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Research Article

Acceleration of Material Waves in Fermi Accelerator

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Cold Atoms bouncing on modulated atomic mirror exhibits acceleration and dynamical localization subject to modulated strength. We explain characteristics of acceleration and define control parameters in terms of effective Planck's constant. We show that the effective Planck's constant plays a vital role in limiting classical-like overall linear growth of the variance of accelerated atoms with time. For large values of the effective Planck's constant the atomic quantum acceleration is seized as localization window overlaps the accelerated window.

1. Introduction

Fermi Accelerator is a system well investigated to study Hamiltonian chaos and its manifestations in quantum mechanical systems [1, 2]. In 1961, Ulam suggested the accelerator as a two-wall system, one fixed and the other oscillating with a frequency, named as Fermi-Ulam Accelerator. At the end of the seventies, Pustylńikov replaced fixed wall by gravity and discussed unbounded acceleration for it. The classical dynamics in Fermi accelerator is described by the Standard Map which explains that the stochasticity in the phase space increases with the driving strength, and when the latter is sufficiently strong global diffusion takes place. In latter accelerator model, for particular set of initial data in phase space and modulation strength [3], onset of unbounded acceleration modes takes place [4–7]. In this paper we explain Fermi-Pustylńikov accelerator and show that by increasing effective Planck's constant the acceleration of a bouncing particle in the accelerator system is modified, due to the onset of dynamical localization [8].

A classical system subject to time-periodic modulation, in general, becomes globally chaotic for increasing modulation strengths and absorbs energy from the external field in a diffusive way. However, in corresponding quantum domain the diffusive dynamics may be suppressed by quantum interference effect. It is a manifestation of dynamical localization phenomenon in the system, which is an analogous to Anderson localization of solid state

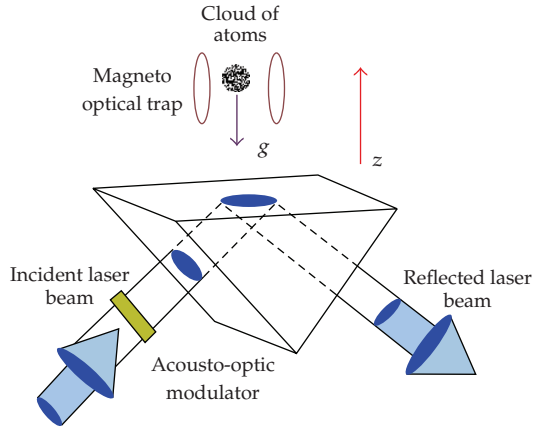

Figure 1: A cloud of atoms is trapped and cooled in a magneto-optical trap upto a few micro-Kelvin. The magneto-optical trap (MOT) is placed at a height at the start of the experiment. On switching off the MOT, atoms move with constant gravitational acceleration towards the exponential decaying field.

physics. The phenomenon has been discussed in model systems in quantum chaos, such as, kicked rotator [9], modulated quantum bouncer [10], atoms in modulated standing wave fields [11], ion in a Paul trap [12, 13], and molecular systems in the presence of electric and magnetic fields [14]. Dynamical localization is a general phenomenon in periodically driven systems [15, 16]. The *delocalization* in such quantum systems is a purely quantum effect since the long-time unbounded propagation is not related to the corresponding classical diffusion [2, 17].

In Section 2, we describe the physical system and develop its Hamiltonian. In Section 3, we discuss the diffusion versus acceleration in Fermi accelerator with the help of acceleration windows. In Section 4, we discuss quantum control on accelerator dynamics, discuss results, and explain numerical data based on analytical calculations in Section 5.

