Review Article

Some Relatively New Techniques for Nonlinear Problems

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This paper outlines a detailed study of some relatively new techniques which are originated by He for solving diversified nonlinear problems of physical nature. In particular, we will focus on the variational iteration method (VIM) and its modifications, the homotopy perturbation method (HPM), the parameter expansion method, and exp-function method. These relatively new but very reliable techniques proved useful for solving a wide class of nonlinear problems and are capable to cope with the versatility of the physical problems. Several examples are given to reconfirm the efficiency of these algorithms. Some open problems are also suggested for future research work.

1. Introduction

With the rapid development of nonlinear sciences, many analytical and numerical techniques have been developed by various scientists. Most of the developed techniques have their limitations like limited convergence, divergent results, linearization, discretization, unrealistic assumptions and noncompatibility with the versatility of physical problems [1–100]. He [16–40] developed a number of efficient and reliable techniques for solving a wide class of nonlinear problems. These relatively new but very powerful methods proved to be fully synchronized with the complexities of the physical problems, see [1–7, 11–40, 49–80, 84–100] and the references therein. In the present study, we will focus our attention on He's variational iteration (VIM), homotopy perturbation (HPM), modified variational iteration (MVIMS), parameter expansion, and exp-function methods. The variational iteration method (VIM) was suggested and proposed by He [17–24] in its preliminary form in 1999. The method has been used to solve nonlinear differential

equations. In a subsequent work [23, 24] the VIM was developed into a full theory for solving diversified physical problems of versatile nature. It is to be highlighted that the variational iteration method (VIM) is also very effective for solving differential-difference equations, see [49, 95] and the references therein. Moreover, He [17, 18, 27–38] introduced another wonderful technique, namely, homotopy perturbation (HPM) by merging the standard homotopy and perturbation. The HPM is independent of the drawbacks of the coupled techniques and absorbs all their positive features. It is to be noted that homotopy perturbation is a kind of perturbation method which can take full advantage of various perturbation methods, while using the homotopy technique to guarantee simple solution procedure. It is to worth mentioning that if the initial solution is suitably chosen, then only one or two iterations are enough to get the appropriate result, see [17, 18, 27–38]. The exp-function method was first proposed by He and Wu [39] in 2006. The method was originally suggested to search for solitary solutions and periodic solutions of nonlinear wave equations. It always leads to a generalized solution with free parameters which can be determined by using the initial/boundary conditions. The most interesting part is transformation between periodic and solitary solutions by using the so-called He-Wu transformations. The method [17, 39, 40, 84–87, 97, 100] is always used as a tool to find exact solutions, but it can be utilized also for finding solutions approximately including the solutions for boundary value problems. It is to be highlighted that the present study would also outline the He parameter-expansion technique [18, 25–27]. The parameter expansion technique includes the modified Lindstedt-Pioncare and book keeping parameter methods and previously called the parameter-expanding method. He [18] in his review article in 2006 also explained that the method does not require to construct a homotopy. These efficient techniques have been applied to a wide class of nonlinear problems, see [1–7, 11–40, 49–80, 84–100] and the references therein. With the passage of time some modifications in He's variational iteration method (VIM) has been introduced by various authors. Abbasbandy [1, 2] made the coupling of Adomian's polynomials with the correction functional (VIMAP) of the VIM and applied this reliable version for solving Riccati differential and Klein Gordon equations. In a later work, Noor and Mohyud-Din [62, 64, 74] exploited this concept for solving various singular and nonsingular boundary and initial value problems. Recently, Ghorbani et al. [13, 14] introduced He's polynomials (which are calculated from He's homotopy perturbation method) by splitting the nonlinear term and also proved that He's polynomials are fully compatible with Adomian's polynomials but are easier to calculate and are more user friendly. More recently, Noor and Mohyud-Din [60, 66–69, 72, 73] combined He's polynomials and correction functional of the VIM and applied this reliable version (VIMHP) to a number of physical problems. It has been observed [60, 66–69, 72, 73] that the modification based on He's polynomials (VIMHP) which was developed by Noor and Mohyud-Din is much easier to implement as compared to the one (VIMAP) where the so-called Adomian's polynomials along with their complexities are used. The basic motivation of the present study is the review of these very powerful and reliable techniques which have been originated by He for solving various nonlinear initial and boundary value problems of diversified physical nature. Several examples are given to reveal the efficiency and potential of these relatively new techniques. We have also pointed out that the techniques discussed in this paper can be extended for solving obstacle, free, moving, and contact problems, which arise in various fields of pure and engineering sciences. This is another aspect of future research work. The interested readers are advised to explore this avenue for innovative and novel applications of these techniques.

2. Exp-Function Method

Consider the general nonlinear partial differential equation of the type

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xxxx}, \ldots) = 0. \tag{2.1}$$

Using a transformation

$$\eta = kx + \omega t, \tag{2.2}$$

where k and ω are constants, we can rewrite (2.1) in the following nonlinear ODE;

$$Q\left(u, u', u'', u''', u^{(\mathrm{iv})}, \ldots\right) = 0, \tag{2.3}$$

according to the exp-function method, which was developed by He and Wu [39], we assume that the wave solutions can be expressed in the following form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp[n\eta]}{\sum_{m=-p}^{q} b_m \exp[m\eta]}, \tag{2.4}$$

where p, q, c, and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite (2.4) in the following equivalent form:

$$u(\eta) = \frac{a_c \exp[c\eta] + \cdots + a_{-d} \exp[-d\eta]}{b_p \exp[p\eta] + \cdots + b_{-q} \exp[-q\eta]}. \tag{2.5}$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems [5, 17, 39, 40, 53, 54, 58, 59, 75, 76, 84–87, 97–100]. To determine the value of c and p, we balance the linear term of highest order of (2.4) with the highest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of lowest order of (2.3) with lowest order nonlinear term.

