\,\text{mH}$, $C_1 = 22\,\text{nF}$, $C_2 = 1\,\text{nF}$, $C_3 = 1\,\mu\text{F}$, $R_1 = 1.5\,\text{k}\Omega$, $R_2 = 80\,\Omega$, $R_3 = 470\,\text{k}\Omega$, $R_4 = 1\,\text{k}\Omega$, $R_5 = R_6 = R_7 = 100\,\text{k}\Omega$, $R_8 = 2\,\text{k}\Omega$, $R_9 = 1\,\text{k}\Omega$. Using (3.2), the corresponding parameters of (2.2) are $\sigma = 4.2$, $b = 0.3$, $\gamma = 1.7$. We worked with $E = 15\,\text{V}$, giving $r = 28$. (Note: we verified that with an independent DC power source it is possible to use $E$ values up to 30 V, or $r = 56$, without waveform clipping.) The six projections of the experimental hyperchaotic attractor are shown in Figure 9(b); the calculated attractor is shown in Figure 9(a). For the experiments on synchronization, bidirectional diffusive coupling can be obtained simply by connecting a resistor $R$ linking the capacitor $C_1$ of the first circuit with the capacitor $C_1'$ of the second circuit, since in this work we have tested only $x$-$x$, or $v_1$-$v_1$, coupling. For unidirectional coupling a voltage follower is added in series with the resistor $R$, as sketched in Figure 10. The coupling strength is given by the relation $k = \sigma R_1/R$, as can be easily verified by adding the term $(v_1' - v_1)/R$ to the first

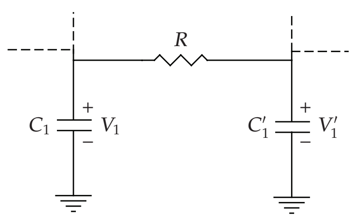


(a)

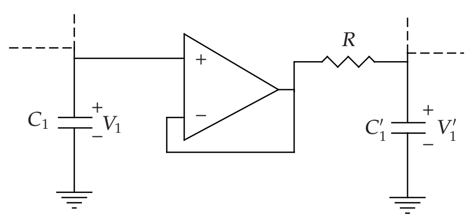


(b)

**Figure 9:** (a) Projections of a calculated hyperchaotic attractor described by (2.2) for $\sigma = 4.2$, $b = 0.3$, $\gamma = 1.7$, and $r = 28$. From left to right, top: $x \times z$; $x \times y$; $y \times z$; bottom: $w \times z$; $x \times w$; $y \times w$. (b) Experimental hyperchaotic Lorenz attractor generated by the circuit of Figure 8, as projected on the oscilloscope screen. From left to right, top: $v_1 \times v_2$; $v_1 \times i$; $i \times v_2$; bottom: $v_3 \times v_2$; $v_1 \times v_3$; $i \times v_3$. The same parameter values as those of Figure 9(a)—see text.

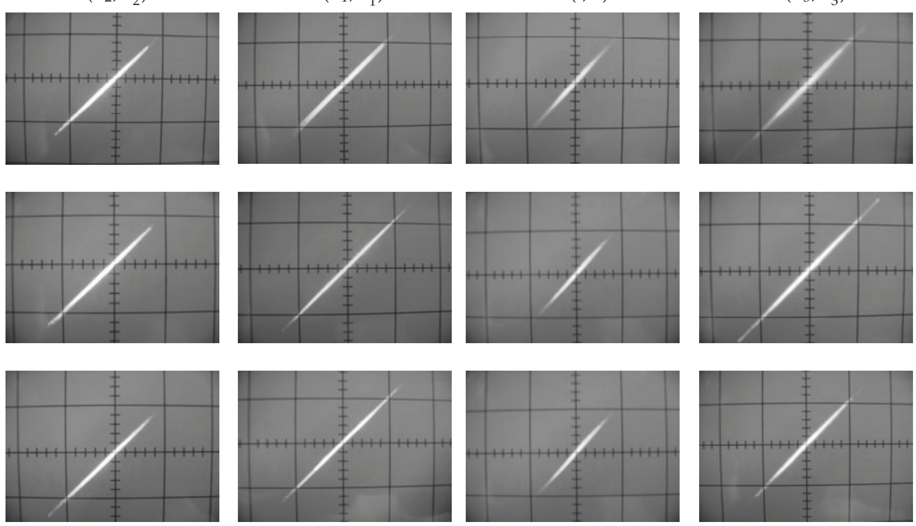



(a)

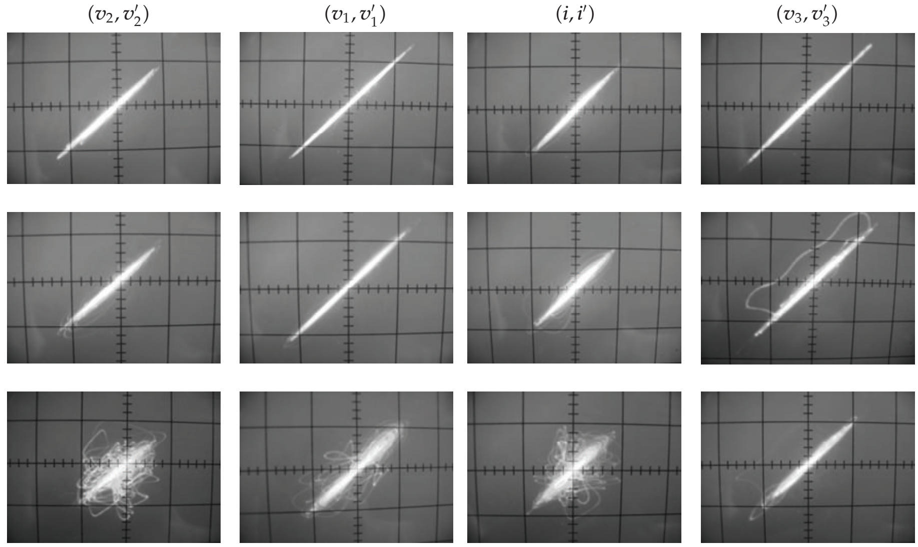



(b)