2. The Experimental System

Thirty years after the first suggestion of Fermi, Pustylńikov provided detailed study of the accelerator model, which we call Fermi Accelerator or modulated quantum bouncer [4, 18] in this paper. In his work, Pustylńikov proved that a particle bouncing in the accelerator system attains modes, where it always gets unbounded acceleration. The feature makes the Fermi-Pustylńikov model richer in its dynamical beauty. In the atomic Fermi accelerator, atoms move under the influence of gravitational field towards an atomic mirror made up of an evanescent wave field. The atomic mirror is provided a spatial modulation by means of an acousto-optic modulator which provides intensity modulation to the incident laser light field [10]. Hence, an ultra-cold two-level atom, after a normal incidence with the modulated atomic mirror, bounces off and travels in the gravitational field, as shown in Figure 1. In order to avoid any atomic momentum along the plane of the mirror the laser light which undergoes total internal reflection is reflected back. Therefore, we find a standing wave in the plane of the mirror which avoids any specular reflection [19]. The periodic modulation in the intensity of the evanescent wave optical field may lead to the spatial modulation of the atomic mirror as

$$\overline{I}\left(\overline{z},\overline{t}\right) = I_0 e^{-2\kappa\overline{z}+a\cos(\omega\overline{t})}. \tag{2.1}$$

Thus, the motion of the atom in $\overline{z}$-direction follows effectively the Hamiltonian

$$\overline{H} = \frac{\overline{p}_z^{\,2}}{2m} + mg\overline{z} + \hbar\Omega_{\text{eff}} e^{-2\kappa\overline{z}+a\cos(\omega\overline{t})}, \tag{2.2}$$

where Ω_{eff} denotes the effective Rabi frequency [5]. Moreover, a and ω express the amplitude and the frequency of the external modulation, respectively.

3. Diffusion versus Acceleration in Fermi Accelerator

In case the decay constant κ of the evanescent wave field is large, simplified Hamiltonian of our system in moving coordinates becomes

$$\overline{H} = \frac{\overline{p}_z^{\,2}}{2m} + mg\widetilde{z} + V\widetilde{z}\cos\omega\overline{t}, \quad \widetilde{z} \geq 0, \tag{3.1}$$

where $\widetilde{z} = \overline{z} - (\epsilon/2\kappa)\cos\omega\overline{t}$. The Hamiltonian given in (3.1) describes a particle of mass m bouncing off an oscillating hard surface in the presence of gravitational field. We proceed onward by introducing the dimensionless position and momentum coordinates. Here, we define $H = \overline{H}(\omega^2/mg^2)$ as dimensionless Hamiltonian, $V = \hbar\omega^2\Omega/4mg^2$ potential of the external field and then other parameters for this Hamiltonian system as, $\epsilon = \omega^2 a/2\kappa g$, $t = \omega\overline{t}$, and dimensionless Planck's constant, $\hbar = \hbar(\omega^3/mg^2)$. Hence, the Hamiltonian takes the dimensionless form as

$$H(z,p,t) = \frac{p^2}{2} + z + \epsilon z\cos t, \quad z \geq 0. \tag{3.2}$$

The Hamiltonian system is integrable in the absence of time dependent term. We may express the time development of the particle moving in time dependent system by the impact map which gives the evolution immediately after a bounce in terms of immediately after the previous bounce [1, 2], that is,

$$\begin{aligned} \wp_{i+1} &= \wp_i + K\sin(\phi_i), \\ \phi_{i+1} &= \phi_i + \wp_{i+1}. \end{aligned} \tag{3.3}$$

The map obtained in (3.3) is the Standard Map, where $\wp_i = 2p_i$ and $\phi_i = \omega t_i$. The onset of diffusive excitation in the system takes place as the chaos parameter $K = 4\epsilon$ takes a value larger than $K_{cr} \approx 0.96$, or when the perturbation amplitude exceeds the critical value $\epsilon_{cr} = 0.24$ [20], while the quantum evolution remains localized until a larger value λ_u of the modulation. Above that point both the classical and the quantum dynamics are diffusive. However, for specific set of initial conditions that originate within phase space disks of radius ρ, accelerating modes appear for values of the modulation strength ϵ within the windows [4–6]

$$s\pi\left(\frac{1-\xi}{1+\xi}\right) \leq \epsilon < \sqrt{1 + (s\pi)^2\left(\frac{1-\xi}{1+\xi}\right)^2}, \tag{3.4}$$

