Research Article

Computation of All Stabilizing PID Gain for Second-Order Delay System

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The problem of stabilizing a second-order delay system using classical proportional-integral-derivative (PID) controller is considered. An extension of the Hermite-Biehler theorem, which is applicable to quasipolynomials, is used to seek the set of complete stabilizing PID parameters. The range of admissible proportional gains is determined in closed form. For each proportional gain, the stabilizing set in the space of the integral and derivative gains is shown to be either a trapezoid or a triangle.

1. Introduction

Dead times are often encountered in various engineering systems and industry processes such as electrical and communication network, chemical process, turbojet engine, nuclear reactor, and hydraulic system. In fact, delays are caused by many phenomena like the time required to transport mass, energy or information, the time processing for sensors, the time needed for controllers to execute a complicated algorithm control, and the accumulation of time lags in a number of simple plants connected in series [1]. Delay makes system analysis and control design much more complex which is explained as follows. First, the effect of the control action takes some time to be felt in the controlled variable. Second, the control action that is applied based on the actual error tries to correct a situation that originated previously. Finally, in the frequency domain, the time delay introduces an extra decrease in the systems phase, which may cause instability [1, 2].

Today, proportional-integral (PI) and proportional-integral-derivative (PID) controller types are the most widely used control strategy. It is estimated that over 90% of process control applications employ PID control thanks to its essential functionality and structural

simplicity [3]. Since the minimal requirement for PID controller is to guarantee the system stability. It is desirable to know the complete set of stabilizing PID parameters before tuning and design. Surveys have reported that poor tuning, configuration errors, traditional and empirical techniques such as Ziegler Nichols or Cohen-Coon tuning methods are found in the industrial application of PID [4]. A great deal of academic and industrial effort has focused on improving PID control in the areas of tuning rules to decrease the rising gap between engineering practice and control theory. Recent studies have made use of the generalization of Hermite-Biehler theorem to compute the set of all stabilizing PID controllers for a given plant [5]. Since almost all plants encountered in process control contain time delays, so computing the complete set of PID controllers that stabilize time delay system is of considerable importance. A robust PI/PID design via numerical optimization for delay processes was proposed in [6]. Padma Sree et al. [7] propose a PI/PID controllers design for first-order delay system by extracting the coefficients of the numerator and denominator of the closed-loop transfer function. In the work presented in [8], the authors have developed a graphical approach which is based on D-partition method [9], the borders of absolute and relative stability regions are described in the parameter space. A unified approach and Delta operator are used to obtain unified stability boundaries for PI, PD, and PID controller for an arbitrary order delay system; where the boundaries can be found when only the frequency response and not the parameters of the plant are known [10]. Eriksson and Johansson [11] consider a design of PID controller for system with varying time delay using multi-objectives' optimization. Neural network has been widely applied for nonlinear dynamical system identification via PID controller [12]. Zhang et al. [13] present an intelligent nonlinear PID controller based on the recurrent neural networks (RNNs) to control multivariable plants by using two predictive control schemes. In [14, 15], the authors propose a new PID neural networks (PIDNN) which is a dynamic multilayer network based on P, I, and D neurons. This PIDNN is used to control multivariable plants [15] and has also the ability to control time delay system [14]. In [16], Roy and Iqbal have explored PID tuning of first-order delay system using a first-order Padé approximation and the Hermite-Biehler stabilization framework. The set's characterization of all stabilizing PI/PID parameters using a version of the Hermit-Biehler theorem for first-order delay system is presented in [5, 17–20]. However, these results are not applicable to the second-order delay system. Although, the PID stabilization for a given time delay plant has been developed [21] using the generalized Nyquist criterion, the stabilizing set of the proportional gain cannot be found. In [22–24], the characterization of the set of all stabilizing P/PI/PID parameters is given by using the Hermit-Biehler Theorem for polynomials for a class of time delay system which verify the interlacing property at high frequencies. This method, however, is complex and there are difficulties to achieve the stabilizing range of proportional parameter. As shown in [23, Example 2], this latter approach does not determine the precise range of proportional gain which is larger than the exact stabilizing range given by [18]. Furthermore, a known stabilizing value of (K_p, K_i, K_d) is first determined using Nyquist criterion, then the entire stability region is established. In our work, the stabilizing problem of PID controller for second order delay system is analyzed using the extended Hermit-Biehler theorem for quasi-polynomials. The exact range of stabilizing proportional gain is first determined. The set of stabilizing integral and derivative constant values are then derived. Our algorithm is more simple and faster in time computing than the other presented in [24].

2. Preliminary Results for Analyzing Time Delay System

Several problems in process control engineering are related to the presence of delays. These delays intervene in dynamic models whose characteristic equations are of the following form [5, 25]

$$\delta(s) = d(s) + e^{-L_1 s} n_1(s) + e^{-L_2 s} n_2(s) + \cdots + e^{-L_m s} n_m(s), \tag{2.1}$$

where: $d(s)$ and $n_i(s)$ are polynomials with real coefficients and L_i represent time delays. These characteristic equations are recognized as quasi-polynomials. Under the following assumptions:

(A_1) $\deg(d(s)) = n$, $\deg(n_i(s)) < n$, for $i = 1, 2, \ldots, m$,

(A_2) $L_1 < L_2 < \cdots < L_m$.