**Figure 10:** Coupling schemes: (a) bidirectional; (b) unidirectional.




(a)




(b)

**Figure 11:** (a) Experimental trajectories near the synchronization hyperplane for several values of the unidirectional coupling resistor. A: 0 Ω; B: 100 Ω; C: 220 Ω. (b) Continuation of Figure 11(a). D: 470 Ω; E: 1.0 kΩ; F: 2.2 kΩ.

of (3.1). Note that each variable of the second circuit is represented by the same symbol as the corresponding one of the first circuit, but with an uppercase prime. The four projections of the trajectories near the synchronization hyperplane are shown in Figures 11(a) and 11(b) for several values of the coupling resistor $R$. The waveforms $v_2(t)$ and $v_2'(t)$ are shown in Figure 12. All these results refer to unidirectional coupling through the variables $v_1$ and $v_1'$.

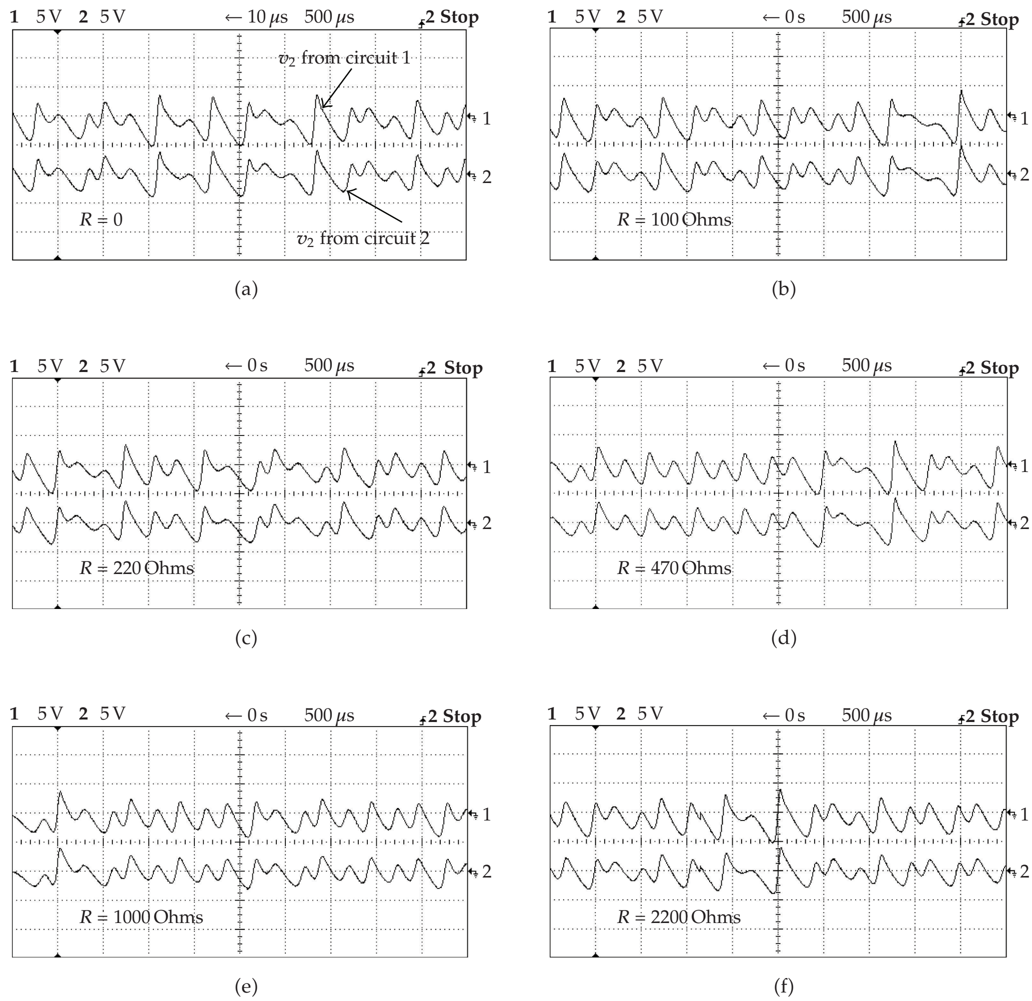



**Figure 12:** Experimental waveforms $v_2$ and $v_2'$ corresponding to the first column of Figure 11.

## 4. Conclusion

In this work the synchronizing properties of diffusively coupled hyperchaotic Lorenz 4D systems, described by (2.2), have been studied both numerically and experimentally. The numerical investigation was realized by calculating the transverse Lyapunov exponents as a function of the coupling strength $k$, and also by visually inspecting the phase space trajectories near the synchronization hyperplane. We concluded that using a single coupling variable, either $x$ or $y$ (but neither $z$ nor $w$), guarantees stable synchronization. Although $z$ and $w$ are not good choices for the single-variable scheme, double coupling with both $z$ and $w$ easily provides synchronization. We also verified that a small degree of parameter mismatch seems tolerable. For the laboratory work a very simple electrical circuit described by the Lorenz 4D system was proposed and described here for the first time. The experiments confirmed the qualitative behavior predicted by the numerical approach.

## Acknowledgments

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