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Research Article

MultiPoint BVPs for Second-Order Functional Differential Equations with Impulses

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This paper is concerned about the existence of extreme solutions of multipoint boundary value problem for a class of second-order impulsive functional differential equations. We introduce a new concept of lower and upper solutions. Then, by using the method of upper and lower solutions introduced and monotone iterative technique, we obtain the existence results of extreme solutions.

1. Introduction

In this paper, we consider the multipoint boundary value problems for the impulsive functional differential equation:

$$\begin{aligned} -u''(t) &= f(t,u(t),u(\theta(t))), \quad t \in J = [0,1], t \neq t_k, \\ \Delta u'(t_k) &= I_k(u(t_k)), \quad k = 1,\dots,m, \\ u(0) - au'(0) &= cu(\eta), \qquad u(1) + bu'(1) = du(\xi), \end{aligned} \tag{1.1}$$

where $f \in C(J \times \mathbf{R}^2, \mathbf{R})$, $0 \leq \theta(t) \leq t$, $t \in J$, $\theta \in C(J)$, $a \geq 0$, $b \geq 0$, $0 \leq c \leq 1$, $0 \leq d \leq 1$, $0 < \eta$, $\xi < 1$. $0 < t_1 < t_2 < \cdots < t_m < 1$, f is continuous everywhere except at $\{t_k\} \times R^2$; $f(t_k^+,\cdot,\cdot)$, and $f(t_k^-,\cdot,\cdot)$ exist with $f(t_k^-,\cdot,\cdot) = f(t_k,\cdot,\cdot)$; $I_k \in C(R,R)$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$. Denote $J^- = J \setminus \{t_i,\ i = 1,2,\dots,m\}$. Let $PC(J,R) = \{u : J \to R; u(t)|_{J^-}$ is continuous, $u(t_k^+)$ and

$u(t_k^-)$ exist with $u(t_k^-) = u(t_k),\ k = 1, 2, \ldots, m\}$; $PC^1(J, R) = \{u : J \to R;\ u(t)|_{J^-}$ is continuous differentiable, $u'(t_k^+)$ and $u'(t_k^-)$ exist with $u'(t_k^-) = u'(t_k),\ k = 1, 2, \ldots, m\}$. Let $E = PC^1(J, R) \cap C^2(J, R)$. A function $u \in E$ is called a solution of BVP(1.1) if it satisfies (1.1).

The method of upper and lower solutions combining monotone iterative technique offers an approach for obtaining approximate solutions of nonlinear differential equations [1–3]. There exist much literature devoted to the applications of this technique to general boundary value problems and periodic boundary value problems, for example, see [1, 4–6] for ordinary differential equations, [7–11] for functional differential equations, and [12] for differential equations with piecewise constant arguments. For the studies about some special boundary value problems, for example, Lidston boundary value problems and antiperiodic boundary value problems, one may see [13, 14] and the references cited therein.

Here, we hope to mention some papers where existence results of solutions of certain boundary value problems of impulsive differential equations were studied [11, 15] and certain multipoint boundary value problems also were studied [6, 16–21]. These works motivate that we study the multipoint boundary value problems for the impulsive functional differential equation (1.1).

We also note that when $I_k = 0$ and $\theta(t) = t$, the boundary value problem (1.1) reduces to multi-point boundary value problems for ordinary differential equations which have been studied in many papers, see, for example, [6, 16–18] and the references cited therein. To our knowledge, only a few papers paid attention to multi-point boundary value problems for impulsive functional differential equations.

In this paper, we are concerned with the existence of extreme solutions for the boundary value problem (1.1). The paper is organized as follows. In Section 2, we establish two comparison principles. In Section 3, we consider a linear problem associated to (1.1) and then give a proof for the existence theorem. In Section 4, we first introduce a new concept of lower and upper solutions. By using the method of upper and lower solutions with a monotone iterative technique, we obtain the existence of extreme solutions for the boundary value problem (1.1).

2. Comparison Principles

In the following, we always assume that the following condition (H) is satisfied:

(H) $\quad a \geq 0,\ b \geq 0,\ 0 \leq c \leq 1, 0 \leq d \leq 1,\ 0 < \eta,\ \xi < 1,\ a + c > 0,\ b + d > 0.$

For any given function $g \in E$, we denote

$$A_g = \max\left\{\frac{g(0) - ag'(0) - cg(\eta)}{a\pi + c\sin\pi\eta}, \frac{g(1) + bg'(1) - dg(\xi)}{b\pi + d\sin\pi\xi}\right\},$$
$$B_g = \max\{A_g, 0\}, \quad c_g(t) = B_g\sin(\pi t), \quad r = \pi^2. \tag{2.1}$$

We now present main results of this section.

Theorem 2.1. *Assume that* $u \in E$ *satisfies*

$$
\begin{gathered}
-u''(t) + Mu(t) + Nu(\theta(t)) \le 0, \quad t \in J,\ t \ne t_k, \\
\Delta u'(t_k) \ge L_k u(t_k), \quad k = 1, \dots, m, \\
u(0) - au'(0) \le cu(\eta), \qquad u(1) + bu'(1) \le du(\xi),
\end{gathered}
\tag{2.2}
$$

where $a \ge 0,\ b \ge 0, 0 \le c \le 1,\ 0 \le d \le 1,\ 0 < \eta,\ \xi < 1, L_k \ge 0$ *and constants* $M,\ N$ *satisfy*

$$
M > 0,\ N \ge 0, \quad \frac{M+N}{2} + \sum_{k=1}^{m} L_k \le 1. \tag{2.3}
$$

Then $u(t) \le 0$ *for* $t \in J$.

Proof. Suppose, to the contrary, that $u(t) > 0$ for some $t \in J$.

If $u(1) = \max_{t\in J} u(t) > 0$, then $u'(1) \ge 0$, $u(1) \ge u(\xi)$, and

$$
du(\xi) \le u(1) \le u(1) + bu'(1) \le du(\xi). \tag{2.4}
$$

So $d = 1$ and $u(\xi)$ is a maximum value.