One can consider the quasi-polynomials $\delta^*(s)$ described by

$$\delta^*(s) = e^{s L_m} \delta(s),$$

$$\delta^*(s) = e^{s L_m} d(s) + e^{s(L_m - L_1)} n_1(s) + e^{s(L_m - L_2)} n_2(s) + \cdots + n_m(s). \tag{2.2}$$

The zeros of $\delta(s)$ are identical to those of $\delta^*(s)$ since $e^{s L_m}$ does not have any finite zeros in the complex plane. However, the quasi-polynomial $\delta^*(s)$ has a principal term since the coefficient of the term containing the highest powers of s and e^s is nonzero. If $\delta^*(s)$ does not have a principal term, then it has an infinity roots with positive real parts [5].

The stability of the system with the characteristic equation (2.1) is equivalent to the condition that all the zeros of $\delta^*(s)$ must be in the open left half of the complex plan. We said that $\delta^*(s)$ is Hurwitz or is stable. The following theorem gives a necessary and sufficient condition for the stability of $\delta^*(s)$.

Theorem 2.1 (see [5]). *Let $\delta^*(s)$ be given by (2.2), and write*

$$\delta^*(j\omega) = \delta_r(\omega) + j\delta_i(\omega), \tag{2.3}$$

where $\delta_r(\omega)$ and $\delta_i(\omega)$ represent, respectively, the real and imaginary parts of $\delta^*(j\omega)$.
  Under conditions (A_1) and (A_2), $\delta^*(s)$ is stable if and only if

(1) $\delta_r(\omega)$ and $\delta_i(\omega)$ have only simple, real roots and these interlace,

(2) $\delta'_i(\omega_0)\delta_r(\omega_0) - \delta_i(\omega_0)\delta'_r(\omega_0) > 0$ for some ω_0 in $[-\infty, +\infty]$,

where $\delta'_i(\omega)$ and $\delta'_r(\omega)$ denote the first derivative with respect to ω of $\delta_r(\omega)$ and $\delta_i(\omega)$, respectively.

A crucial stage in the application of the precedent theorem is to make sure that $\delta_r(\omega)$ and $\delta_i(\omega)$ have only real roots. Such a property can be checked while using the following theorem.

Theorem 2.2 (see [5]). *Let M and N designate the highest powers of s and e^s which appear in $\delta^*(s)$. Let η be an appropriate constant such that the coefficient of terms of highest degree in $\delta_r(\omega)$ and $\delta_i(\omega)$ do not vanish at $\omega = \eta$. Then a necessary and sufficient condition that $\delta_r(\omega)$ and $\delta_i(\omega)$ have only real roots is that in each of the intervals $-2l\pi + \eta\ < \omega <\ 2l\pi + \eta$, $l = l_0, l_0 + 1, l_0 + 2 \cdots \delta_r(\omega)$ or $\delta_i(\omega)$ have exactly $4lN + M$ real roots for a sufficiently large l_0.*

3. PID Control for Second Order Delay System

A second order system with delay can be mathematically expressed by a transfer function having the following form:

$$G(s) = \frac{K}{s^2 + a_1 s + a_0} e^{-Ls}, \tag{3.1}$$

where K is the static gain of the plant, L is the time delay, and a_0 and a_1 are the plant parameter's which are always positive. The characteristic equation of the closed-loop system is given by

$$\delta(s) = K\left(K_i + K_p s + K_d s^2\right) e^{-Ls} + \left(s^2 + a_1 s + a_0\right) s, \tag{3.2}$$

we deduce the quasi-polynomial $\delta^*(s)$:

$$\delta^*(s) = e^{Ls}\delta(s) = K\left(K_i + K_p s + K_d s^2\right) + s\left(s^2 + a_1 s + a_0\right) e^{Ls}, \tag{3.3}$$

by replacing s by $j\omega$, we get

$$\delta^*(j\omega) = \delta_r(\omega) + j\delta_i(\omega), \tag{3.4}$$

with:

$$\delta_r(\omega) = KK_i - KK_d\omega + \left(\omega^3 - a_0\omega\right)\sin(L\omega) - a_1\omega^2\cos(L\omega),$$
$$\delta_i(\omega) = \omega\left[KK_p + \left(a_0 - \omega^2\right)\cos(L\omega) - a_1\omega\sin(L\omega)\right]. \tag{3.5}$$

Clearly, the parameters K_i and K_d only affect the real part of $\delta^*(j\omega)$ whereas the parameter K_p affects the imaginary part.

Let's put $z = L\omega$, we get

$$\delta_r(z) = KK_i - KK_d\frac{z^2}{L^2} + \sin(z)\left(\frac{z^3}{L^3} - a_0\frac{z}{L}\right) - a_1\frac{z^2}{L^2}\cos(z),$$
$$\delta_i(z) = \frac{z}{L}\left(KK_p + \cos(z)\left(a_0 - \frac{z^2}{L^2}\right) - a_1\frac{z}{L}\sin(z)\right). \tag{3.6}$$

The application of the second condition of Theorem 2.1, led us to

$$E(z_0) = \delta_i'(z_0)\delta_r(z_0) - \delta_i(z_0)\delta_r'(z_0) > 0, \qquad (3.7)$$

from (3.6) we have

$$\delta_i'(z) = \frac{KK_p}{L} - \sin(z)\left(a_0 + \frac{2a_1 z}{L^2} - \frac{z^3}{L^3}\right) + \cos(z)\left(\frac{a_0}{L} - \frac{3z^2}{L^3} - a_1\frac{z^2}{L^2}\right), \qquad (3.8)$$

for $z_0 = 0$ (a value that annul $\delta_i(z)$), and we get:

$$E(z_0) = \delta_i'(z_0)\delta_r(z_0) = \left(\frac{KK_p + a_0}{L}\right)KK_i > 0, \qquad (3.9)$$

which implies $K_p > -a_0/K$ since $K > 0$ and $K_i > 0$.

