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Research Article

Pressure Drop Equations for a Partially Penetrating Vertical Well in a Circular Cylinder Drainage Volume

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Taking a partially penetrating vertical well as a uniform line sink in three-dimensional space, by developing necessary mathematical analysis, this paper presents unsteady-state pressure drop equations for an off-center partially penetrating vertical well in a circular cylinder drainage volume with constant pressure at outer boundary. First, the point sink solution to the diffusivity equation is derived, then using superposition principle, pressure drop equations for a uniform line sink model are obtained. This paper also gives an equation to calculate pseudoskin factor due to partial penetration. The proposed equations provide fast analytical tools to evaluate the performance of a vertical well which is located arbitrarily in a circular cylinder drainage volume. It is concluded that the well off-center distance has significant effect on well pressure drop behavior, but it does not have any effect on pseudoskin factor due to partial penetration. Because the outer boundary is at constant pressure, when producing time is sufficiently long, steady-state is definitely reached. When well producing length is equal to payzone thickness, the pressure drop equations for a fully penetrating well are obtained.

1. Introduction

For both fully and partially penetrating vertical wells, steady-state and unsteady-state pressure-transient testings are useful tools for evaluating in situ reservoir and wellbore parameters that describe the production characteristics of a well. The use of transient well testing for determining reservoir parameters and well productivity has become common, in the past years, analytic solutions have been presented for the pressure behavior of partially penetrating vertical wells.

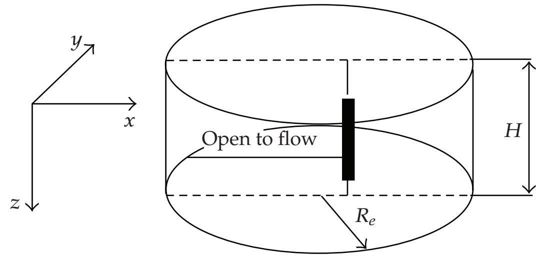

Figure 1: Partially penetrating vertical well in circular cylinder drainage volume.

The problem of fluid flow into wells with partial penetration has received much attention in the past years in petroleum engineering [1–7].

In many oil and gas reservoirs the producing wells are completed as partially penetrating wells; that is, only a portion of the pay zone is perforated. This may be done for a variety of reasons, but the most common one is to prevent or delay the unwanted fluids into the wellbore. The exact solution of the partial penetration problem presents great analytical problems because the boundary conditions that the solutions of the partial differential equations must satisfy are mixed; that is, on one of the boundaries the pressure is specified on one portion and the flux on the other. This difficult occurs at the wellbore, for the flux over the nonproductive section of the well is zero, the potential over the perforated interval must be constant.

This problem may be overcome in the case of constant rate production by making the assumption that the flux into the well is uniform over the entire perforated interval, so that on the wellbore the flux is specified over the total formation thickness. This approximation naturally leads to an error in the solution since the potential (pressure) will not be uniform over the perforated interval, but it has been shown that this occurrence is not too significant.

Many different techniques have been used for solving the partial penetration problem, namely, finite difference method [2], Fourier, Hankel and Laplace transforms [3–5], Green's functions [6]. The analytical expressions and the numerical results obtained for reservoir pressures by different methods were essentially identical, however, there are some differences between the values of wellbore pressures computed from numerical models and those obtained from analytical solutions [7].

The primary goal of this study is to present unsteady state pressure drop equations for an off-center partially penetrating vertical well in a circular cylinder drainage volume. Analytical solutions are derived by making the assumption of uniform fluid withdrawal along the portion of the wellbore open to flow. Taking the producing portion of a partially penetrating well as a uniform line sink, using principle of potential superposition, pressure drop equations for a partially penetrating well are obtained.

2. Partially Penetrating Vertical Well Model

Figure 1 is a schematic of an off-center partially penetrating vertical well. A partially penetrating well of drilled length L drains a circular cylinder porous volume with height H and radius R_e.

The following assumptions are made.

(1) The porous media volume is circular cylinder which has constant K_x, K_y, K_z permeabilities, thickness H, porosity ϕ. And the porous volume is bounded by top and bottom impermeable boundaries.

(2) The pressure is initially constant in the cylindrical body, during production the pressure remains constant and equal to the initial pressure P_i at the lateral surface.