If $u(0) = \max_{t\in J} u(t) > 0$, then $u'(0) \le 0$, $u(0) \ge u(\eta)$, and

$$
cu(\eta) \le u(0) \le u(0) - bu'(0) \le cu(\eta). \tag{2.5}
$$

So $c = 1$ and $u(\eta)$ is a maximum value.

Therefore, there is a $\rho \in (0,1)$ such that

$$
u(\rho) = \max_{t\in J} u(t) > 0, \quad u'(\rho^+) \le 0. \tag{2.6}
$$

Suppose that $u(t) \ge 0$ for $t \in J$. From the first inequality of (2.2), we obtain that $u''(t) \ge 0$ for $t \in J$. Hence

$$
u(0) = \max_{t\in J} u(t) \quad \text{or} \quad u(1) = \max_{t\in J} u(t). \tag{2.7}
$$

If $u'(0) \ge 0$, then $u''(t) \ge 0$, $t \in (t_i, t_{i+1}]$, it is easy to obtain that $u(t)$ is nondecreasing. Since $u(1) \le du(\xi) \le u(1)$, it follows that $u(t) \equiv K$ $(K > 0)$ for $t \in [\xi, 1]$. From the first inequality of (2.2), we have that when $t \in [\xi, 1]$,

$$
0 < MK \le Mu(t) + Nu(\theta(t)) \le u''(t) = 0, \tag{2.8}
$$

which is a contradiction.

If $u'(0) \le 0$, then $u(0) = \max_{t\in J} u(t) > 0$, or $u(1) = \max_{t\in J} u(t) > 0$. If $u(0) = \max_{t\in J} u(t) > 0$, then $u(t) \equiv K$ $(K > 0)$ for $t \in [0, \eta]$. If $u(1) = \max_{t\in J} u(t) > 0$, then $u(t) \equiv K$ for $t \in [\xi, 1]$.

From the first inequality of (2.2), we have that when $t \in [\xi, 1]$,

$$0 < MK \leq Mu(t) + Nu(\theta(t)) \leq u''(t) = 0, \tag{2.9}$$

which is a contradiction.

Suppose that there exist $t_1,\ t_2 \in J$ such that $u(t_1) > 0$ and $u(t_2) < 0$. We consider two possible cases.

Case 1 ($u(0) > 0$). Since $u(t_2) < 0$, there is $\kappa > 0,\ \varepsilon > 0$ such that $u(\kappa) = 0$, $u(t) \geq 0$ for $t \in [0, \kappa)$ and $u(t) < 0$ for all $t \in (\kappa, \kappa + \varepsilon]$. It is easy to obtain that $u''(t) \geq 0$ for $t \in [0, \kappa]$. If $t^* < \kappa$, then $0 < Mu(t^*) \leq u''(t^*) \leq 0$, a contradiction. Hence $t^* > \kappa + \varepsilon$. Let $t_* \in [0, t^*)$ such that $u(t_*) = \min_{t\in[0,t^*)} u(t)$, then $u(t_*) < 0$. From the first inequality of (2.2), we have

$$\begin{gathered} u''(t) \geq (M+N)u(t_*), \quad t \in [0, t^*), \quad t \neq t_k, \\ \Delta u'(t_k) \geq L_k u(t_k), \quad k = 1, \ldots, m. \end{gathered} \tag{2.10}$$

Integrating the above inequality from $s (t_* \leq s \leq t^*)$ to t^*, we obtain

$$\begin{aligned} u'(t^*) - u'(s) &\geq (t^* - s)(M+N)u(t_*) + \sum_{s<t_k<t^*} L_k u(t_k) \\ &\geq (t^* - s)(M+N)u(t_*) + \sum_{k=1}^{m} L_k u(t_*). \end{aligned} \tag{2.11}$$

Hence

$$-u'(s) \geq [\,(t^* - s)(M+N) + \sum_{k=1}^{m} L_k u(t_*), \quad t_* \leq s \leq t^*, \tag{2.12}$$

and then integrate from t_* to t^* to obtain

$$\begin{aligned} -u(t_*) &< u(t^*) - u(t_*) \\ &\leq \int_{t_*}^{t^*} (s - t^*)(M+N)u(t_*)ds - \sum_{k=1}^{m} L_k u(t_*) \\ &\leq -\left(\frac{M+N}{2}(t^* - t_*)^2 + \sum_{k=1}^{m} L_k\right) u(t_*) \\ &\leq -\left(\frac{M+N}{2} + \sum_{k=1}^{m} L_k\right) u(t_*). \end{aligned} \tag{2.13}$$

From (2.3), we have that $u(t_*) > 0$. This is a contradiction.

Case 2 ($u(0) \le 0$). Let $t_* \in [0, t^*)$ such that $u(t_*) = \min_{t\in[0,t^*)} u(t) \le 0$. From the first inequality of (2.2), we have

$$u''(t) \ge (M+N)u(t_*), \quad t \in [0, t^*),\ t \ne t_k,$$
$$\Delta u'(t_k) \ge L_k u(t_k), \quad k = 1, \dots, m. \tag{2.14}$$

The rest proof is similar to that of Case 1. The proof is complete. □

Theorem 2.2. *Assume that* (H) *holds and* $u \in E$ *satisfies*

$$-u''(t) + Mu(t) + Nu(\theta(t)) + [(M+r)c_u(t) + Nc_u(\theta(t))] \le 0, \quad t \in J, t \ne t_k,$$
$$\Delta u'(t_k) \ge L_k u(t_k) + L_k c_u(t_k), \quad k = 1, \dots, m, \tag{2.15}$$

where constants $M,\ N$ *satisfy* (2.3)*, and* $L_k \ge 0$*, then* $u(t) \le 0$ *for* $t \in J$.