We pass to the verification of the interlacing condition of $\delta_r(z)$ and $\delta_i(z)$ roots. For such purpose, we are going to determine the roots from the imaginary part, which is translated by the following relation:

$$\delta_i(z) = 0 \Longrightarrow z = 0 \quad \text{or} \quad KK_p + \cos(z)\left(a_0 - \frac{z^2}{L^2}\right) - a_1\frac{z}{L}\sin(z) = 0,$$

$$\Longrightarrow z = 0 \quad \text{or} \quad KK_p + \cos(z)\left(a_0 - \frac{z^2}{L^2}\right) = a_1\frac{z}{L}\sin(z), \qquad (3.10)$$

$$\Longrightarrow z = 0 \quad \text{or} \quad f(z) = g(z).$$

We notice the $z_0 = 0$ is a trivial root of the imaginary part. The others are difficult to solve analytically. However, this can be made graphically. The following two cases are presented.

Case 1 ($K_p > -a_0/K$). In this case, we graph the curves of $f(z)$ and of $g(z)$ which are presented in Figure 1.

$$\begin{aligned}
z_1 &\in [0, \pi/2], & z_2 &\in [\pi/2, \pi], \\
z_3 &\in [3\pi/2, 2\pi], & z_4 &\in [5\pi/2, 3\pi], \\
z_5 &\in [7\pi/2, 4\pi], & z_6 &\in [9\pi/2, 5\pi], \\
&\vdots & &\vdots
\end{aligned} \qquad (3.11)$$
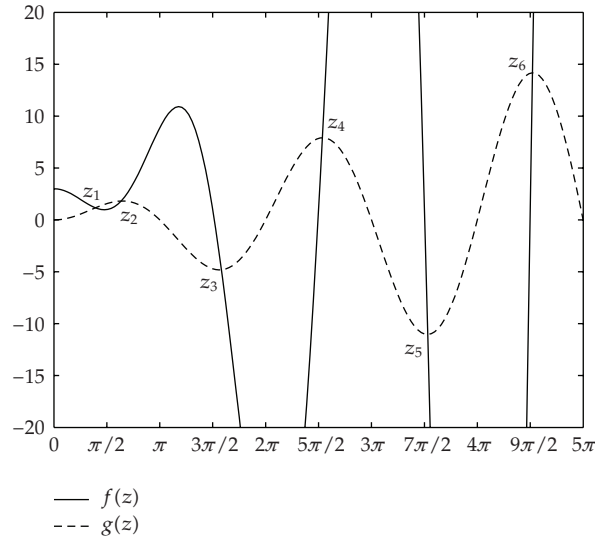
Figure 1: Representation of the curves of $f(z)$ and of $g(z)$ ($K = L = a_1 = 1$, $a_0 = 2$, $K_p = 1.3$, and $K_u = 1.5884$) (Case: $K_p < K_u$).
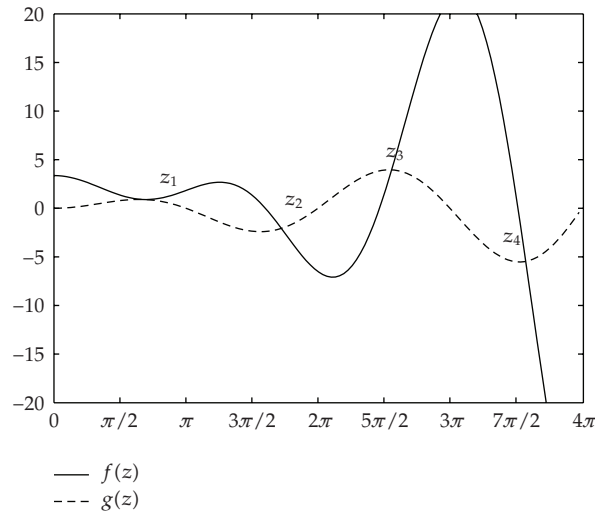
Figure 2: Representation of the curves of $f(z)$ and of $g(z)$ ($K = L = a_1 = 1$, $a_0 = 2$, $K_p = K_u = 1.5884$) (Case: $K_p = K_u$).

that is, z_j verifies

$$z_1 \in \left[0, \frac{\pi}{2}\right],$$
$$z_j \in \left[(2j-3)\frac{\pi}{2}, (j-1)\pi\right], \quad \text{for } j \geq 2. \tag{3.12}$$

Case 2 ($K_p \geq K_u$). Figure 2 represents the case where the curves of $f(z) = KK_u + \cos(z)(a_0 - (z^2/L^2))$ and of $g(z)$ are tangent in $[0, \pi]$, where K_u is the largest number of K_p.

The plot in Figure 3 corresponds to the case where $K_p > K_u$ and the plot of $f(z)$ does not intersect $g(z)$ twice in the interval $[0, \pi]$.