(3) The production occurs through a partially penetrating vertical well of radius R_w, represented in the model by a uniform line sink which is located at R_0 away from the axis of symmetry of the cylindrical body. The drilled well length is L, the producing well length is L_{pr}.

(4) A single-phase fluid, of small and constant compressibility C_f, constant viscosity μ, and formation volume factor B, flows from the porous media to the well. Fluids properties are independent of pressure. Gravity forces are neglected.

The porous media domain is

$$\Omega = \left\{(x, y, z) \mid x^2 + y^2 < R_e^2,\ 0 < z < H\right\}, \tag{2.1}$$

where R_e is cylinder radius, Ω is the cylindrical body.

Located at R_0 away from the center of the cylindrical body, the coordinates of the top and bottom points of the well line are $(R_0, 0, 0)$ and $(R_0, 0, L)$, respectively, while point $(R_0, 0, L_1)$ and point $(R_0, 0, L_2)$ are the beginning point and end point of the producing portion of the well, respectively. The well is a uniform line sink between $(R_0, 0, L_1)$ and $(R_0, 0, L_2)$, and there holds

$$L_{pr} = L_2 - L_1, \quad L_{pr} \le L \le H. \tag{2.2}$$

We assume

$$K_x = K_y = K_h, \qquad K_z = K_v \tag{2.3}$$

and define average permeability

$$K_a = (K_x K_y K_z)^{1/3} = K_h^{2/3} K_v^{1/3}. \tag{2.4}$$

Suppose point $(R_0, 0, z')$ is on the producing portion, and its point convergence intensity is q, in order to obtain the pressure at point (x, y, z) caused by the point $(R_0, 0, z')$, according to mass conservation law and Darcy's law, we have to obtain the basic solution of the diffusivity equation in Ω [8]:

$$K_h \frac{\partial^2 P}{\partial x^2} + K_h \frac{\partial^2 P}{\partial y^2} + K_v \frac{\partial^2 P}{\partial z^2} = \phi \mu C_t \frac{\partial P}{\partial t} + \mu q B \delta(x - R_0)\delta(y)\delta(z - z'), \quad \text{in } \Omega, \tag{2.5}$$

where C_t is total compressibility coefficient of porous media, $\delta(x-R_0)$, $\delta(y)$, $\delta(z-z')$ are Dirac functions.

The initial condition is

$$P(t,x,y,z)|_{t=0} = P_i, \quad \text{in } \Omega. \tag{2.6}$$

The lateral boundary condition is

$$P(t,x,y,z) = P_i, \quad \text{on } \Gamma, \tag{2.7}$$

where Γ is the cylindrical lateral surface:

$$\Gamma = \left\{(x,y,z) \mid x^2+y^2 = R_e^2,\ 0<z<H\right\}. \tag{2.8}$$

The porous media domain is bounded by top and bottom impermeable boundaries, so

$$\left.\frac{\partial P}{\partial z}\right|_{z=0} = 0; \qquad \left.\frac{\partial P}{\partial z}\right|_{z=H} = 0. \tag{2.9}$$

In order to simplify the above equations, we take the following dimensionless transforms:

$$x_D = \frac{2x}{L}, \qquad y_D = \frac{2y}{L}, \qquad z_D = \left(\frac{2z}{L}\right)\left(\frac{K_h}{K_v}\right)^{1/2}, \tag{2.10}$$

$$L_D = 2\left(\frac{K_h}{K_v}\right)^{1/2}, \qquad H_D = \left(\frac{2H}{L}\right)\left(\frac{K_h}{K_v}\right)^{1/2}, \tag{2.11}$$

$$L_{prD} = L_{2D} - L_{1D} = \left[\frac{2(L_2-L_1)}{L}\right]\left(\frac{K_h}{K_v}\right)^{1/2}, \tag{2.12}$$

$$R_{0D} = \frac{2R_0}{L}, \qquad R_{eD} = \frac{2R_e}{L}, \qquad R_{wD} = \frac{2R_w}{L}, \tag{2.13}$$

$$t_D = \frac{4K_h t}{\phi\mu C_t L^2}. \tag{2.14}$$

Assuming q is the point convergence intensity at the point sink $(R_0, 0, z')$, the partially penetrating well is a uniform line sink, the total flow rate of the well is Q, and there holds