Proof. Assume that $u(0) - au'(0) \le cu(\eta), u(1) + bu'(1) \le du(\xi)$, then $c_u(t) \equiv 0$. By Theorem 2.1, $u(t) \le 0$.

Assume that $u(0) - au'(0) \le cu(\eta), u(1) + bu'(1) > du(\xi)$, then

$$c_u(t) = \frac{\sin(\pi t)}{b\pi + d\sin(\pi\xi)}\left(u(1) + bu'(1) - du(\xi)\right). \tag{2.16}$$

Put $y(t) = u(t) + c_u(t),\ t \in J$, then $y(t) \ge u(t)$ for all $t \in J$, and

$$y'(t) = u'(t) + \frac{\pi\cos(\pi t)}{b\pi + d\sin(\pi\xi)}\left(u(1) + bu'(1) - du(\xi)\right), \quad t \in J,$$
$$y''(t) = u''(t) - rc_u(t), \quad t \in J. \tag{2.17}$$

Hence

$$y(0) = u(0), \qquad y(1) = u(1),$$
$$y(\xi) = u(\xi) + \frac{\sin(\pi\xi)}{b\pi + d\sin(\pi\xi)}\left(u(1) + bu'(1) - du(\xi)\right),$$
$$y'(0) = u'(0) + \frac{\pi}{b\pi + d\sin(\pi\xi)}\left(u(1) + bu'(1) - du(\xi)\right),$$
$$y'(1) = u'(1) - \frac{\pi}{b\pi + d\sin(\pi\xi)}\left(u(1) + bu'(1) - du(\xi)\right),$$
$$-y''(t) + My(t) + Ny(\theta(t)) = -u''(t) + Mu(t) + Nu(\theta(t)) + [(M+r)c_u(t) + Nc_u(\theta(t))] \le 0,$$
$$y(0) - ay'(0) = u(0) - au'(0) - \frac{a\pi}{b\pi + d\sin(\pi\xi)}\left(u(1) + bu'(1) - du(\xi)\right) \le cu(\eta) \le cy(\eta),$$

$$
\begin{aligned}
&y(1) + by'(1) - dy(\xi) \\
&\quad = u(1) + bu'(1) - du(\xi) - \frac{b\pi}{b\pi + d\sin(\pi\xi)}\left(u(1) + bu'(1) - du(\xi)\right) \\
&\qquad - \frac{d\sin(\pi\xi)}{b\pi + d\sin\pi\xi}\left(u(1) + bu'(1) - du(\xi)\right) \le 0, \\
&\Delta y'(t_k) = \Delta u'(t_k)\Delta c_u'(t_k) \ge L_k u(t_k) + L_k c_u(t_k) = L_k y(t_k).
\end{aligned}
\tag{2.18}
$$

By Theorem 2.1, $y(t) \le 0$ for all $t \in J$, which implies that $u(t) \le 0$ for $t \in J$.

Assume that $u(0) - au'(0) > cu(\eta), u(1) + bu'(1) \le du(\xi)$, then

$$
c_u(t) = \frac{\sin\pi t}{a\pi + c\sin(\pi\eta)}\left(u(0) - au'(0) - cu(\eta)\right). \tag{2.19}
$$

Put $y(t) = u(t) + c_u(t),\ t \in J$, then $y(t) \ge u(t)$ for all $t \in J$, and

$$
\begin{gathered}
y'(t) = u'(t) + \frac{\pi\cos(\pi t)}{a\pi + c\sin(\pi\eta)}\left(u(0) - au'(0) - cu(\eta)\right), \quad t \in J, \\
y''(t) = u''(t) - rc_u(t), \quad t \in J.
\end{gathered}
\tag{2.20}
$$

Hence

$$
\begin{aligned}
&y(0) = u(0), \quad y(1) = u(1), \\
&y(\eta) = u(\eta) + \frac{\sin(\pi\eta)}{a\pi + c\sin(\pi\eta)}\left(u(0) - au'(0) - cu(\eta)\right), \\
&y'(0) = u'(0) + \frac{\pi}{a\pi + c\sin(\pi\eta)}\left(u(0) - au'(0) - cu(\eta)\right), \\
&y'(1) = u'(1) - \frac{\pi}{a\pi + c\sin(\pi\eta)}\left(u(0) - au'(0) - cu(\eta)\right), \\
&-y''(t) + My(t) + Ny(\theta(t)) = -u''(t) + Mu(t) + Nu(\theta(t)) + \left[(M + r)c_u(t) + Nc_u(\theta(t))\right] \le 0, \\
&y(0) - ay'(0) - cy(\eta) \\
&\quad = u(0) - au'(0) - cu(\eta) - \frac{a\pi}{a\pi + c\sin(\pi\eta)}\left(u(0) - au'(0) - cu(\eta)\right) \\
&\qquad - \frac{c\sin(\pi\eta)}{a\pi + c\sin(\pi\eta)}\left(u(0) - au'(0) - cu(\eta)\right) \le 0, \\
&y(1) + by'(1) = u(1) + bu'(1) - \frac{b\pi}{a\pi + c\sin(\pi\eta)}\left(u(0) - au'(0) - cu(\eta)\right) \le du(\xi) \le dy(\xi), \\
&\Delta y'(t_k) = \Delta u'(t_k) + \Delta c_u'(t_k) \ge L_k u(t_k) + L_k c_u(t_k) = L_k y(t_k).
\end{aligned}
\tag{2.21}
$$

By Theorem 2.1, $y(t) \le 0$ for all $t \in J$, which implies that $u(t) \le 0$ for $t \in J$.
Assume that $u(0) - au'(0) > cu(\eta), u(1) + bu'(1) > du(\xi)$, then $c_u(t) = A_u \sin(\pi t)$.
Put $y(t) = u(t) + c_u(t),\ t \in J$, then $y(t) \ge u(t)$ for all $t \in J$, and

$$\begin{aligned} y'(t) &= u'(t) + A_u\pi\cos(\pi t), \quad t \in J, \\ y''(t) &= u''(t) - rc_u(t), \quad t \in J. \end{aligned} \tag{2.22}$$