Theorem 2.2 is used to verify that $\delta_i(z)$ possess only simple roots. By replacing Ls by s_1 in (2.2), we rewrite $\delta^*(s)$ as follows:

$$
\begin{aligned}
\delta^*(s) = e^{Ls}\delta(s) &= e^{s_1}\delta(s_1) \\
&= e^{s_1}\left(\left(\frac{s_1}{L}\right)^3 + a_1\left(\frac{s_1}{L}\right)^2 + a_0\frac{s_1}{L}\right) + K\left(K_p\frac{s_1}{L} + K_i\right).
\end{aligned}
\tag{3.13}
$$

For this new quasi-polynomial, we see that $M = 3$ and $N = 1$ where M and N designate the highest powers of s and e^s which appear in $\delta^*(s)$. We choose $\eta = \pi/4$ that satisfies the condition giving by theorem 3 as $\delta_r(\eta) \neq 0$ and $\delta_i(\eta) \neq 0$. According to Figure 1, we notice that for $-a_0/K < K_p < K_u$, $\delta_i(z)$ possess four roots in the interval $[0, 2\pi - \pi/4] = [0, 7\pi/4]$ including the root at origin. As $\delta_i(z)$ is odd function of z, so it possesses seven roots in $[-2\pi + \pi/4, 2\pi - \pi/4] = [-7\pi/4, 7\pi/4]$. Hence, we can affirm that $\delta_i(z)$ has exactly $4N + M = 7$ in $[-2\pi + \pi/4, 2\pi + \pi/4] = [-7\pi/4, 9\pi/4]$. In addition, it can be shown that $\delta_i(z)$ has two real roots in each of the intervals $[2l\pi + \pi/4, 2(l+1)\pi + \pi/4]$ and $[-2(l+1)\pi + \pi/4, -2l\pi + \pi/4]$ for $l = 1, 2, \ldots$. It fallows that $\delta_i(z)$ has exactly $4lN + M$ real roots in $[-2l\pi + \pi/4, 2l\pi + \pi/4]$ for $-a_0/K < K_p < K_u$. At the end, according to Theorem 2.2, $\delta_i(z)$ has only real roots for every K_p in $[-a_0/K, K_u]$. For $K_p \geq K_u$, corresponding to Figures 2 and 3, the roots of $\delta_i(z)$ are not real. We pass to determine the superior value of K_p. According to the definition of K_u, if $K_p = K_u$, then the curves of $f(z)$ and $g(z)$ are tangent in the point α. Which is translated by

$$
\begin{aligned}
KK_u + \cos(\alpha)&\left(a_0 - \frac{\alpha^2}{L^2}\right) = a_1\frac{\alpha}{L}\sin(\alpha), \\
\frac{d}{dz}&\left[KK_u + \cos(z)\left(a_0 - \frac{z^2}{L^2}\right)\right]_{z=\alpha} \\
&= \frac{d}{dz}\left[a_1\frac{z}{L}\sin(z)\right]_{z=\alpha}, \\
\Longrightarrow\ &-2\alpha\cos(\alpha)(1 + a_1L) + \sin(\alpha)\left(\alpha^2 - a_0L^2 - a_1L\right) = 0 \\
\Longrightarrow\ &\tan(\alpha) = \frac{\alpha(2 + a_1L)}{(\alpha^2 - a_0L^2 - a_1L)},
\end{aligned}
\tag{3.14}
$$

once α is determined, the parameter K_u is expressed by the relation:

$$
K_u = \frac{1}{K}\left(a_1\frac{\alpha}{L}\sin(\alpha) - \cos(\alpha)\left(a_0 - \frac{\alpha^2}{L^2}\right)\right).
\tag{3.15}
$$
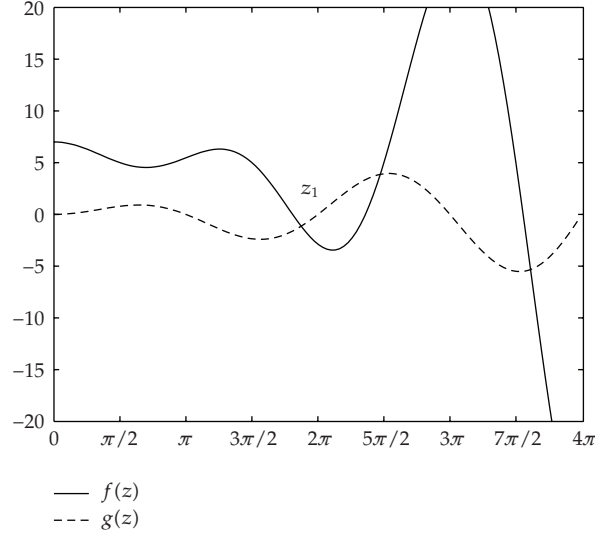
Figure 3: Representation of the curves of $f(z)$ and of $g(z)$ ($K = L = a_1 = 1$, $a_0 = 2$, $K_p = 2$, and $K_u = 1.5884$ (Case: $K_p > K_u$).

Theorem 3.1. *The $\delta_i(z)$ have real roots if and only if:*

$$-\frac{a_0}{K} < K_p < \frac{1}{K}\left(a_1\frac{\alpha}{L}\sin(\alpha) - \cos(\alpha)\left(a_0 - \frac{\alpha^2}{L^2}\right)\right), \tag{3.16}$$

where α is the solution of the equation $\tan(\alpha) = \alpha(2 + a_1 L)/(\alpha^2 - a_1 L - a_0 L^2)$ in the interval $[0, \pi]$.