Example 2.1 (see [58]). Consider the ZK-MEW (2.6)

$$u_t + a(u)^3{}_x + \left(bu_{xt} + ru_{yy}\right)_x = 0. \tag{2.6}$$

Introducing a transformation as $\eta = kx + \omega y + \rho t$, we can covert (2.6) into ordinary differential equations

$$\rho u' + 3aku^2 u' + \left(bk^2\rho + rk\omega^2\right)u''' = 0, \tag{2.7}$$

where the prime denotes the derivative with respect to η. The trial solution of the (2.7) can be expressed as follows, as shown in (2.6):

$$u(\eta) = \frac{a_c \exp[c\eta] + \cdots + a_{-d} \exp[-d\eta]}{b_p \exp[p\eta] + \cdots + b_{-q} \exp[-q\eta]}. \tag{2.8}$$

To determine the value of c and p, we balance the linear term of highest order of (2.7) with the highest order nonlinear term, we obtain

$$p = c, \quad d = q. \tag{2.9}$$

Case 1. We can freely choose the values of p, c, d but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$, then the trial solution yields

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}. \tag{2.10}$$

Substituting (2.10) into (2.7), we have

$$\frac{1}{A}\left[c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1}\exp(-\eta) + c_{-2}\exp(-2\eta) + c_{-3}\exp(-3\eta)\right] = 0, \tag{2.11}$$

where $A = (b_1 \exp(\eta) + b_0 + b_{-1}\exp(-\eta))^4$, $c_i (i = -3, \ldots, 0, \ldots, 3)$ are constants obtained by Maple 11. Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$\{c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0\}. \tag{2.12}$$

Solution of (2.12) will yield

$$b_1 = \frac{1}{8}\frac{a_0{}^2 a(bk^2 + 1)}{b_{-1}r\omega}, \quad b_{-1} = b_{-1}, \quad a_0 = a_0, \quad \rho = -\frac{kr\omega^2}{(bk^2 + 1)}, \quad a_{-1} = 0, \quad a_1 = 0, \quad b_0 = 0. \tag{2.13}$$

We, therefore, obtained the following generalized solitary solution $u(x, y, t)$ of (2.6) as follows:

$$u(x, y, t) = \frac{a_0}{1/8 \ (a_0{}^2 a(1 + bk^2))/(b_{-1} \ r\omega^2)e^{(kx+\omega y+\rho t)} + b_{-1}e^{-(kx+\omega y+\rho t)}}, \tag{2.14}$$

where $\rho = -kr\omega^2/(bk^2 + 1)$, a_0, b_{-1}, a, b, k, r, and ω are real numbers.
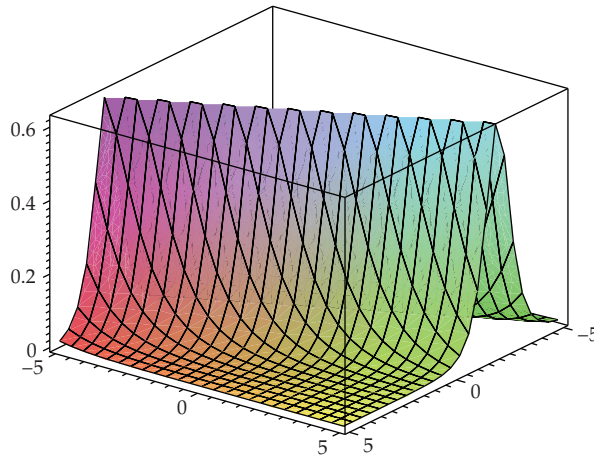
Figure 1

Figure 1 depicts the soliton solutions of (2.6), when $a_0 = b_{-1} = a = b = r = k = 1$. In case k and ω are imaginary numbers, the obtained soliton solutions can be converted into periodic solutions or compact-like solutions. Therefore, we write $k = iK$ and $\omega = iW$ consequently, (2.14) becomes

$$u(x, y, t) = \frac{a_0}{-1/8 \ (a_0{}^2 \ a(1 - bK^2 \ ))/(b_{-1}rW^2)e^{(iKx+iWy+\rho t)} + b_{-1}e^{-(iKx+iWy+\rho t)}}. \tag{2.15}$$

The above expression can be rewritten in expanded form:

$$u(x, y, t) = \frac{-8a_0b_{-1}rW^2 \left[\begin{array}{l} \cos\left(\dfrac{\mathscr{A}}{bK^2 - 1}\right)\left[-aba_0{}^2K^2 + aa_0{}^2 - 8rW^2b_{-1}{}^2\right] \\ +i\sin\left(\dfrac{\mathscr{A}}{bK^2 - 1}\right)\left[-aba_0{}^2K^2 + aa_0{}^2 + 8b_{-1}{}^2rW^2\right] \end{array} \right]}{\left[\begin{array}{l} 32aba_0{}^2rb_{-1}W^2\cos(\mathscr{A})^2\left[K^2 - 1\right] + a^2b^2a_0{}^4K^4 \\ -2a^2ba_0{}^4K^2 - 16aba_0{}^2b_{-1}{}^2rK^2W^2 + a^2a_0{}^4 + 16aa_0{}^2b_{-1}{}^2rW^2 + 64b_{-1}{}^4r^2W^4 \end{array} \right]}, \tag{2.16}$$

where $\mathscr{A} = -K^3bx + Kx - WbK^2y + Wy + KrW^2t$. If we search for periodic solutions or compact-like solutions, the imaginary part in (2.16) must be zero, hence

$$u(x, y, t) = \frac{-8a_0b_{-1}rW^2\left[\cos(\mathscr{A})\left[-aba_0{}^2K^2 + aa_0{}^2 - 8rW^2b_{-1}{}^2\right]\right]}{\left[\begin{array}{l} 32aba_0{}^2rb_{-1}W^2\cos(\mathscr{A})^2\left[K^2 - 1\right] + a^2b^2a_0{}^4K^4 \\ -2a^2ba_0{}^4K^2 - 16aba_0{}^2b_{-1}{}^2rK^2W^2 + a^2a_0{}^4 + 16aa_0{}^2b_{-1}{}^2rW^2 + 64b_{-1}{}^4r^2W^4 \end{array} \right]}, \tag{2.17}$$