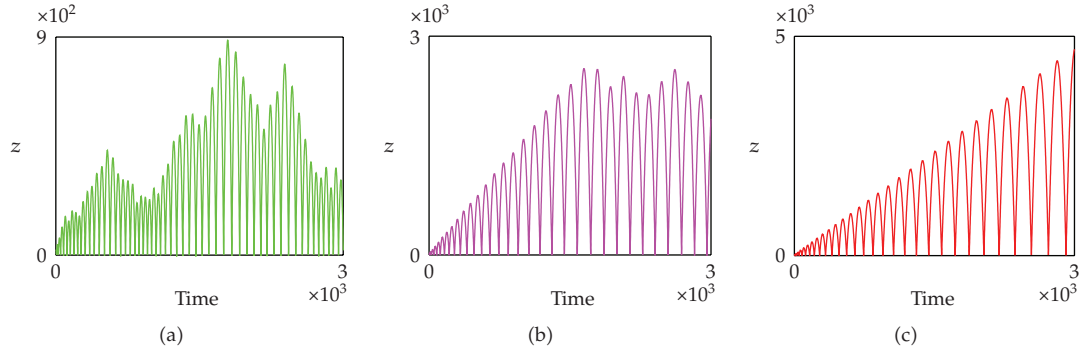

Figure 2: We plot the trajectories of a bouncing particle in time domain for three different phase points (z, p), which represent stochastic near acceleration and acceleration regions. For these three phase points we take $V = 60$ and $\epsilon = 1.7$. (a) Our particle is in a chaotic region with phase point $(23.239, 0)$, so there is no consistent acceleration of the particle. (b) Particle originates from the phase point $(21.898, 0)$ is on the boundary of the accelerated region of phase space. Initially the position of the particle increase gradually but after some time it decreases as we move in time domain. (c) Classical particle is in accelerated region of phase space with phase point $(19.739, 0)$. The position of the particle increases consistently as we move in time domain.

where, s can take integer and half-integer values for the sinusoidal modulation of the reflecting surface considered here. As the fundamental requirement for the acceleration is met by choosing a modulation strength within the acceleration windows and the bouncing particle from supporting areas of phase the particle displays acceleration discussed in Figure 2. We found numerically that for a modulation strength outside the windows of (3.4) the dynamics is dominantly diffusive as shown in Figure 3(a). However, for the ensemble originating from areas of phase space of radius ρ, exhibit unbounded acceleration for modulation strength from acceleration windows, as shown in Figure 3(b). A small diffusive background results from a small part of the initial distribution which is residing outside the area of phase space supporting acceleration. This coherent acceleration restricts the momentum space variance Δp which then remains very small indicating the absence of diffusive dynamics.

4. Quantum Control on Acceleration Dynamics

As a matter of fact, the variables $\wp$ and ϕ in (3.3) do not form a conjugate pair in the full Hamiltonian formulation of the model. The variable conjugate to the phase ϕ is the quantity $N = E$, where E is unperturbed energy [16] and its value can be determined as $E = (3\pi I)^{2/3}/2$. By introducing $\wp_{i+1} = \sqrt{2(N_{i+1} - z)}$ and $\wp_i = \sqrt{2(N_i - z)}$ in (3.3), the impact map in conjugate variables (N, ϕ) takes the form

$$\begin{aligned} N_{i+1} &= N_i + 2\epsilon\sqrt{2N_i}\sin\phi_i, \\ \phi_{i+1} &= \phi_i + 2\sqrt{2N_{i+1}} + O(\epsilon), \end{aligned} \tag{4.1}$$

here we neglected higher-order terms. The preservation of phase space volume for Hamiltonian systems has a consequence that there are no attractors, that is, no subregions of lower phase-space dimension to which the motion is confined asymptotically. The map given