Hence

$$\begin{aligned}
& y(0) = u(0), \qquad y(1) = u(1), \\
& y(\eta) = u(\eta) + A_u \sin(\pi\eta), \qquad y(\xi) = u(\xi) + A_u \sin(\pi\xi), \\
& y'(0) = u'(0) + A_u\pi, \qquad y'(1) = u'(1) - A_u\pi, \\
& -y''(t) + My(t) + Ny(\theta(t)) \\
& \qquad = -u''(t) + Mu(t) + Nu(\theta(t)) + [(M+r)c_u(t) + Nc_u(\theta(t))] \le 0, \\
& y(0) - ay'(0) - cy(\eta) = u(0) - au'(0) - cu(\eta) - aA_u\pi - cA_u\sin(\pi\eta) \le 0, \\
& y(1) + by'(1) - dy(\xi) = u(1) + bu'(1) - du(\xi) - bA_u\pi - dA_u\sin(\pi\xi) \le 0, \\
& \Delta y'(t_k) = \Delta u'(t_k) + \Delta c_u'(t_k) \ge L_k u(t_k) + L_k c_u(t_k) = L_k y(t_k).
\end{aligned} \tag{2.23}$$

By Theorem 2.1, $y(t) \le 0$ for all $t \in J$, which implies that $u(t) \le 0$ for $t \in J$. The proof is complete. □

3. Linear Problem

In this section, we consider the linear boundary value problem

$$\begin{aligned} -u''(t) + Mu(t) + Nu(\theta(t)) &= \sigma(t), \quad t \in J, t \ne t_k, \\ \Delta u'(t_k) &= L_k u(t_k) + e_k, \quad k = 1, \dots, m, \\ u(0) - au'(0) = cu(\eta), &\qquad u(1) + bu'(1) = du(\xi). \end{aligned} \tag{3.1}$$

Theorem 3.1. *Assume that* (H) *holds,* $\sigma \in C(J)$*,* $e_k \in R$*, and constants* $M,\ N$ *satisfy* (2.3) *with*

$$\mu = \left(\frac{a(1+2b)}{2(a+b+1)} + \frac{1}{8}\left(\frac{1+2b}{a+b+1}\right)^2\right)(M+N) + \left(1 + \frac{(1+b)^2}{a+b+1}\right)\sum_{k=1}^{m} L_k < 1. \tag{3.2}$$

Further suppose that there exist $\alpha, \beta \in E$ such that

(h_1) $\alpha \le \beta$ *on* J,

(h_2)

$$\begin{gathered}
-\alpha''(t) + M\alpha(t) + N\alpha(\theta(t)) + [(M+r)c_\alpha(t) + Nc_\alpha(\theta(t))] \le \sigma(t), \quad t \in J,\ t \ne t_k, \\
\Delta\alpha'(t_k) \ge L_k\alpha(t_k) + L_k c_\alpha(t_k) + e_k, \quad k = 1, \dots, m,
\end{gathered} \tag{3.3}$$

(h_3)

$$\begin{gathered}
-\beta''(t) + M\beta(t) + N\beta(\theta(t)) - \left[(M+r)c_{-\beta}(t) + Nc_{-\beta}(\theta(t))\right] \ge \sigma(t), \quad t \in J, t \ne t_k, \\
\Delta\beta'(t_k) \le L_k\beta(t_k) - L_k c_{-\beta}(t_k) + e_k, \quad k = 1, \dots, m.
\end{gathered} \tag{3.4}$$

Then the boundary value problem (3.1) *has one unique solution* $u(t)$ *and* $\alpha \le u \le \beta$ *for* $t \in J$.

Proof. We first show that the solution of (3.1) is unique. Let $u_1,\ u_2$ be the solution of (3.1) and set $v = u_1 - u_2$. Thus,

$$\begin{gathered}
-v''(t) + Mv(t) + Nv(\theta(t)) = 0, \quad t \in J, t \ne t_k, \\
\Delta v'(t_k) = L_k v(t_k), \quad k = 1, \dots, m, \\
v(0) - av'(0) = cv(\eta), \qquad v(1) + bv'(1) = dv(\xi).
\end{gathered} \tag{3.5}$$

By Theorem 2.1, we have that $v \le 0$ for $t \in J$, that is, $u_1 \le u_2$ on J. Similarly, one can obtain that $u_2 \le u_1$ on J. Hence $u_1 = u_2$.

Next, we prove that if u is a solution of (3.1), then $\alpha \le u \le \beta$. Let $p = \alpha - u$. From boundary conditions, we have that $c_\alpha(t) = c_p(t)$ for all $t \in J$. From (h_2) and (3.1), we have

$$\begin{gathered}
-p''(t) + Mp(t) + Np(\theta(t)) + \left[(M+r)c_p(t) + Nc_p(\theta(t))\right] \le 0, \quad t \in J, t \ne t_k, \\
\Delta p'(t_k) \ge L_k p(t_k) + L_k c_p(t_k), \quad k = 1, \dots, m.
\end{gathered} \tag{3.6}$$

By Theorem 2.1, we have that $p = \alpha - u \le 0$ on J. Analogously, $u \le \beta$ on J.

Finally, we show that the boundary value problem (3.1) has a solution by five steps as follows.