Figure 2 depicts the periodic solutions of (2.6) when $a = b = a_0 = b_{-1} = K = W = r = t = 1$.
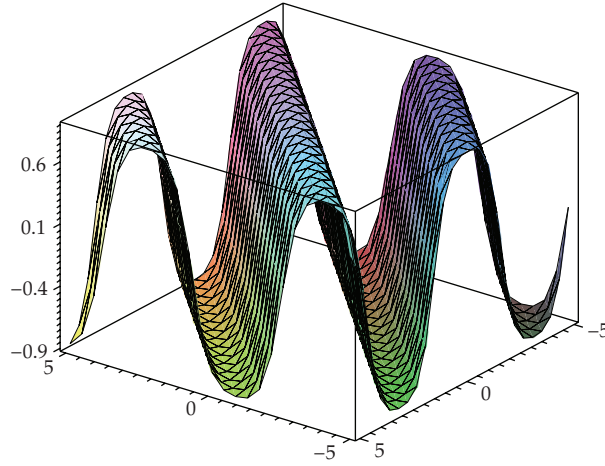
Figure 2

Case 2. If $p = c = 2$, and $q = d = 1$, then the trial solution, (2.6) reduces to

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0 + b_{-1} \exp[-\eta]}. \tag{2.18}$$

Proceeding as before, we obtain

$$b_2 = b_2, \quad b_1 = 0, \quad b_0 = 0, \quad a_1 = 0, \quad \omega = \omega, \quad b_{-1} = b_{-1}, \quad \rho = \frac{9kr\omega^2}{(9bk^2 - 2)},$$

$$a_0 = 0, \quad a_2 = a_2, \quad b_0 = 0, \quad a_{-1} = -\frac{b_{-1}a_2}{b_2}, \quad \alpha = \frac{9b_2{}^2 r\omega^2}{(9bk^2 - 2)a_2{}^2}. \tag{2.19}$$

Hence, we get the generalized solitary solutions $u(x, y, t)$ of (2.6) as follows:

$$u(x, y, t) = -1 + \frac{2b_{-1}}{b_1 e^{2(kx + \omega y - (9kr\omega^2 \, t)/(9bk^2 - 2))} + b_{-1}}, \tag{2.20}$$

where b_{-1}, b_1, ω, and k are real numbers.

Remark 2.2. It is worth mentioning that the transformation $k = ik$ which is used to transform the solitary solutions to periodic or compacton-like solutions was first proposed by He and Wu [39] and is called the He-Wu transformation. Moreover, the interpretation of this transformation is given by He [17].

3. Variational Iteration Method (VIM) and its Modifications

To illustrate the basic concept of the He's VIM, we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{3.1}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term. According to variational iteration method [1–7, 11, 17–24, 44, 49, 50, 60–64, 66–74, 77–80, 88, 90], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left(Lu_n(s) + N\tilde{u}_n(s) - g(s) \right) ds, \tag{3.2}$$

where λ is a Lagrange multiplier [17–24], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta \tilde{u}_n = 0$; (3.2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [17–24]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation $u_{n+1}, n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n \to \infty} u_n$. We summarize some useful iteration formulae [23, 24] which would be used in the subsequent section:

$$u' + f(u, u') = 0,$$

$$u_{n+1}(t) = u_n(t) - \int_0^t \left(u_n^I(s) + f\left(u_n, u_n^I\right) \right) ds.$$

$$u'' + f(u, u', u'') = 0,$$

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\left(u_n^{II}(s) + f\left(u_n, u_n^I, u_n^{II}\right) \right) ds.$$

$$u''' + f(u, u', u'', u''') = 0,$$

$$u_{n+1}(t) = u_n(t) - \int_0^t \frac{1}{2!}(s-t)^2 \left(u_n^{III}(s) + f\left(u_n, u_n^I, u_n^{II}, u_n^{III}\right) \right) ds.$$

$$u^{(iv)} + f\left(u, u', u'', u''', u^{(iv)}\right) = 0,$$

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{3!}(s-t)^3 \left(u_n^{(iv)}(s) + f\left(u_n, u_n^I, u_n^{II}, u_n^{III}, u_n^{(iv)}\right) \right) ds.$$

$$u^{(n)} + f\left(u, u', u'', u''', u^{(iv)}, \ldots, u^{(n)}\right) = 0,$$

$$u_{n+1}(t) = u_n(t) + (-1)^n \int_0^t \frac{1}{(n-1)!}(s-t)^{n-1}\left(u_n^{(n)}(s) + \left(u_n, u_n^I, u_n^{II}, u_n^{III}, u_n^{(iv)}, \ldots, u_n^{(n)}\right)\right)ds.$$

$$(3.3)$$

3.1. Variational Iteration Method Using He's Polynomials (VIMHP)

This modified version of variational iteration method [60, 66–69, 72, 73] is obtained by the elegant coupling of correction functional (2.7) of variational iteration method (VIM) with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p \int_0^x \lambda(s)\left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n)\right)ds - \int_0^x \lambda(s)g(s)ds. \qquad (3.4)$$

comparisons of like powers of p give solutions of various orders.

3.2. Variational Iteration Method Using Adomian's Polynomials (VIMAP)

This modified version of VIM is obtained by the coupling of correction functional (2.3) of VIM with Adomian's polynomials [1, 2, 62, 64, 70, 74] and is given by

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda\left(Lu_n(x) + \sum_{n=0}^{\infty} A_n - g(x)\right)dx, \qquad (3.5)$$

where A_n are the so-called Adomian's polynomials and are calculated for various classes of nonlinearities by using the specific algorithm developed in [81–83].