$$q = \frac{Q}{L_{prD}}. \tag{2.15}$$

Define dimensionless pressures

$$P_D = \frac{4\pi L (K_h K_v)^{1/2} (P_i - P)}{(\mu q B)}, \tag{2.16}$$

$$P_{wD} = \frac{4\pi L (K_h K_v)^{1/2} (P_i - P_w)}{(\mu q B)}. \tag{2.17}$$

Note that if c is a positive constant, there holds [9]

$$\delta(cx) = \frac{\delta(x)}{c}, \tag{2.18}$$

consequently, (2.5) becomes [8, 9]

$$\frac{\partial^2 P_D}{\partial x D^2} + \frac{\partial^2 P_D}{\partial y D^2} + \frac{\partial^2 P_D}{\partial z D^2} = \frac{\partial P_D}{\partial t_D} - 8\pi \delta(x_D - R_{0D}) \delta(y_D) \delta(z_D - z'_D), \quad \text{in } \Omega_D, \tag{2.19}$$

where

$$\Omega_D = \left\{ (x_D, y_D, z_D) \mid x_D^2 + y_D^2 < R_{eD}^2,\ 0 < z_D < H_D \right\}. \tag{2.20}$$

If point $\mathbf{r}_0$ and point $\mathbf{r}$ are with distances ρ_0 and ρ, respectively, from the circular center, then the dimensionless off-center distances are

$$\rho_{0D} = \frac{2\rho_0}{L}, \qquad \rho_D = \frac{2\rho}{L}, \tag{2.21}$$

There holds

$$\begin{aligned} \left(\frac{\pi}{H_D}\right)\left(2R_{eD} - \rho_{0D} - \rho_D - \sqrt{\rho_{0D}\rho_D}\right) &= \left(\frac{K_v}{K_h}\right)^{1/2} \left(\frac{\pi L}{2H}\right) \left(\frac{4R_e}{L} - \frac{2\rho_0}{L} - \frac{2\rho}{L} - \frac{2\sqrt{\rho_0 \rho}}{L}\right) \\ &= \left(\frac{K_v}{K_h}\right)^{1/2} \left(\frac{\pi R_e}{H}\right) \left(2 - \frac{\rho_0}{R_e} - \frac{\rho}{R_e} - \frac{\sqrt{\rho_0 \rho}}{R_e}\right) \\ &= \left(\frac{K_v}{K_h}\right)^{1/2} \left(\frac{\pi R_e}{H}\right) \left(2 - \vartheta_0 - \vartheta - \sqrt{\vartheta_0 \vartheta}\right), \end{aligned} \tag{2.22}$$

where

$$\vartheta_0 = \frac{\rho_0}{R_e}, \qquad \vartheta = \frac{\rho}{R_e}. \tag{2.23}$$

Since the reservoir is with constant pressure outer boundary (edge water), in order to delay water encroachment, a producing well must keep a sufficient distance from the outer boundary. Thus in this paper, it is reasonable to assume

$$\vartheta_0 \le 0.6, \qquad \vartheta \le 0.6. \tag{2.24}$$

If

$$\vartheta_0 = \vartheta = 0.6, \qquad \frac{K_v}{K_h} = 0.25, \qquad \frac{R_e}{H} = 15 \tag{2.25}$$

then

$$\begin{gathered}\left(\frac{K_v}{K_h}\right)^{1/2}\left(\frac{\pi R_e}{H}\right)\left(2-\vartheta_0-\vartheta-\sqrt{\vartheta_0\vartheta}\right)= 0.25^{1/2}\times(\pi\times 15)\times\left(2.0-0.6-0.6-\sqrt{0.6\times 0.6}\right),\\ \exp(-4.7124) = 8.983\times 10^{-3};\end{gathered} \tag{2.26}$$

and if

$$\vartheta_0 = \vartheta = 0.5, \qquad \frac{K_v}{K_h} = 0.5, \qquad \frac{R_e}{H} = 10, \tag{2.27}$$

then

$$\begin{gathered}\left(\frac{K_v}{K_h}\right)^{1/2}\left(\frac{\pi R_e}{H}\right)\left(2-\vartheta_0-\vartheta-\sqrt{\vartheta_0\vartheta}\right)= 0.5^{1/2}\times(\pi\times 10)\times\left(2.0-0.5-0.5-\sqrt{0.5\times 0.5}\right) = 11.107,\\ \exp(-11.107) = 1.501\times 10^{-5}.\end{gathered} \tag{2.28}$$