After determination of the roots of the imaginary part $\delta_i(z)$, we pass to the evaluation of these roots by the real part $\delta_r(z)$

$$
\begin{aligned}
\delta_r(z) &= KK_i - KK_d\frac{z^2}{L^2} + \sin(z)\left(\frac{z^3}{L^3} - a_0\frac{z}{L}\right) - a_1\frac{z^2}{L^2}\cos(z) \\
&= K\frac{z^2}{L^2}\left[-K_d + K_i\frac{L^2}{z^2} + \frac{L}{Kz}\left[-a_1\frac{z}{L}\cos(z) + \sin(z)\left(\frac{z^2}{L^2} - a_0\right)\right]\right] \\
&= K\frac{z^2}{L^2}[-K_d + m(z)K_i + b(z)],
\end{aligned}
\tag{3.17}
$$

where:

$$
\begin{aligned}
m(z) &= \frac{L^2}{z^2}, \\
b(z) &= \frac{L}{Kz}\left[-a_1\frac{z}{L}\cos(z) + \sin(z)\left(\frac{z^2}{L^2} - a_0\right)\right].
\end{aligned}
\tag{3.18}
$$

From condition 1 of **Theorem 2.1**, the roots of $\delta_r(z)$ and $\delta_i(z)$ have to interlace for $\delta^*(s)$ to be stable. We evaluate $\delta_r(z)$ at the roots of the imaginary part $\delta_i(z)$.

For $z_0 = 0$, we have

$$\delta_r(z_0) = K K_i > 0, \tag{3.19}$$

for $z_j \neq z_0$, where $j = 1, 2, 3, \ldots$, we get:

$$
\begin{aligned}
\delta_r(z_j) &= K \frac{z_j^2}{L^2} \left[-K_d + m(z_j) K_i + b(z_j) \right] \\
&= K \frac{z_j^2}{L^2} \left[-K_d + m_j K_i + b_j \right],
\end{aligned}
\tag{3.20}
$$

where:

$$
\begin{aligned}
m(z_j) &= m_j, \\
b(z_j) &= b_j.
\end{aligned}
\tag{3.21}
$$

Interlacing the roots of $\delta_r(z)$ and $\delta_i(z)$ is equivalent to $\delta_r(z_0) > 0$ (since $K_i > 0$), $\delta_r(z_1) < 0$, $\delta_r(z_2) > 0 \ldots$.

We can use the interlacing property and do it as $\delta_i(z)$ which has only real roots to reach that $\delta_r(z)$ possess real roots too.

From the previous equations we get the following inequalities:

$$
\begin{cases}
\delta_r(z_0) > 0 \\
\delta_r(z_1) < 0 \\
\delta_r(z_2) > 0 \\
\delta_r(z_3) < 0 \\
\delta_r(z_4) > 0 \\
\vdots
\end{cases}
\implies
\begin{cases}
K_i > 0 \\
K_d > m_1 K_i + b_1 \\
K_d < m_2 K_i + b_2 \\
K_d > m_3 K_i + b_3 \\
K_d < m_4 K_i + b_4 \\
\vdots
\end{cases}
\tag{3.22}
$$

Thus intersecting all these regions in the (K_i, K_d) space, we get the set of (K_i, K_d) values for which the roots of $\delta_r(z)$ and $\delta_i(z)$ interlace for a fixed value of K_p. We notice that all these regions are half planes with their boundaries being lines with positive slopes m_j.

Example 3.2. Consider the plant given by relation (3.1) with the following parameters $K = L = a_1 = 1$ and $a_0 = 2$:

$$G(s) = \frac{1}{s^2 + s + 2} e^{-s}. \tag{3.23}$$

The imaginary part $\delta_i(z)$ has only simple real roots if and only if $-1 < K_p < 1.58$.

We set the controller parameter K_p to 1.3, which is inside the previous range. For this K_p, $\delta_i(z)$ takes the form:

$$z\left(1.3 + \cos(z)\left(2 - z^2\right) - z\,\sin(z)\right) = 0, \tag{3.24}$$

We next compute some of the positive real roots of this equation and arrange them in increasing order of magnitude:

$$z_0 = 0, \qquad z_1 = 1.3608, \qquad z_2 = 1.8905, \qquad z_3 = 4.9829,$$

$$z_4 = 7.9619, \qquad z_5 = 11.0976, \qquad z_6 = 14.2017. \tag{3.25}$$

Using (3.18) we calculate the parameters m_j and b_j for $j = 1, \ldots, 6$:

$$\begin{cases} m_1 = 0.54, \\ m_2 = 0.2798, \\ m_3 = 0.0403, \\ m_4 = 0.0158, \end{cases} \qquad \begin{cases} b_1 = 0.54, \\ b_2 = -0.3150, \\ b_3 = 1.1047, \\ b_4 = -4.6822. \end{cases} \tag{3.26}$$

Interlacing the roots of the real and the imaginary part occurs for $K_p = 1.3$, if and only if the following inequalities are satisfied:

$$K_i > 0,$$

$$K_d > 0.54K_i - 0.3150,$$

$$K_d < 0.2798K_i + 1.1047, \tag{3.27}$$

$$K_d > 0.0403K_i - 4.6822,$$

$$K_d < 0.0158K_i + 7.7736.$$

The boundaries of these regions are illustrated in Figure 4.

In this case, the stability region is defined by only two boundaries:

$$K_d = m_1 K_i + b_1,$$

$$K_d = m_2 K_i + b_2, \tag{3.28}$$

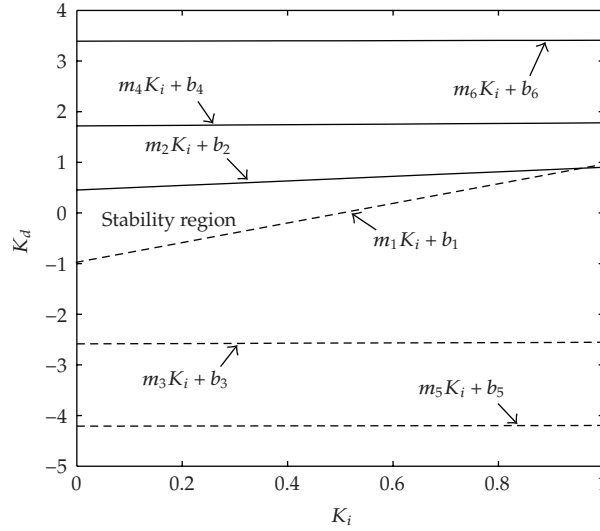**Figure 4:** Region boundaries of Example 3.2.