Example 3.1 (see [61]). Consider the following singularly perturbed sixth-order Boussinesq equation

$$u_{tt} = u_{xx} + (u^2)_{xx} - u_{xxxx} + \frac{1}{2}u_{xxxxxx}, \qquad (3.6)$$

with initial conditions

$$u(x,0) = -\frac{105}{169}\operatorname{sech}^4\left(\frac{x}{\sqrt{26}}\right), \quad u_t(x,0) = \frac{-210\sqrt{194/13}\operatorname{sech}^4\left(x/\sqrt{26}\right)\tanh\left(x/\sqrt{26}\right)}{2197}.$$

$$(3.7)$$

The exact solution of the problem is given as

$$u(x,t) = -\frac{105}{169}\,\text{sech}^4\left[\sqrt{\frac{1}{26}}\left(x - \sqrt{\frac{97}{169}}t\right)\right]. \tag{3.8}$$

The correction functional is given by

$$u_{n+1}(x,t) = -\frac{105}{169}\,\text{sech}^4\left(\frac{x}{\sqrt{26}}\right) + \frac{-210\sqrt{194/13}\,\text{sech}^4\left(x/\sqrt{26}\right)\tanh\left(x/\sqrt{26}\right)}{2197}t$$

$$+ \int_0^t \lambda(s)\left(\frac{\partial^2 u_n}{\partial t^2} - \left((\tilde{u}_n)_{xx} + (\tilde{u}_n^2)_{xx} - (\tilde{u}_n)_{xxxx} + \frac{1}{2}(\tilde{u}_n)_{xxxxxx}\right)\right)ds. \tag{3.9}$$

Making the correction functional stationary, the Lagrange multiplier can easily be identified as $\lambda(s) = (s - x)$, we get the following iterative formula

$$u_{n+1}(x,t) = -\frac{105}{169}\,\text{sech}^4\left(\frac{x}{\sqrt{26}}\right) + \frac{-210\sqrt{194/13}\,\text{sech}^4\left(x/\sqrt{26}\right)\tanh\left(x/\sqrt{26}\right)}{2197}t$$

$$+ \int_0^t (s - x)\left(\frac{\partial^2 u_n}{\partial t^2} - \left((u_n)_{xx} + (u_n^2)_{xx} - (u_n)_{xxxx} + \frac{1}{2}(u_n)_{xxxxxx}\right)\right)ds. \tag{3.10}$$

Consequently, following approximants are obtained

$$u_0(x,t) = -\frac{105}{169}\,\text{sech}^4\left(\frac{x}{\sqrt{26}}\right),$$

$$u_1(x,t) = -\frac{105}{169}\,\text{sech}^4\left(\frac{x}{\sqrt{26}}\right) - \frac{105\sqrt{194/13}\,\text{sech}^6\left(x/\sqrt{26}\right)\sinh\left(\sqrt{2}\,x/\sqrt{13}\right)}{2197}t$$

$$- \frac{105}{371293}\left(-291 + 194\cosh\left(\frac{\sqrt{2}\,x}{\sqrt{13}}\right)\right)\text{sech}^6\frac{x}{\sqrt{26}}t^2, \tag{3.11}$$

$$\vdots$$

Table 1: Error estimates.

x_i	t_j					
	0.01	0.02	0.04	0.1	0.2	0.5
−1	7.77156 E-16	1.36557 E-14	8.57869 E-13	2.09264 E-10	1.33823 E-8	3.25944 E-6
−0.8	1.11022 E-16	1.99840 E-15	1.12688 E-13	2.73880 E-11	1.74288 E-9	4.14094 E-7
−0.6	2.22045 E-16	1.09912 E-14	7.28861 E-13	1.78030 E-10	1.14025 E-8	2.79028 E-6
−0.4	1.11022 E-16	2.32037 E-14	1.50302 E-12	3.67002 E-10	2.34944 E-8	5.74091 E-6
−0.2	6.66134 E-16	3.23075 E-14	2.04747 E-12	4.99918 E-10	3.19983 E-9	7.81509 E-6
0	4.44089 E-16	3.49720 E-14	2.24365 E-12	5.47741 E-10	3.50559 E-8	8.55935 E-6
0.2	5.55112 E-16	3.19744 E-14	2.04714 E-12	4.99820 E-10	3.19858 E-8	7.80749 E-6
0.4	3.33067 E-16	2.32037 E-14	1.50324 E-12	3.66815 E-10	2.34706 E-8	5.72641 E-6
0.6	3.33067 E-16	1.12133 E-14	7.28528 E-12	1.77772 E-10	1.13695 E-8	2.77022 E-6
0.8	3.33067 E-16	1.99840 E-15	1.13132 E-13	2.76944 E-11	1.78208 E-9	4.41936 E-7
1	7.77156 E-16	1.38778 E-14	8.58313 E-13	2.09593 E-10	1.34244 E-8	3.28504 E-6

The series solution is given by

$$
u(x,t) = -\frac{105}{169}\operatorname{sech}^4\left(\frac{x}{\sqrt{26}}\right) - \frac{105\sqrt{194/13}\operatorname{sech}^6\left(x/\sqrt{26}\right)\sinh\left(\sqrt{2}\,x/\sqrt{13}\right)}{2197}t
$$

$$
- \frac{105}{371293}\left(-291 + 194\cosh\left(\frac{\sqrt{2}\,x}{\sqrt{13}}\right)\right)\operatorname{sech}^6\frac{x}{\sqrt{26}}t^2
$$

$$
+ \frac{395\operatorname{sech}^7\dfrac{x}{\sqrt{26}}}{52206766144}\left(10816\sqrt{2522}\sinh\frac{x}{\sqrt{26}} - 1664\sqrt{2522}\sinh\frac{3x}{\sqrt{26}}\right)t^3
$$

$$
+ \left(-334200\operatorname{sech}^5\left(\frac{x}{\sqrt{26}}\right) + 354247\cosh\left(\frac{2}{\sqrt{13}}x\right)\operatorname{sech}^5\left(\frac{x}{\sqrt{26}}\right)\right.
$$

$$
\left. -47164\cosh\left(\frac{2\sqrt{2}}{\sqrt{13}}x\right)\operatorname{sech}^5\left(\frac{x}{\sqrt{26}}\right)\right)t^4
$$

$$
+ \left(3201\cosh^3\left(\frac{3\sqrt{2}}{\sqrt{13}}x\right)\operatorname{sech}^5\left(\frac{x}{\sqrt{26}}\right) - 388\cosh\left(\frac{4\sqrt{2}}{\sqrt{13}}x\right)\operatorname{sech}^5\left(\frac{x}{\sqrt{26}}\right)\right)t^4
$$

$$
+ \cdots,
$$

$$(3.12)$$

Table 1 exhibits the absolute error between the exact and the series solutions. Higher accuracy can be obtained by introducing some more components of the series solution.