Recall (2.22), according to the above calculations, without losing generality, there holds

$$\exp\left[-\left(\frac{\pi}{H_D}\right)\left(2R_{eD}-\rho_{0D}-\rho_D-\sqrt{\rho_{0D}\rho_D}\right)\right] \approx 0. \tag{2.29}$$

In the same manner, we have

$$\exp\left[-\left(\frac{\pi}{H_D}\right)\left(2R_{eD}-\rho_{0D}-\rho_D\right)\right] \approx 0. \tag{2.30}$$

3. Point Sink Solution

For convenience in the following reference, we use dimensionless transforms given by (2.10) through (2.17), every variable, domain, initial and boundary conditions below should be taken as dimensionless, but we drop the subscript D.

Thus, if the point sink is at $(x', 0, z')$, (2.19) can be written as

$$\frac{\partial P}{\partial t} - \Delta P = 8\pi\delta(x - x')\delta(y)\delta(z - z'), \quad \text{in } \Omega, \tag{3.1}$$

where

$$\begin{gathered}\Omega = \left\{(x, y, z) \mid x^2 + y^2 < R_e^2,\ 0 < z < H\right\}, \\ \Delta P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}.\end{gathered} \tag{3.2}$$

The equation of initial condition is changed to

$$P(t, x, y, z)|_{t=0} = 0, \quad \text{in } \Omega. \tag{3.3}$$

The equation of lateral boundary condition is changed to

$$P(t, x, y, z) = 0, \quad \text{on } \Gamma, \tag{3.4}$$

where

$$\Gamma = \left\{(x, y, z) \mid x^2 + y^2 = R_e^2,\ 0 < z < H\right\}. \tag{3.5}$$

The problem under consideration is that of fluid flow toward a point sink from an off-center position within a circular of radius R_e. We want to determine the pressure change at an observation point with a distance ρ from the center of circle.

Figure 2 is a geometric representation of the system. In Figure 2, the point sink $\mathbf{r}_0$ and the observation point $\mathbf{r}$, are with distances ρ_0 and ρ, respectively, from the circular center; and the two points are separated at the center by an angle θ. The inverse point of the point sink $\mathbf{r}_0$ with respect to the circle is point $\mathbf{r}_*$. Point $\mathbf{r}_*$ with a distance ρ_* from the center, and ρ_1 from the observation point. The inverse point is the point outside the circle, on the extension of the line connecting the center and the point sink, and such that

$$\rho_* = \frac{R_e^2}{\rho_0}. \tag{3.6}$$

Assume R' is the distance between point $\mathbf{r}$ and point $\mathbf{r}_0$, then [9, 10]

$$R' = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\theta}. \tag{3.7}$$

If the observation point $\mathbf{r}$ is on the drainage circle, $\rho = R_e$, then

$$R' = \sqrt{R_e^2 + \rho_0^2 - 2R_e\rho_0 \cos\theta}, \quad R_e > \rho_0 > 0. \tag{3.8}$$

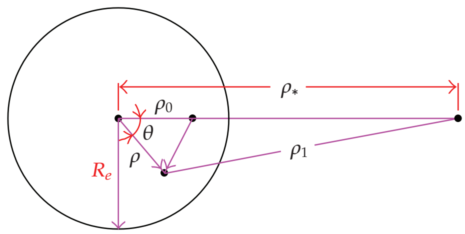

Figure 2: Geometric representation of a circular system.