because we have the following inequalities:

$$
\begin{aligned}
b_j &< b_{j+2}, \quad \text{for } j = 2, 4, 6, \ldots, \\
b_j &> b_{j+2}, \quad \text{for } j = 1, 3, 5, 7, \ldots, \\
m_j &< m_{j+2}, \quad \text{for } j \geq 1.
\end{aligned}
\tag{3.29}
$$

Example 3.3. Consider the plant (3.1) with the following parameters $K = 2, a_0 = 3, a_1 = 1$, $L = 2$:

$$
G(s) = \frac{2}{s^2 + s + 3} e^{-2s}.
\tag{3.30}
$$

The imaginary part $\delta_i(z)$ has only simple real roots if and only if $-1.5 < K_p < 0.98$.

We now set the controller parameter K_p to 0.5, which is inside the previous range. For this K_p, $\delta_i(z)$ takes the form

$$
z\left(0.5 + \cos(z)\left(2 - \frac{z^2}{4} \right) - 0.5\, z\, \sin(z) \right) = 0.
\tag{3.31}
$$

We next compute some of the positive real roots of this equation and arrange them in increasing order of magnitude

$$
\begin{aligned}
z_0 = 0, \quad z_1 = 1.6480, \quad z_2 = 2.9830, \quad z_3 = 5.4560, \quad z_4 = 8.077, \\
z_5 = 11.22, \quad z_6 = 14.26, \quad z_7 = 17.41.
\end{aligned}
\tag{3.32}
$$

Using (3.18), we calculate the parameters m_j and b_j for $j = 1, \ldots, 7$:

$$
\begin{cases}
m_1 = 1.4728, \\
m_2 = 0.4495, \\
m_3 = 0.1344, \\
m_4 = 0.0613, \\
m_5 = 0.0318, \\
m_6 = 0.0197, \\
m_7 = 0.0132,
\end{cases}
\qquad
\begin{cases}
b_1 = -1.3656, \\
b_2 = 0.4527, \\
b_3 = -0.9377, \\
b_4 = 1.7176, \\
b_5 = -2.5853, \\
b_6 = 3.3906, \\
b_7 = -4.2097.
\end{cases}
\tag{3.33}
$$

Interlacing the roots of the real and imaginary part occurs for $K_p = 0.5$, if and only if the following inequalities are satisfied:

$$
\begin{aligned}
& K_i > 0, \\
& K_d > 1.4728 K_i - 1.3656, \\
& K_d < 0.4495 K_i + 0.4527, \\
& K_d > 0.1344 K_i - 0.9377, \\
& K_d < 0.0613 K_i + 1.7176, \\
& K_d > 0.0318 K_i - 2.5853, \\
& K_d < 0.0197 K_i + 3.3906, \\
& K_d > 0.0132 K_i - 4.2097.
\end{aligned}
\tag{3.34}
$$

The boundaries of these regions are illustrated in Figure 5.

In this case, the stability region is defined by only two boundaries:

$$
\begin{aligned}
K_d &= m_1 K_i + b_1, \\
K_d &= m_2 K_i + b_2, \\
K_d &= m_3 K_i + b_3,
\end{aligned}
\tag{3.35}
$$

because we have the following inequalities:

$$
\begin{aligned}
& b_j < b_{j+2}, \quad \text{for } j = 2, 4, 6, \ldots, \\
& b_j > b_{j+2}, \quad \text{for } j = 3, 5, 7, \ldots, \\
& b_1 < b_3, \\
& m_j < m_{j+2}, \quad \text{for } j \geq 1.
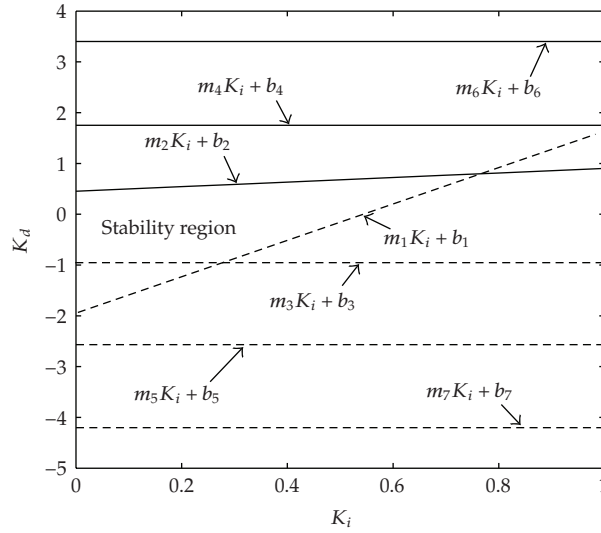\end{aligned}
\tag{3.36}
$$

Figure 5: Region boundaries of Example 3.3.

As pointed out in Examples 3.2 and 3.3, the inequalities given by (3.22) represent half planes in the space of K_i and K_d. Their boundaries are given by lines with the following equations:

$$K_d = m_j K_i + b_j, \quad \text{for } j = 1, 2, 3, \dots . \tag{3.37}$$

We notice that the difference between Examples 3.2 and 3.3 resides in change of the sign of $(b_1 - b_3)$ which is explained by the localization of z_1 and z_3 in different intervals according to (3.12).