Example 3.2 (see [72]). Consider the following Helmholtz equation:

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} - u(x,y) = 0,
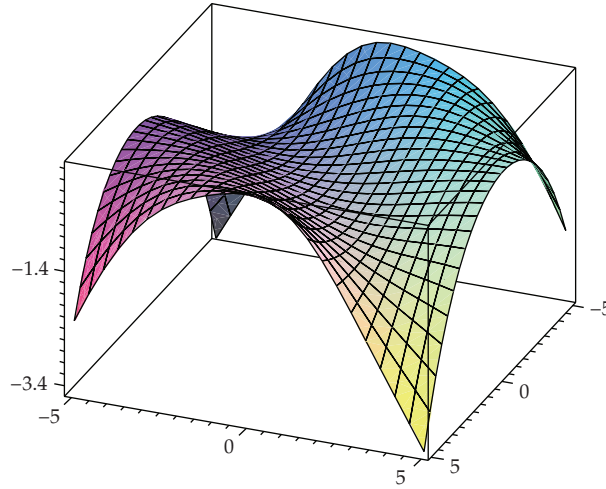$$

$$(3.13)$$

Figure 3

with initial conditions

$$u(0, y) = y, \qquad u_x(0, y) = y + \cosh y \; . \tag{3.14}$$

The correction functional is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 \tilde{u}_n(x, y)}{\partial y^2} - \tilde{u}_n(x, y) \right) ds. \tag{3.15}$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = (s - x)$, we obtained

$$y_{n+1}(x) = y_n(x) + \int_0^x (s - x) \left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n(x, y)}{\partial y^2} - u_n(x, y) \right) ds. \tag{3.16}$$

Applying the variational iteration method using He's polynomials (VIMHP), we get

$$y_0 + p y_1 + p^2 y_2 + \cdots = y_n(x) + p \int_0^x (s - x) \left(\frac{\partial^2 u_0}{\partial x^2} + p \frac{\partial^2 u_1}{\partial x^2} + p^2 \frac{\partial^2 u_2}{\partial x^2} + \cdots \right) ds$$

$$+ \; p \int_0^x (s - x) \left(\left(\frac{\partial^2 u_0}{\partial y^2} + p \frac{\partial^2 u_1}{\partial y^2} + p^2 \frac{\partial^2 u_2}{\partial y^2} + \cdots \right) \right. \tag{3.17}$$

$$\left. - \left(u_0 + p u_1 + p^2 u_2 + \cdots \right) \right) ds.$$

Table 2: Error estimates.

X	Y	Exact solution	HPM	ADM	VIMHP	Absolute error
−1	−1	−1.9109600760	−1.9097472990	−1.9097472990	−1.9097472990	0.0012127770
−0.8	−0.8	−1.4294111280	−1.491500900	−1.491500900	−1.491500900	0.0002610380
−0.6	−0.6	−1.0405661130	−1.0405303310	−1.0405303310	−1.0405303310	0.0000357820
−0.4	−0.4	−.7005569670	−0.7005548150	−0.7005548150	−0.7005548150	0.0000021520
−0.2	−0.2	−.367755010	−0.3677594840	−0.3677594840	−0.3677594840	0.0000000170
0.0	−0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.2	0.2	0.4482939020	0.4482938840	0.4482938840	0.4482938840	0.0000000180
0.4	0.4	1.0291588280	1.0291564150	1.0291564150	1.0291564150	0.0000024130
0.6	0.6	1.8045504110	1.8045079310	1.8045079310	1.8045079310	0.0000424800
0.8	0.8	2.850380700	2.8500524900	2.8500524900	2.8500524900	0.0003282100
1.0	1.0	4.2613624630	4.2597472990	4.2597472990	4.2597472990	0.0016151640

Comparing the coefficient of like powers of p, following approximants are obtained:

$$p^{(0)} : u_0(x, y) = y(1 + x) + x \cosh y,$$

$$p^{(1)} : u_1(x, y) = y(1 + x) + x \cosh y + \frac{1}{2!}x^2 y + \frac{1}{3!}x^3 y,$$

$$p^{(2)} : u_2(x, y) = y(1 + x) + x \cosh y + \frac{1}{2!}x^2 y + \frac{1}{3!}x^3 y + \frac{1}{4!}x^4 y + \frac{1}{5!}x^5 y,$$

$$\vdots$$

(3.18)

The solution is given as

$$u(x, y) = y \exp(x) + x \cosh(y).$$

(3.19)

Table 2 exhibits the approximate solution obtained by using the HPM, ADM, and VIMHP. It is clear that the obtained results are in high agreement with the exact solutions. Higher accuracy can be obtained by using more terms.

Example 3.3 (see [62]). Consider the following nonlinear Schrödinger equation

$$iu_t + u_{xx} - 2 u |u|^2 = 0,$$

(3.20)

with initial conditions

$$u(x, 0) = e^{ix}.$$

(3.21)

The correction functional is given as

$$u_{n+1}(x, t) = e^{ix} + \int_0^t \lambda(s) \left(\frac{\partial u_n}{\partial s} - i\left((\tilde{u}_n)_{xx} - 2 \tilde{u}_n |\tilde{u}_n|^2 \right) \right) ds.$$
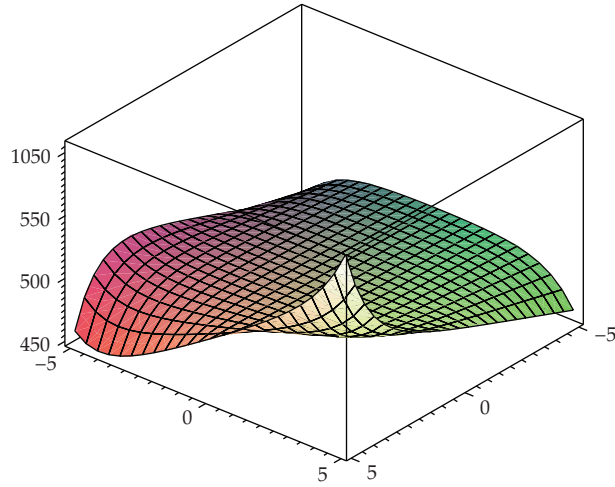
(3.22)

Figure 4: Depicts series solutions $u(x, y)$.