Step 1. Let $\overline{\alpha}(t) = \alpha(t) + c_\alpha(t),\ \overline{\beta}(t) = \beta(t) - c_{-\beta}(t)$. We claim that

(1)

$$\begin{gathered}
-\overline{\alpha}''(t) + M\overline{\alpha}(t) + N\overline{\alpha}(\theta(t)) + [(M+r)c_{\overline{\alpha}}(t) + Nc_{\overline{\alpha}}(\theta(t))] \le \sigma(t) \quad \text{for } t \in J, t \ne t_k, \\
\Delta\overline{\alpha}'(t_k) \ge L_k\overline{\alpha}(t_k) + e_k, \quad k = 1, \dots, m,
\end{gathered} \tag{3.7}$$

(2)

$$
\begin{gathered}
-\overline{\beta}''(t) + M\overline{\beta}(t) + N\overline{\beta}(\theta(t)) - \left[(M+r)c_{-\overline{\beta}}(t) + Nc_{-\overline{\beta}}(\theta(t))\right] \geq \sigma(t) \quad \text{for } t \in J, t \neq t_k, \\
\Delta\overline{\beta}'(t_k) \leq L_k\overline{\beta}(t_k) + e_k, k = 1, \ldots, m,
\end{gathered} \tag{3.8}
$$

(3) $\alpha(t) \leq \overline{\alpha}(t) \leq \overline{\beta}(t) \leq \beta(t)$ for $t \in J$.

From (h_2) and (h_3), we have

$$
\begin{gathered}
-\overline{\alpha}''(t) + M\overline{\alpha}(t) + N\overline{\alpha}(\theta(t)) \leq \sigma(t), \quad t \in J, t \neq t_k, \\
\Delta\overline{\alpha}'(t_k) \geq L_k\overline{\alpha}(t_k) + e_k, \quad k = 1, \ldots, m.
\end{gathered} \tag{3.9}
$$

$$
\begin{gathered}
-\overline{\beta}''(t) + M\overline{\beta}(t) + N\overline{\beta}(\theta(t)) \geq \sigma(t), \quad t \in J, t \neq t_k, \\
\Delta\overline{\beta}'(t_k) \leq L_k\overline{\beta}(t_k) + e_k, \quad k = 1, \ldots, m,
\end{gathered} \tag{3.10}
$$

$$
\overline{\alpha}(0) - a\overline{\alpha}'(0) - c\overline{\alpha}(\eta) = \alpha(0) - a\alpha'(0) - c\alpha(\eta) - (a\pi + c\sin(\pi\eta))B_\alpha \leq 0, \tag{3.11}
$$

$$
\overline{\alpha}(1) + b\overline{\alpha}'(1) - d\overline{\alpha}(\xi) = \alpha(1) + b\alpha'(0) - d\alpha(\xi) - (b\pi + d\sin(\pi\xi))B_\alpha \leq 0, \tag{3.12}
$$

$$
-\left[\overline{\beta}(0) - a\overline{\beta}'(0) - c\overline{\beta}(\eta)\right] = -\beta(0) + a\beta'(0) + c\beta(\eta) - (a\pi + c\sin(\pi\eta))B_{-\beta} \leq 0, \tag{3.13}
$$

$$
-\left[\overline{\beta}(1) + b\overline{\beta}'(1) - d\overline{\beta}(\xi)\right] = -\beta(1) - b\beta'(0) + d\beta(\xi) - (b\pi + d\sin(\pi\xi))B_{-\beta} \leq 0. \tag{3.14}
$$

From (3.9)–(3.14), we obtain that $c_{\overline{\alpha}}(t) = c_{-\overline{\beta}}(t) \equiv 0,\ t \in J$. Combining (3.9) and (3.10), we obtain that (1) and (2) hold.

It is easy to see that $\alpha \leq \overline{\alpha},\ \overline{\beta} \leq \beta$ on J. We show that $\overline{\alpha} \leq \overline{\beta}$ on J. Let $p = \overline{\alpha} - \overline{\beta}$, then $p(t) = \alpha(t) - \beta(t) + c_\alpha(t) + c_{-\beta}(t)$. From (3.9)–(3.14), we have

$$
\begin{aligned}
& -p''(t) + Mp(t) + Np(\theta(t)) \leq 0,\ t \in J, \quad t \neq t_k, \\
& \Delta p'(t_k) \geq L_k p(t_k), \quad k = 1, \ldots, m, \\
& p(0) - ap'(0) - cp(\eta) = \alpha(0) - a\alpha'(0) - c\alpha(\eta) - (a\pi + c\sin I(\pi\eta))B_\alpha \\
& \qquad\qquad - \beta(0) + a\beta'(0) + c\beta(\eta) - (a\pi + c\sin(\pi\eta))B_{-\beta} \leq 0, \\
& p(1) + bp'(1) - dp(\xi) = \alpha(1) + b\alpha'(1) - d\alpha(\xi) - (b\pi + d\sin(\pi\xi))B_\alpha \\
& \qquad\qquad - \beta(1) - b\beta'(1) + d\beta(\eta) - (b\pi + d\sin(\pi\xi))B_{-\beta} \leq 0, \\
& \Delta p'(t_k) = \Delta\alpha'(t_k) - \Delta\beta'(t_k) + \Delta c_a'(t_k) + \Delta c_{-\beta}'(t_k) \geq L_k(\alpha(t_k) - \beta(t_k)) \\
& \qquad + L_k(c_\alpha(t_k) + c_{-\beta}(t_k)) = L_k p(t_k).
\end{aligned} \tag{3.15}
$$

By Theorem 2.1, we have that $p \leq 0$ on J, that is, $\overline{\alpha} \leq \overline{\beta}$ on J. So (3) holds.

Step 2. Consider the boundary value problem

$$\begin{gathered}
-u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J, \quad t \neq t_k, \\
\Delta u'(t_k) = L_k u(t_k) + e_k, \quad k = 1, \ldots, m, \\
u(0) - au'(0) = \lambda, \qquad u(1) + bu'(1) = \delta,
\end{gathered} \tag{3.16}$$

where $\lambda \in \mathbf{R}$, $\delta \in \mathbf{R}$. We show that the boundary value problem (3.16) has one unique solution $u(t, \lambda, \delta)$.