We now state an important technical lemma that allows us to develop an algorithm for solving the PID stabilization problem. This lemma shows the behavior of the parameter b_j, $j = 2, 3, 4, \dots$ for different values of the parameter K_p inside the range proposed by Theorem 3.1.

Lemma 3.4. *If*

$$-\frac{a_0}{K} < K_p < \frac{1}{K}\left(a_1 \frac{\alpha}{L} \sin(\alpha) - \cos(\alpha)\left(a_0 - \frac{\alpha^2}{L^2} \right) \right), \tag{3.38}$$

where α is the solution of the equation $\tan(\alpha) = \alpha(2 + a_1 L)/(\alpha^2 - a_1 L - a_0 L^2)$ in the interval $[0, \pi]$. Then:

(1) *$b_j < b_{j+2}$ for $j = 2, 4, 6, \dots,$*

(2) *$b_j > b_{j+2}$ for $j = 3, 5, 7, \dots,$*

(3) *$m_j > m_{j+2}$ for $j \geq 1$, and $\lim_{j \to \infty} m_j = 0$.*

Proof. From (3.18) we have

$$m(z_j) = \frac{L^2}{z_j^2}, \quad 0 < z_j < z_{j+2} \Longrightarrow \frac{L^2}{z_j^2} > \frac{L^2}{z_{j+2}^2}, \quad \text{for } j \geq 1, \tag{3.39}$$

then

$$m(z_j) > m(z_{j+2}), \qquad \lim_{j \to \infty} m(z_j) = 0. \tag{3.40}$$

$\square$

Remark 3.5. As we can see from Figure 1, the odd roots of $\delta_i(z)$, that is, z_j for $j = 3, 5, 7, \ldots$, verify $z_j \in [(2j - 3)\pi/2, (j - 1)\pi]$ and are getting closer to $(2j - 3)\pi/2$ as j increase, so we have:

$$\lim_{j \to \infty} \cos(z_j) = 0, \qquad \lim_{j \to \infty} \sin(z_j) = -1. \tag{3.41}$$

Moreover, since cosine function and sine function are monotonically increasing between $(2j - 3)\pi/2$ and $(j - 1)\pi$, we have

$$\begin{cases} \cos(z_3) > \cos(z_5) > \cos(z_7) > \cdots \\ \sin(z_3) > \sin(z_5) > \sin(z_7) > \cdots \end{cases} \Longrightarrow \begin{cases} \cos(z_j) > \cos(z_{j+2}) > 0, \\ \sin(z_{j+2}) < \sin(z_j) < 0. \end{cases} \tag{3.42}$$

Remark 3.6. As we can see from Figure 1, the even roots of $\delta_i(z)$ that is, z_j for $j = 2, 4, 6, \ldots$ verify $z_j \in [(2j - 3)\pi/2, (j - 1)\pi]$ and are getting closer to $(2j - 3)\pi/2$ as j increase, so we have:

$$\lim_{j \to \infty} \cos(z_j) = 0, \qquad \lim_{j \to \infty} \sin(z_j) = 1. \tag{3.43}$$

Moreover, since cosine function and sine function are monotonically decreasing between $(2j - 3)\pi/2$ and $(j - 1)\pi$, we have:

$$\begin{cases} \cos(z_2) < \cos(z_4) < \cos(z_6) < \cdots \\ \sin(z_2) < \sin(z_4) < \sin(z_6) < \cdots \end{cases} \Longrightarrow \begin{cases} \cos(z_j) < \cos(z_{j+2}) < 0, \\ 0 < \sin(z_j) < \sin(z_{j+2}). \end{cases} \tag{3.44}$$

The change of $b(z_j)$ can be found with graphical approach. Given a stabilizing K_p value inside the admissible range by using Theorem 3.7, the curve of $\delta_i(z)$ is plotted to obtain their roots denoted by $z_j, j = 1, 2, 3, \ldots$, the curves of $(-a_1(z/L)) \cos(z)$ and $\sin(z)(z^2/L^2 - a_0)$ are plotted too in order to understand the behavior of $b(z_j)$.

As far as the odd roots of $\delta_i(z)$ are concerned, the corresponding $\sin(z)(z^2/L^2 - a_0)$ is decreasing by large magnitude, and for the even ones, the corresponding $\sin(z)(z^2/L^2 - a_0)$ is increasing by large magnitude. However, compared to the change of $\sin(z_j)(z_j^2/L^2 - a_0)$, the difference between the values of $(-a_1(z_{j+2}/L)) \cos(z_{j+2})$ and $(-a_1(z_j/L)) \cos(z_j)$ is
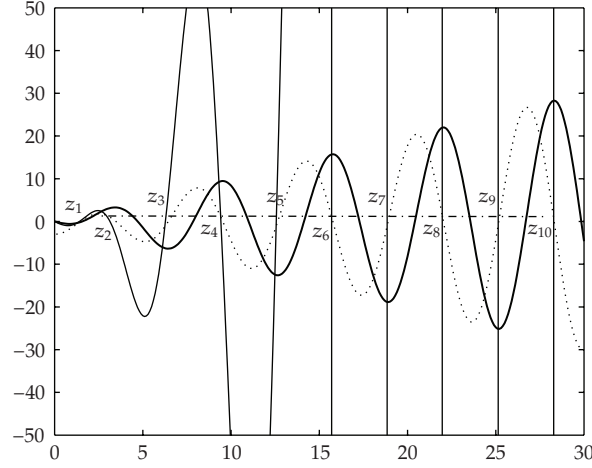
Figure 6: Plots of $\delta_i(z)$ (dotted line), $(-a_1(z/L))\cos(z)$ (thick solid line) and $\sin(z)(z^2/L^2 - a_0)$ (thin solid line).