Making the correction functional stationary, the Lagrange multipliers can be identified as $\lambda(s) = -1$, consequently

$$u_{n+1}(x, t) = e^{ix} - \int_0^t \left(\frac{\partial u_n}{\partial s} - i\left((u_n)_{xx} - 2\, u_n |u_n|^2 \right) \right) ds. \tag{3.23}$$

Applying the variational iteration method using Adomian's polynomials (VIMAP):

$$u_{n+1}(x, t) = e^{ix} - \int_0^t \left(\frac{\partial u_n}{\partial s} - i\left((u_n)_{xx} - 2 \sum_{n=0}^{\infty} A_n \right) \right) ds, \tag{3.24}$$

where A_n are the so-called Adomian's polynomials. First few Adomian's polynomials for nonlinear Schrödinger equation are as follows:

$$A_0 = u_o^2\, \bar{u}_0,$$

$$A_1 = 2u_0 u_1\, \bar{u}_0 + u_o^2\, \bar{u}_1,$$

$$A_2 = 2\, u_0 u_2\, \bar{u}_0 + u_1^2\, \bar{u}_0 + 2u_0 u_1\, \bar{u}_2 + u_o^2 \bar{u}_2,$$

$$A_3 = 2u_0 u_3\, \bar{u}_0 + u_1^2\, \bar{u}_1 + 2u_1 u_2\, \bar{u}_0 + u_o^2 \bar{u}_3 + 2\, u_0 u_2\, \bar{u}_1 + 2\, u_0 u_1\, \bar{u}_2, \tag{3.25}$$

$$\vdots$$

Employing these polynomials in the above iterative scheme, following approximants are obtained:

$$u_0(x,t) = e^{ix},$$

$$u_1(x,t) = e^{ix}\,(1 - 3it),$$

$$u_2(x,t) = e^{ix}\,\left(1 - 3it - \frac{9}{2!}t^2\right), \psi$$

$$u_3(x,t) = e^{ix}\,\left(1 - 3it - \frac{9}{2!}t^2 + \frac{9}{2!}it^3\right), \psi$$

(3.26)

$$\vdots$$

The solution in a series form is given by

$$u(x,t) = e^{ix}\,\left(1 - 3it + \frac{(3it)^2}{2!}t^2 - \frac{(3it)^3}{3!}t^3 + \frac{(3it)^4}{4!}t^4 - \cdots\right),$$

(3.27)

and in a closed form by

$$u(x,t) = e^{i\,(x-3\,t)}.$$

(3.28)

Remark 3.4. It is worth mentioningthat although both the modified versions of variational iteration method (VIM) are compatible yet the modification based upon He's polynomials (VIMHP) is much easier to implement and is more user friendly as compared to VIMAP where Adomian's polynomials along with their complexities are used.

4. Homotopy Perturbation Method (HPM) and He's Polynomials

To explain the He's homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0,$$

(4.1)

where L is any integral or differential operator. We define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1 - p)F(u) + pL(u),$$

(4.2)

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u,p) = 0,$$

(4.3)

we have

$$H(u, 0) = F(u), \qquad H(u, 1) = L(u). \tag{4.4}$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(f, 1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [13, 14, 17, 18, 26–38, 51, 52, 55–57, 60, 65–69, 72, 73, 89, 91–93, 96]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [17, 18, 26–38] to obtain

$$u = \sum_{i=0}^{\infty} p^i \, u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{4.5}$$

if $p \to 1$, then (4.5) corresponds to (4.2) and becomes the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{4.6}$$

It is well known that series (4.5) is convergent for most of the cases and also the rate of convergence is dependent on L (u); see [17, 18, 26–38]. We assume that (4.6) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to [13, 14], He's HPM considers the nonlinear term $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} p^i \, H_i = H_0 + p \, H_1 + p^2 H_2 + \cdots, \tag{4.7}$$

where H_n's are the so-called He's polynomials [13, 14], which can be calculated by using the formula

$$H_n(u_0, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left(N\left(\sum_{i=0}^{n} p^i u_i \right) \right)_{p=0}, \quad n = 0, 1, 2, \dots \tag{4.8}$$

Example 4.1 (see [51]). Consider the following seventh-order generalized KdV (SOG-KdV) equation

$$u_t + u \, u_x + u_{xxx} - u_{xxxxx} + \sigma \, u_{xxxxxxx} = 0, \tag{4.9}$$

where $\sigma = \beta \delta / \gamma^2$, with initial conditions

$$u(x, 0) = a_0 + a_6 \mathrm{sech}^6(kx), \tag{4.10}$$

where $K = 5/\sqrt{1538}$, $a_0 = c - 18000/769^2$, $a_6 = 519750/769^2$, and c is an arbitrary parameter. Applying the convex homotopy, we get

$$
u_0 + pu_1 + p^2 u_2 + \cdots = u_0 - p \int_0^t \left(\left(u_0 + pu_1 + p^2 u_2 + \cdots \right) \left(\frac{\partial u_0}{\partial x} + p \frac{\partial u_1}{\partial x} + p^2 \frac{\partial u_2}{\partial x} + \cdots \right) \right) ds
$$

$$
- p \int_0^t \left(\left(\frac{\partial^3 u_0}{\partial x^3} + p \frac{\partial^3 u_1}{\partial x^3} + \cdots \right) - \left(\frac{\partial^5 u_0}{\partial x^5} + p \frac{\partial^5 u_1}{\partial x^5} + \cdots \right) \right.
$$

$$
\left. + \sigma \left(\frac{\partial^7 u_0}{\partial x^7} + p \frac{\partial^7 u_1}{\partial x^7} + \cdots \right) \right) ds.
$$

(4.11)