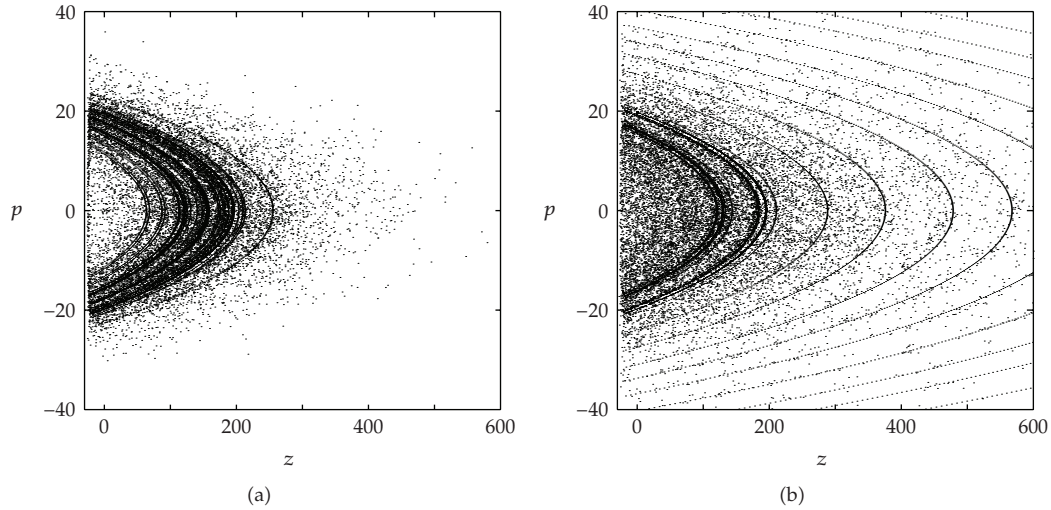

Figure 3: The above diagram of the phase space evolution of a classical ensemble of particles initially in a narrowly peaked Gaussian distribution originating from the area of phase space that supports accelerated trajectories. The initial distribution, centered at $\overline{z} = 2\pi^2$ and $\overline{p} = 0$ with $\Delta p(0) = \Delta z(0) = 0.1$, is propagated for $\epsilon = 1$ (a) and $\epsilon = 1.7$ (b) for time $t = 1000$. The numerical calculations correspond to Cesium atoms of mass $m = 2.2 \times 10^{-25}$ Kg bouncing off an atomic mirror with an intensity modulation of $a = 0.55$. The modulation frequencies extend to the megahertz range, and $\kappa^{-1} = 0.55\,\mu$m.

in (4.1) is the analogue of the Kepler map which was found very helpful in the hydrogen atom problem [21]. A classical analysis of mapping given in (4.1) leads to predict the onset of chaos under the same conditions found for the Standard Map description. Above the chaotic threshold a diffusive growth of N is observed. Consider the phase space defined in conjugate coordinates (N, ϕ) and introduce the initial distribution of phase-space points $f(N, \phi, t = 0)$. The time evolution of f is described by a Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial N}\left(D_N \frac{\partial f}{\partial N}\right), \tag{4.2}$$

where t is time measured in the number of iterations of the map, that is, in the numbers of the bounces. The Fokker-Planck (4.2) can be solved by the method of characteristics [22]. We consider the conjugate pair mapping as in (4.1), switching parameters and with the help of (4.2), we get the interesting relation between, ΔN, and $\bar{k}$, that is,

$$\Delta N^2 = \frac{4Nj\epsilon^2}{\bar{k}}. \tag{4.3}$$

Here, ΔN is the dimensionless energy and j describes the number of bounces. The diffusion coefficient D_N is