It is easy to check that the boundary value problem (3.16) is equivalent to the integral equation:

$$\begin{aligned}
u(t) = {} & \frac{\delta t + (1-t)\lambda + b\lambda + a\delta}{a+b+1} + \int_0^1 G(t,s)[\sigma(s) - Mu(s) - Nu(\theta(s))]ds \\
& + \sum_{0<t_k<t} (t-t_k)[L_k u(t_k) + e_k] - \frac{1}{a+b+1}(t+b)\sum_{k=1}^{m}[(1-t_k)+b][L_k u(t_k) + e_k],
\end{aligned} \tag{3.17}$$

where

$$G(t,s) = \frac{1}{a+b+1}\begin{cases} (a+t)(1+b-s), & 0 \le t \le s \le 1, \\ (a+s)(1+b-t), & 0 \le s \le t \le 1. \end{cases} \tag{3.18}$$

It is easy to see that $PC(J,R)$ with norm $\|u\| = \max_{t\in J}|u(t)|$ is a Banach space. Define a mapping $\Phi : PC(J,R) \to PC(J,R)$ by

$$\begin{aligned}
(\Phi u)(t) = {} & \frac{\delta t + (1-t)\lambda + b\lambda + a\delta}{a+b+1} + \int_0^1 G(t,s)[\sigma(s) - Mu(s) - Nu(\theta(s))]ds \\
& + \sum_{0<t_k<t} (t-t_k)[L_k u(t_k) + e_k] - \frac{1}{a+b+1}(t+b)\sum_{k=1}^{m}[(1-t_k)+b][L_k u(t_k) + e_k].
\end{aligned} \tag{3.19}$$

For any $x,\ y \in PC(J,R)$, we have

$$\begin{aligned}
&\left|(\Phi x)(t) - (\Phi y)(t)\right| \\
&\quad \le \int_0^1 G(t,s)\left[M\left(y(s) - x(s)\right) + N\left(y(\theta(s)) - x(\theta(s))\right)\right]ds + \left(1 + \frac{(1+b)^2}{a+b+1}\right)\sum_{k=1}^{m} L_k\|x-y\| \\
&\quad \le \int_0^1 G(t,s)ds\|x-y\|(M+N) + \left(1 + \frac{(1+b)^2}{a+b+1}\right)\sum_{k=1}^{m} L_k\|x-y\|.
\end{aligned} \tag{3.20}$$

Since

$$\max_{t\in J}\int_0^1 G(t,s)ds = \frac{a(1+2b)}{2(a+b+1)} + \frac{1}{8}\left(\frac{1+2b}{a+b+1}\right)^2, \tag{3.21}$$

the inequality (3.2) implies that $\Phi : PC(J) \to PC(J)$ is a contraction mapping. Thus there exists a unique $u \in PC(J)$ such that $\Phi u = u$. The boundary value problem (3.16) has a unique solution.

Step 3. We show that for any $t \in J$, the unique solution $u(t,\lambda,\delta)$ of the boundary value problem (3.16) is continuous in λ and δ. Let $u(t,\lambda_i,\delta_i)$, $i = 1,2$, be the solution of

$$\begin{gathered} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J,\ t \neq t_k, \\ \Delta u'(t_k) = L_k u(t_k) + e_k, \quad k = 1,\dots,m, \\ u(0) - au'(0) = \lambda_i, \quad u(1) + bu'(1) = \delta_i, \quad i = 1,2. \end{gathered} \tag{3.22}$$

Then

$$\begin{aligned} u(t,\lambda_i,\delta_i) = {} & \frac{\delta_i t + (1-t)\lambda_i + b\lambda_i + a\delta_i}{a+b+1} + \int_0^1 G(t,s)[\sigma(s) - Mu(s,\lambda_i,\delta_i) - Nu(\theta(s),\lambda_i,\delta_i)]ds \\ & + \sum_{0<t_k<t}(t-t_k)[L_k u(t_k) + e_k] - \frac{1}{a+b+1}(t+b) \\ & \times \sum_{k=1}^{m}[(1-t_k)+b][L_k u(t_k)+e_k], \quad i = 1,2. \end{aligned} \tag{3.23}$$

From (3.23), we have that

$$\begin{aligned} \|u(t,\lambda_1,\delta_1) - u(t,\lambda_2,\delta_2)\| \le {} & |\lambda_1-\lambda_2| + |\delta_1-\delta_2| \\ & + (M+N)\|u(t,\lambda_1,\delta_1) - u(t,\lambda_2,\delta_2)\|\max_{t\in J}\int_0^1 G(t,s)ds \\ & + \|u(t,\lambda_1,\delta_1) - u(t,\lambda_2,\delta_2)\|\left(1 + \frac{(1+b)^2}{a+b+1}\right)\sum_{k=1}^{m} L_k\|x-y\| \\ \le {} & |\lambda_1-\lambda_2| + |\delta_1-\delta_2| + \mu\|u(t,\lambda_1,\delta_1) - u(t,\lambda_2,\delta_2)\|. \end{aligned} \tag{3.24}$$

Hence

$$\|u(t,\lambda_1,\delta_1) - u(t,\lambda_2,\delta_2)\|_0 \le \frac{1}{1-\mu}(|\lambda_1-\lambda_2| + |\delta_1-\delta_2|). \tag{3.25}$$

Step 4. We show that

$$\overline{\alpha}(t) \leq u(t,\lambda,\delta) \leq \overline{\beta}(t) \tag{3.26}$$

for any $t \in J$, $\lambda \in [c\overline{\alpha}(\eta),\ c\overline{\beta}(\eta)]$, and $\delta \in [d\overline{\alpha}(\xi),\ d\overline{\beta}(\xi)]$, where $u(t,\lambda,\delta)$ is unique solution of the boundary value problem (3.16).