Comparing the coefficient of like powers of p

$$
p^{(0)} : u_0(x, t) = a_0 + a_6 \operatorname{sech}^6(kx),
$$

$$
p^{(1)} : u_1(x, t) = \frac{3a_6 kt}{16} \left\{ 10a_0 + 32a_6 - 312k^2 - 26400k^4 - 9866112\delta k^6 \right.
$$

$$
+ \left(15a_0 - 256k^2 - 10480k^4 + 9932224\delta k^6 \right) \cosh(2kx)
$$

$$
+ \left(6a_0 - 8k^2 + 14624k^4 - 1443968\delta k^6 \right) \cosh(4kx)
$$

$$
+ \left(a_0 + 36k^2 - 1296k^4 + 46656\delta k^6 \right) \cosh(6kx) \right\} \operatorname{sech}^{12}(kx) \tanh(kx),
$$

$$
\vdots
$$

(4.12)

where $p^i s$ are the He's polynomials. The series solution is given by

$$
u(x, t) = a_0 + a_6 \operatorname{sech}^6(kx) + \frac{3a_6 kt}{16} \left\{ 10a_0 + 32a_6 - 312k^2 - 26400k^4 - 9866112\delta k^6 \right.
$$

$$
+ \left(15a_0 - 256k^2 - 10480k^4 + 9932224\delta k^6 \right) \cosh(2kx)
$$

$$
+ \left(6a_0 - 8k^2 + 14624k^4 - 1443968\delta k^6 \right) \cosh(4kx)
$$

$$
+ \left(a_0 + 36k^2 - 1296k^4 + 46656\delta k^6 \right) \cosh(6kx) \right\} \operatorname{sech}^{12}(kx) \tanh(kx)
$$

$$
- \frac{3a_6 kt^2}{4096} \left[1320a_0^2 + 6400a_0 a_6 + 40960a_6^2 - 93120a_0 k^2 - 642560a_6 k^2 \right.
$$

$$
+ 3592320k^4 - 718464a_0 k^4 - 223897600a_6 k^4 + 1066859520k^6
$$

$$
- 1066859520a_0 \delta k^6 - 371744399360a_6 \delta k^6 + 151760209920k^8
$$

$$
+ 303520419840\delta k^{10} + 186385174609920\delta^2 k^{12}
$$

$$+ \left(2079a_0^2 + 6912a_0a_6 - 36864a_6^2 - 124488a_0k^2 - 291840a_6k^2 \right.$$

$$+ 3662064k^4 - 7324128a_0k^4 + 83202048a_6k^4 + 643886208k^6 - 643886208a_0\delta k^6$$

$$+ 520393113600a_6\delta k^6 + 10077306624k^8 + 20154613248\delta k^8 - 1154272125572672\delta k^{10}$$

$$\left. -3181670012646236166\delta^2 k^{12}\right) \cosh(2kx) + \left(924a_0^2 - 1536a_0a_6 - 14112a_0k^2 \right.$$

$$+ 592896a_6k^2 - 1544256k^4 + 3088512a_0k^4 + 208736256a_6k^4 - 1003580928k^6$$

$$+ 1003580928a_0\delta k^6 - 173675003904a_6\delta k^6 - 180653147136k^8$$

$$- 361306294272\delta k^8 - 1590450116935688\delta k^{10}$$

$$\left. +12914876783576678\delta^2 k^{12}\right) \cosh(4kx) + \left(77a_0^2 - 2816a_0a_6 + 36904a_0k^2 \right.$$

$$- 2283568k^4 + 4567136a_0\delta k^6 + 4399738112k^8 + 87994762248\delta k^8$$

$$\left. +11465095777376\delta k^{10} - 276184346637230008\delta^2 k^{12}\right) \cosh(6kx) - \left(168a_0^2 \right.$$

$$+ 768a_0a_6 - 25536a_0k^2 + 69120a_6k^2 + 594048k^4 - 1188096a_0k^4 - 8460288a_6k^4$$

$$- 118291504k^6 + 118293504a_0\delta k^6 + 1164533760a_6\delta k^6 - 35208886272k^8$$

$$\left. -704177725448\delta k^8 + 258600494530568\delta k^{10} - 29019893241937928\delta^2 k^{12}\right) \cosh(8kx)$$

$$- \left(105a_0^2 - 5880a_0k^2 - 132720k^4 - 67643520k^6 + 67643520a_0\delta k^6 \right.$$

$$+ 7719962880k^8 + 154399257608\delta k^8 - 21945419059208\delta k^{10}$$

$$\left. +13226030082048088\delta^2 k^{12}\right) \cosh(10kx) - \left(28a_0^2 + 224a_0k^2 - 53312k^4 \right.$$

$$+ 106626a_0k^4 + 11178496k^6 - 11178496a_0\delta k^6 - 436093852k^8 - 8721879048\delta k^8$$

$$\left. +614635356168\delta k^{10} - 206841631539268\delta^2 k^{12}\right) \cosh(12kx) - \left(3a_0^2 + 216a_0k^2 \right.$$

$$+ 3888k^4 - 7776a_0k^4 - 279936k^6 + 279936a\delta k^6 + 5038848k^8 + 100776968\delta k^8$$

$$\left.\left. -3627970568\delta k^{10} + 65303470088\delta^2 k^{12}\right) \cosh(14kx)\right] \mathrm{sech}^{20}(kx) + \cdots .$$

$$(4.13)$$

The closed form solution is given as

$$u(x,t) = a_0 + a_6 \mathrm{sech}^6(k(x - ct)), \tag{4.14}$$

where $k = 5/\sqrt{1538}$, $a_0 = c - 18000/769^2$, $a_6 = 519750/769^2$, and c is an arbitrary parameter.

Table 3: Error Estimates (δ = .01).