$$D_N = \frac{4\epsilon^2 N}{\bar{k}}. \tag{4.4}$$

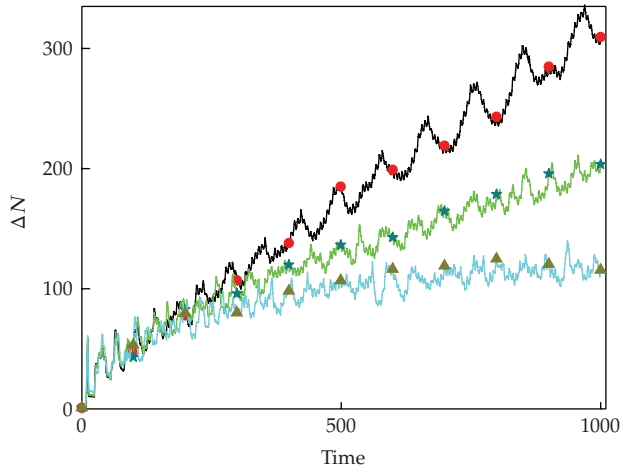

Figure 4: We present ΔN versus time for different values of effective Planck's constant $\hbar$: $\hbar = 1$ (circle and line), $\hbar = 7$ (star and line), and $\hbar = 16$ (triangle and line). We note that as we increase the value of $\hbar$ the dispersion decreases. The remaining parameters are the same as in Figure 2.

We plot ΔN versus time for different values of $\hbar$, shown in Figure 4, which illustrates that, as we increase the value of the $\hbar$, we expand the dynamical localization window, which is $0.24 < \epsilon < \sqrt{\hbar}/2$. In this window, the lower boundary $\epsilon_l = 0.24$ is set by classical dynamics, we can find the ϵ_l by evaluating Lyapunov exponents and from the standard map. For a modulation amplitude $\epsilon < 0.24$, the Lyapunov exponent converges to zero in a vast range of initial conditions, expect in small regions near separatrices. In the simulation, we initially take our wave packet as Gaussian distribution. A comparison between quantum mechanically and classical values of ΔN versus $\hbar$ is shown in Figure 5. We take the average value of ΔN for last 400 to 1000 times. For different values of effective Planck's constant $\hbar$, we extend localization window towards accelerated window. We note that ΔN displaying a decaying behavior for the higher values of effective Planck's constant, $\hbar$.

5. Results and Discussions

In this contribution, we derive a quantum mechanical map for an atom bouncing off a modulated mirror under the influence of gravity by reducing standard map into conjugate pair map. We write the standard mapping as conjugate pair mapping in quantum domain which provides dispersion law and its dependence on system parameters, such as modulation strength, energy, effective Planck's constant, and number of bounces. Our numerical result show a very good agreement with analytically obtained results. We show the dependence of ΔN on $\hbar$ by numerically and by deriving the Fokker Planck equation for our system as shown in Figure 5.

As discussed in Section 4 the localization occurs in the quantum modulated bouncer in a window defined by classical and quantum dispersion laws. The latter is a function of effective Planck's constant and follows a square root law. Whereas the accelerated dynamics takes place in the system for another window on modulation strength in the presence of initial areas in phase space [2]. As we increase the value of the $\hbar$, we extend localization window towards the accelerated window. For larger value of $\hbar$ variance in energy decreases and the

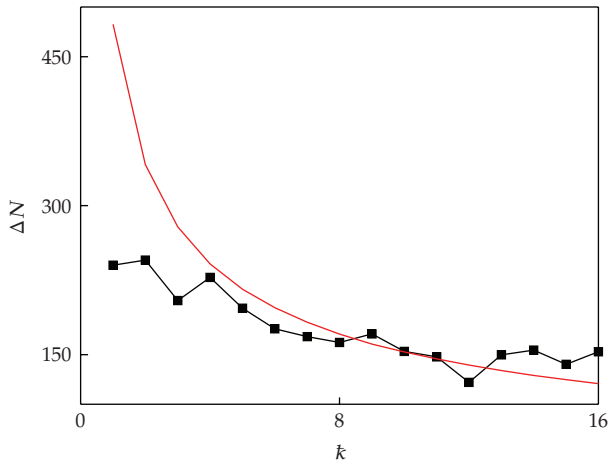

Figure 5: We plot ΔN versus $\bar{k}$ and compare analytical results obtained by (4.3) with the numerical results. Here, $\epsilon = 1.7$, $N = 1235$ and $j = 15$. Black squared line expresses numerical results of ΔN whereas, red line expresses corresponding analytical results obtained in Section 3.

two windows may overlap, where we do not find any accelerated dynamics, but dynamical localization as shown in Figure 4.

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