Let $m(t) = \overline{\alpha}(t) - u(t,\lambda,\delta)$. From (3.9), (3.11), (3.12), and (3.16), we have that $m(0) - am'(0) \leq cm(\eta)$, $m(1) + bm'(1) \leq dm(\xi)$, and

$$\begin{aligned}
&-m''(t) + Mm(t) + Nm(\theta(t)) \\
&\quad = -\overline{\alpha}''(t) + M\overline{\alpha}(t) + N\overline{\alpha}(\theta(t)) + u''(t,\lambda) - Mu(t,\lambda,\delta) - Nu(\theta(t),\lambda,\delta) \leq \sigma(t) - \sigma(t) \leq 0, \\
&\Delta m'(t_k) \geq L_k u(t_k).
\end{aligned} \tag{3.27}$$

By Theorem 2.1, we obtain that $m \leq 0$ on J, that is, $\overline{\alpha}(t) \leq u(t,\lambda,\delta)$ on J. Similarly, $u(t,\lambda,\delta) \leq \overline{\beta}(t)$ on J.

Step 5. Let $D = [c\overline{\alpha}(\eta), c\overline{\beta}(\eta)] \times [d\overline{\alpha}(\xi),\ d\overline{\beta}(\xi)]$. Define a mapping $F : D \to \mathbf{R}^2$ by

$$F(\lambda,\delta) = \big(u(\eta,\lambda,\delta), u(\xi,\lambda,\delta)\big), \tag{3.28}$$

where $u(t,\lambda,\delta)$ is unique solution of the boundary value problem (3.16). From Step 4, we have

$$F(D) \subset D. \tag{3.29}$$

Since D is a compact convex set and F is continuous, it follows by Schauder's fixed point theorem that F has a fixed point $(\lambda_0,\delta_0) \in D$ such that

$$u(\eta,\lambda_0,\delta_0) = \lambda_0, \qquad u(\xi,\lambda_0,\delta_0) = \delta_0. \tag{3.30}$$

Obviously, $u(t,\lambda_0,\delta_0)$ is unique solution of the boundary value problem (3.1). This completes the proof. □

4. Main Results

Let $M \in \mathbf{R}$, $N \in \mathbf{R}$. We first give the following definition.

Definition 4.1. A function $\alpha \in E$ is called a lower solution of the boundary value problem (1.2) if

$$\begin{gathered}
-\alpha''(t) + (M + r)c_\alpha(t) + Nc_\alpha(\theta(t)) \leq f(t,\alpha(t),\alpha(\theta(t))), \quad t \in J,\ t \neq t_k, \\
\Delta\alpha'(t_k) \geq I_k(\alpha(t_k)) + L_k c_\alpha(t_k), \quad k = 1,\ldots,m.
\end{gathered} \tag{4.1}$$

Definition 4.2. A function $\beta \in E$ is called an upper solution of the boundary value problem (1.2) if

$$-\beta''(t) - (M+r)c_{-\beta}(t) - Nc_{-\beta}(\theta(t)) \geq f\big(t,\beta(t),\beta(\theta(t))\big) \quad t \in J,\ t \in J, t \neq t_k,$$
$$\Delta\beta'(t_k) \leq I_k\beta(t_k) - L_k c_{-\beta}(t_k), \quad k = 1,\ldots,m. \tag{4.2}$$

Our main result is the following theorem.

Theorem 4.3. *Assume that* (H) *holds. If the following conditions are satisfied:*

(H_1) *α, β are lower and upper solutions for boundary value problem (1.2) respectively, and $\alpha(t) \leq \beta(t)$ on J,*

(H_2) *the constants M, N in definition of upper and lower solutions satisfy* (2.3), (3.2), *and*

$$f(t,x,y) - f(t,\overline{x},\overline{y}) \geq -M(x-\overline{x}) - N(y-\overline{y}),$$
$$I_k(x) - I_k(y) \geq L_k(x-y), \quad x \leq y, \tag{4.3}$$

for $\alpha(t) \leq \overline{x} \leq x \leq \beta(t)$, $\alpha(\theta(t)) \leq \overline{y} \leq y \leq \beta(\theta(t))$, $t \in J$.

Then, there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n\to\infty}\alpha_n(t) = \rho(t)$, $\lim_{n\to\infty}\beta_n(t) = \varrho(t)$ uniformly on J, and ρ, ϱ are the minimal and the maximal solutions of (1.2), respectively, such that

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \alpha_n \leq \rho \leq x \leq \varrho \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0 \tag{4.4}$$

on J, where x is any solution of (1.1) *such that $\alpha(t) \leq x(t) \leq \beta(t)$ on J.*

Proof. Let $[\alpha,\beta] = \{u \in E : \alpha(t) \leq u(t) \leq \beta(t),\ t \in J\}$. For any $\gamma \in [\alpha,\beta]$, we consider the boundary value problem

$$-u''(t) + Mu(t) + Nu(\theta(t)) = f\big(t,\gamma(t),\gamma(\theta(t))\big) + M\gamma(t) + N\gamma(\theta(t)), \quad t \in J,$$
$$\Delta u'(t_k) = I_k\big(\gamma(t_k)\big) - L_k\big(u(t_k) - \gamma(t_k)\big), \quad k = 1,\ldots,m. \tag{4.5}$$
$$u(0) - ax'(0) = cu(\eta), \qquad u(1) + bu'(1) = du(\xi).$$

Since α is a lower solution of (1.2), from (H_2), we have that

$$\begin{aligned} &-\alpha''(t) + M\alpha(t) + N\alpha(\theta(t)) \\ &\quad\leq f(t,\alpha(t),\alpha(\theta(t))) + M\alpha(t) + N\alpha(\theta(t)) - (M+r)c_\alpha(t) - Nc_\alpha(\theta(t)) \\ &\quad\leq f(t,\gamma(t),\gamma(\theta(t))) + M\gamma(t) + N\gamma(\theta(t)) - (M+r)c_\alpha(t) - Nc_\alpha(\theta(t)), \end{aligned} \tag{4.6}$$
$$\Delta\alpha'(t_k) \geq I_k(\alpha(t_k)) + L_k c_\alpha(t_k) \geq I_k\big(\gamma(t_k)\big) + L_k\alpha(t_k) - L_k\gamma(t_k) + L_k c_\alpha(t_k).$$