x		Error t_j		
	j = .001	j = .01	j = .05	j = .1
−1.0	3.1×10^{-5}	4.6×10^{-4}	5.6×10^{-3}	1.9×10^{-2}
−.8	2.6×10^{-5}	4.0×10^{-4}	5.2×10^{-3}	1.8×10^{-2}
−.6	2.0×10^{-5}	3.4×10^{-4}	4.7×10^{-3}	1.7×10^{-2}
−.4	1.4×10^{-5}	2.7×10^{-4}	4.3×10^{-3}	1.5×10^{-2}
−.2	7.8×10^{-6}	2.0×10^{-4}	3.9×10^{-3}	1.4×10^{-2}
0.0	1.4×10^{-6}	1.4×10^{-4}	3.5×10^{-3}	1.4×10^{-2}
.2	5.0×10^{-6}	7.8×10^{-5}	3.2×10^{-3}	1.3×10^{-2}
.4	1.1×10^{-5}	1.8×10^{-5}	3.0×10^{-3}	1.3×10^{-2}
.6	1.7×10^{-5}	3.6×10^{-5}	2.8×10^{-3}	1.3×10^{-2}
.8	2.3×10^{-5}	8.7×10^{-5}	2.7×10^{-3}	1.3×10^{-2}
1.0	2.8×10^{-5}	1.3×10^{-4}	2.6×10^{-3}	1.3×10^{-2}
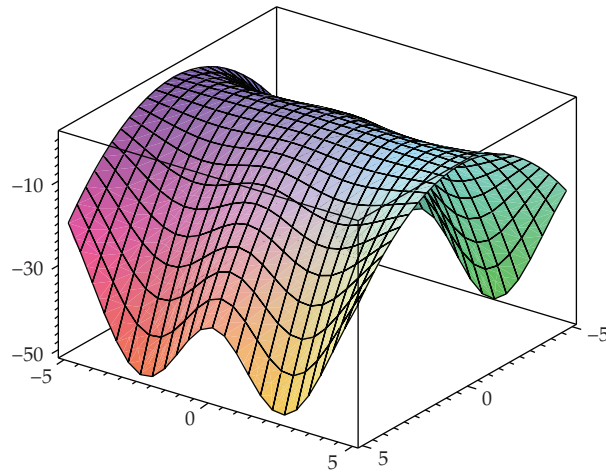
Figure 5: Depicts series solution at δ = .01.

5. Parameter Expansion Method

Consider the following Duffing harmonic oscillator

$$u_{tt} = -\frac{u^3}{1 + u^2}, \qquad u(0) = A, \qquad u_t(0) = 0. \tag{5.1}$$

We rewrite (5.1) in the form

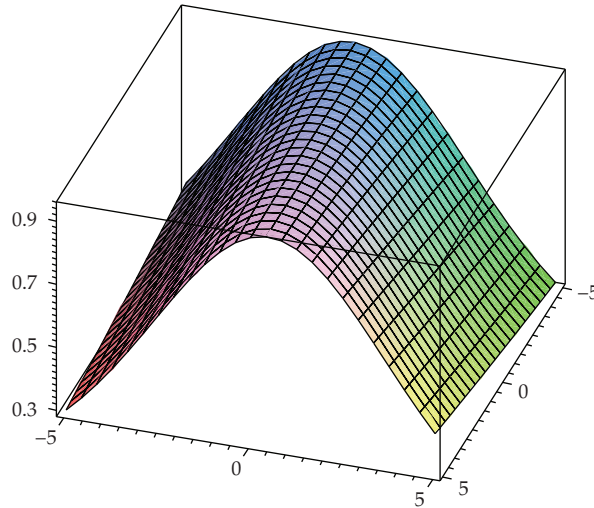$$u_{tt} + o.u + 1.u^2 u_{tt} + 1.u^3 = 0. \tag{5.2}$$

Figure 6: Depicts closed form solution at $\delta = .01$.

Assuming that the solution of (5.1) and coefficient 0, 1 can be expressed as power series in p as follows:

$$0 = \omega^2 + pa_1 + p^2 a_1 + \cdots, \tag{5.3}$$

$$1 = pb_1 + p^2 b_2 + \cdots. \tag{5.4}$$

Consequently, we have

$$u_0'' + \omega^2 u_0 = 0, \qquad u(0) = A, \qquad u_t(0) = 0, \tag{5.5}$$

$$u_1'' + \omega^2 u_1 + a_1 u_0 + b_1 u_0^2 u_0'' + b_1 u_0^3 = 0, \qquad u(0) = 0, \qquad u_t(0) = 0. \tag{5.6}$$

Solving (5.5), we have $u_0 = A\cos \omega t$, substituting u_0 in (5.6) gives

$$u_1'' + \omega^2 u_1 + A\cos \omega t \left(a_1 + \frac{3}{4} b_1 A^2 \left(1 - \omega^2\right) \right) + \frac{1}{4} b_1 A^3 \left(1 - \omega^2\right) \cos 3\omega t = 0. \tag{5.7}$$

Elimination of the secular term requires $a_1 = -(3/4)\, b_1 A^2 (1 - \omega^2)$. If only the first-order approximation is searched for, then $a_1 = -\omega^2 b_1$, is obtained from (5.3) and (5.4) which lead to $\omega^2 = 3A^2/(4 + 3A^2)$.

6. Conclusion and Future Research

In this paper, we made a detailed study of some relatively new techniques along with some of their modifications. In particular, we focused on He's VIM, HPM, MVIMS, exp-function, and expansion of parameters methods and discussed in length their respective

applications in solving various diversified initial and boundary value problems. These proposed methods and their modifications are employed without using linearization, discretization, transformation, or restrictive assumptions, absorb the positive features of the coupled techniques and hence are very much compatible with the diversified and versatile nature of the physical problems. Moreover, the modification of VIM based upon He's polynomials (VIMHP) is easier to implement and is more user friendly as compared to the one where Adomian's polynomials (VIMAP) along with their complexities are used. It is also observed that the coupling of He's or Adomian's polynomials with the correction functional of VIM makes the solution procedure simpler and hence the evaluation of nonlinear term becomes easier. It may be concluded that the relatively new techniques can be treated as alternatives for solving a wide class of nonlinear problems.

We would like to mention that the techniques and ideas presented in this paper can be extended for finding the analytic solution of the obstacle, unilateral, free, moving, and contacts problems which arises in various branches of mathematical, physical, regional, medical, structure analysis, and engineering sciences. These problems can be studied in the general, natural, and unified framework of variational inequalities. In a variational inequality framework of such problems, the location of the contact area (free or moving boundary) becomes an integral part of the solution and no special techniques are needed to obtain it. It is well-known that if the obstacle is known then the variational inequalities can be characterized by system of variational equations. Momani et al. [101] have used Admonian decomposition technique to solve the system of fourth-order obstacle boundary value problems. This area of research is not yet developed and offers a wealth of new opportunities for further research. It is our hope that this brief introduction may inspire and motive the readers to discover new, innovative, and novel applications of these new techniques, which we have discussed in this paper. For the applications, physical formulation, numerical methods and other applications of the variational inequalities, see, Noor [102–109], Noor et al [110] and the references therein.

Acknowledgments

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