much smaller than both for odd and even j. Thus, the $b(z_j)$ have the similar change rules as $\sin(z_j)(z_j^2/L^2 - a_0)$,

$$
\begin{aligned}
b_{j+2} - b_j &= \frac{L}{K}\left[-a_1\frac{z_{j+2}}{L}\cos(z_{j+2}) + \sin(z_{j+2})\left(\frac{z_{j+2}^2}{L^2} - a_0\right)\right] \\
&\quad - \frac{L}{K}\left[-a_1\frac{z_j}{L}\cos(z_j) + \sin(z_j)\left(\frac{z_j^2}{L^2} - a_0\right)\right], \\
b_{j+2} - b_j &= \frac{a_1}{L}\left[\cos(z_j) - \cos(z_{j+2})\right] \\
&\quad + \frac{L}{K}\left[\frac{\sin(z_{j+2})}{z_{j+2}}\left(\frac{z_{j+2}^2}{L^2} - a_0\right) - \frac{\sin(z_j)}{z_j}\left(\frac{z_j^2}{L^2} - a_0\right)\right].
\end{aligned}
\tag{3.45}
$$

Using Remarks 3.5 and 3.6, we can affirm that

  (i) $b_j > b_{j+2}$ and $b_j \to -\infty$ as $j \to +\infty$ for odd values of j,

  (ii) $b_j < b_{j+2}$ and $b_j \to +\infty$ as $j \to +\infty$ for even values of j.

We are ready to state the main results of our work.

Theorem 3.7. *Under the above assumptions on K, L, a_0, and a_1, the range of K_p values for which a solution exists to the PID stabilization problem of an open-loop stable plant with transfer function $G(s)$ is given by*

$$
-\frac{a_0}{K} < K_p < \frac{1}{K}\left(a_1\frac{\alpha}{L}\sin(\alpha) - \cos(\alpha)\left(a_0 - \frac{\alpha^2}{L^2}\right)\right),
\tag{3.46}
$$

where α is the solution of the equation $\tan(\alpha) = \alpha(2 + a_1L)/(\alpha^2 - a_1L - a_0L^2)$ in the interval $[0, \pi]$.

(a)

(b)
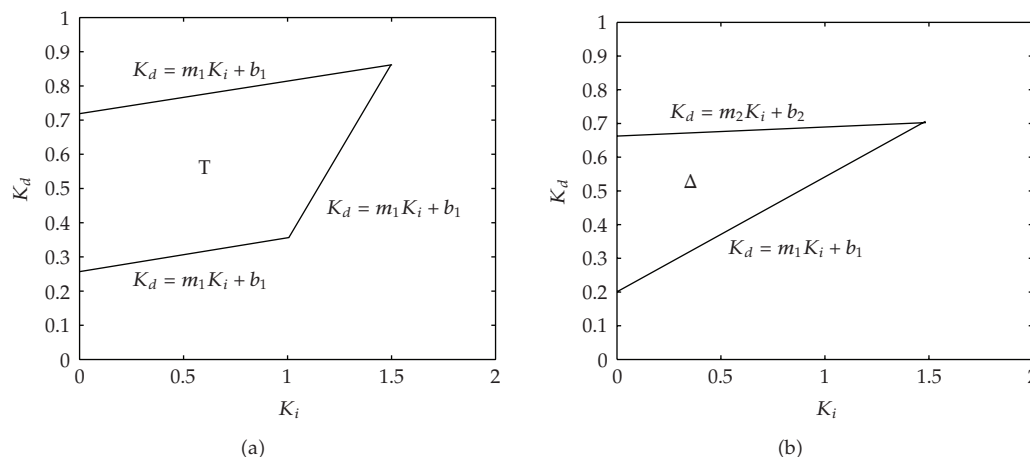
Figure 7: The stabilizing region of (K_p, K_i), (a): $b_3 > b_1$, (b): $b_1 > b_3$.

For K_p values outside this range, there are no stabilizing PID controllers. The complete stabilizing region is given by

(1) the cross-section of the stabilizing region in the (K_i, K_d) space is the Trapezoid T if $b_3 > b_1$,

(2) the cross-section of the stabilizing region in the (K_i, K_d) space is the triangle $\triangle$ if $b_1 > b_3$,

The parameters $b_j, m_j, j = 1, 2, 3$ necessary for determining the boundaries, can be determined using (3.21) where $z_j, j = 1, 2, 3$ are the positive-real solutions of $\delta_i(z)$ arranged in ascending order of magnitude.

In view of Theorem 3.7, we propose an algorithm to determine the set of all stabilizing parameters for a second order delay system.

An algorithm for determining PID parameters is given as follows

(1) Choose K_p in the interval suggested by Theorem 3.7 and initialize $j = 1$,

(2) Find the roots z_j of $\delta_i(z)$,

(3) Compute the parameter b_j, m_j associated with the z_j for $j = 1, 2, 3$ founded,

(4) Determine the stability region in the plane (K_i, K_d) using Figure 7 (Theorem 3.7).

(5) Go to step 1.

4. Conclusion

In this work, we have proposed an extension of Hermit-Biehler theorem to compute the stability region for second-order delay system controlled by PID controller. The procedure is based first on determining the range of proportional gain value K_p for which a solution to PID stabilization exists. Then, it is shown that for a fixed K_p inside this range, the stabilizing integral K_i and derivative gain K_d values lie inside a region with known shape and boundaries.

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