Similarly, we have that

$$
\begin{aligned}
&-\beta''(t)+M\beta(t)+N\beta(\theta(t))\\
&\qquad\geq f\big(t,\gamma(t),\gamma(\theta(t))\big)+M\gamma(t)+N\gamma(\theta(t))+(M+r)c_{-\beta}(t)+Nc_{-\beta}(\theta(t)),\\
&\Delta\beta'(t_k)\leq I_k\big(\beta(t_k)\big)-L_kc_{-\beta}(t_k)\leq I_k\big(\gamma(t_k)\big)+L_k\beta(t_k)-L_k\gamma(t_k)-L_kc_{-\beta}(t_k).
\end{aligned}
\tag{4.7}
$$

By Theorem 3.1, the boundary value problem (4.5) has a unique solution $u\in[\alpha,\beta]$. We define an operator Ψ by $u=\Psi\gamma$, then Ψ is an operator from $[\alpha,\beta]$ to $[\alpha,\beta]$.

We will show that

(a) $\alpha\leq\Psi\alpha,\quad \Psi\beta\leq\beta$,

(b) Ψ is nondecreasing in $[\alpha,\beta]$.

From $\Psi\alpha\in[\alpha,\beta]$ and $\Psi\beta\in[\alpha,\beta]$, we have that (a) holds. To prove (b), we show that $\Psi v_1\leq\Psi v_2$ if $\alpha\leq v_1\leq v_2\leq\beta$.

Let $v_1^*=\Psi v_1,\ v_2^*=\Psi v_2$,and $p=v_1^*-v_2^*$, then by (H_2) and boundary conditions, we have that

$$
\begin{aligned}
&-p''(t)+Mp(t)+Np(\theta(t))\\
&\quad=f(t,v_1(t),v_1(\theta(t)))+Mv_1(t)+Nv_1(\theta(t))\\
&\qquad-f(t,v_2(t),v_2(\theta(t)))-Mv_2(t)-Nv_2(\theta(t))\leq 0,\\
&\Delta p'(t_k)\geq L_kp(t_k),\\
&p(0)-ap'(0)=cp(\eta),\qquad p(1)+pu'(1)=dp(\xi).
\end{aligned}
\tag{4.8}
$$

By Theorem 2.1, $p(t)\leq 0$ on J, which implies that $\Psi v_1\leq\Psi v_2$.

Define the sequences $\{\alpha_n\},\ \{\beta_n\}$ with $\alpha_0=\alpha,\ \beta_0=\beta$ such that $\alpha_{n+1}=\Psi\alpha_n,\ \beta_{n+1}=\Psi\beta_n$ for $n=0,1,2,\ldots$ From (a) and (b), we have

$$
\alpha_0\leq\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_n\leq\beta_n\leq\cdots\leq\beta_2\leq\beta_1\leq\beta_0 \tag{4.9}
$$

on $t\in J$, and each $\alpha_n,\ \beta_n\in E$ satisfies

$$
\begin{gathered}
-\alpha_n''(t)+M\alpha_n(t)+N\alpha_n(\theta(t))=f(t,\alpha_{n-1}(t),\alpha_{n-1}(\theta(t)))+M\alpha_{n-1}(t)+N\alpha_{n-1}(\theta(t)),\ t\in J,t\neq t_k,\\
\Delta\alpha_n'(t_k)=I_k(\alpha_{n-1}(t_k))+L_k(\alpha_n(t_k)-\alpha_{n-1}(t_k)),\quad k=1,2,\ldots,m,\\
\alpha_n(0)-a\alpha_n'(0)=c\alpha_n(\eta),\qquad \alpha_n(1)+b\alpha_n'(1)=d\alpha_n(\xi),\\
-\beta_n''(t)+M\beta_n(t)+N\beta_n(\theta(t))=f\big(t,\beta_{n-1}(t),\beta_{n-1}(\theta(t))\big)+M\beta_{n-1}(t)+N\beta_{n-1}(\theta(t)),\ t\in J,t\neq t_k,\\
\Delta\beta_n(t_k)=I_k\big(\beta_{n-1}(t_k)\big)+L_k\big(\beta_n(t_k)-\beta_{n-1}(t_k)\big),\quad k=1,2,\ldots,m,\\
\beta_n(0)-a\beta_n'(0)=c\beta_n(\eta),\qquad \beta_n(1)+b\beta_n'(1)=d\beta_n(\xi).
\end{gathered}
\tag{4.10}
$$

Therefore, there exist ρ, ϱ such that such that $\lim_{n\to\infty}\alpha_n(t) = \rho(t)$, $\lim_{n\to\infty}\beta_n(t) = \varrho(t)$ uniformly on J. Clearly, ρ, ϱ are solutions of (1.1).

Finally, we prove that if $x \in [\alpha_0, \beta_0]$ is any solution of (1.1), then $\rho(t) \leq x(t) \leq \varrho(t)$ on J. To this end, we assume, without loss of generality, that $\alpha_n(t) \leq x(t) \leq \beta_n(t)$ for some n. Since $\alpha_0(t) \leq x(t) \leq \beta_0(t)$, from property (b), we can obtain

$$\alpha_{n+1}(t) \leq x(t) \leq \beta_{n+1}(t), \quad t \in J. \tag{4.11}$$

Hence, we can conclude that

$$\alpha_n(t) \leq x(t) \leq \beta_n(t), \quad \forall n. \tag{4.12}$$

Passing to the limit as $n \to \infty$, we obtain

$$\rho(t) \leq x(t) \leq \varrho(t), \quad t \in J. \tag{4.13}$$

This completes the proof. □

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