If the observation point $\mathbf{r}$ is on the wellbore, then

$$R' = R_w. \tag{3.9}$$

Recall (2.9), obviously for impermeable upper and lower boundary conditions, there holds [9, 10]

$$\delta(z - z') = \sum_{k=0}^{\infty} \cos\left(\frac{k\pi z'}{H}\right) \cos\frac{k\pi z}{H} / (H d_k), \tag{3.10}$$

where

$$d_k = \begin{cases} 1, & \text{if } k = 0, \\ \dfrac{1}{2}, & \text{if } k > 0. \end{cases} \tag{3.11}$$

Let

$$P(t; x, y, z; x', y', z') = \sum_{k=0}^{\infty} \varphi_k(t, x, y) \cos\left(\frac{k\pi z}{H}\right), \tag{3.12}$$

and substitute (3.12) into (3.1) and compare the coefficients of $\cos(k\pi z/H)$, we obtain

$$\frac{\partial \varphi_k}{\partial t} + \lambda_k^2 \varphi_k - \left(\frac{\partial^2 \varphi_k}{\partial x^2} + \frac{\partial^2 \varphi_k}{\partial y^2}\right) = 8\pi \cos\left(\frac{k\pi z'}{H}\right) \delta(x - x') \delta(y) / (H d_k) \tag{3.13}$$

in circular $\Omega_1 = \{(x, y) \mid x^2 + y^2 < R_e^2\}$, and

$$\varphi_k = 0, \tag{3.14}$$

on circumference $\Gamma_1 = \{(x, y) \mid x^2 + y^2 = R_e^2\}$, and

$$\varphi_k|_{t=0} = 0, \tag{3.15}$$

where

$$\lambda_k = \frac{k\pi}{H}. \tag{3.16}$$

Taking the Laplace transform at the both sides of (3.13), then

$$\left(\frac{\partial^2 \widehat{\varphi_k}}{\partial x^2} + \frac{\partial^2 \widehat{\varphi_k}}{\partial y^2}\right) - \left(s + \lambda_k^2\right)\widehat{\varphi_k} = \frac{\alpha_k \delta(x - x')\delta(y)}{s}, \quad \text{in } \Omega_1, \tag{3.17}$$

$$\widehat{\varphi_k} = 0, \quad \text{on } \Gamma_1, \tag{3.18}$$

where

$$\alpha_k = \left(\frac{-8\pi}{Hd_k}\right)\cos\left(\frac{k\pi z'}{H}\right), \tag{3.19}$$

and s is Laplace transform variable.

Define

$$\beta_k = \left(\frac{1}{2\pi}\right)\alpha_k = \left(\frac{-4}{Hd_k}\right)\cos\left(\frac{k\pi z'}{H}\right). \tag{3.20}$$

Case 1. If $k = 0$, then

$$\frac{\partial^2 \widehat{\varphi_0}}{\partial x^2} + \frac{\partial^2 \widehat{\varphi_0}}{\partial y^2} - s\widehat{\varphi_0} = \frac{\alpha_0 \delta(x - x')\delta(y)}{s}, \quad \text{in } \Omega_1, \tag{3.21}$$

where

$$\begin{gathered} \alpha_0 = \left(\frac{-8\pi}{H}\right), \\ \widehat{\varphi_0} = 0, \quad \text{on } \Gamma_1. \end{gathered} \tag{3.22}$$

Case 2. If $k > 0$, then $\widehat{\varphi_k}$ satisfies (3.17).

Define

$$\zeta_k = \sqrt{\lambda_k^2 + s}. \tag{3.23}$$

Recall (3.8), and $[-\beta_k/s]K_0(\zeta_k R')$ is a basic solution of (3.17), since $k > 0$, we have

$$\begin{gathered} \alpha_k = \left(\frac{-16\pi}{H}\right)\cos\left(\frac{k\pi z'}{H}\right), \\ \beta_k = \left(\frac{-8}{H}\right)\cos\left(\frac{k\pi z'}{H}\right), \end{gathered} \tag{3.24}$$

so let

$$\widehat{\psi_k} = \widehat{\varphi_k} + \widehat{\mu_k}, \tag{3.25}$$

where

$$\widehat{\mu_k} = \frac{\beta_k K_0[\zeta_k R']}{s}, \tag{3.26}$$

thus

$$\widehat{\varphi_k} = \widehat{\psi_k} - \widehat{\mu_k}, \tag{3.27}$$

and $\widehat{\psi_k}$ satisfies homogeneous equation:

$$\begin{gathered} \frac{\partial^2 \widehat{\psi_k}}{\partial x^2} + \frac{\partial^2 \widehat{\psi_k}}{\partial y^2} - \left(s + \lambda_k^2\right)\widehat{\psi_k} = 0, \quad \text{in } \Omega_1, \\ \widehat{\psi_k} = \frac{\beta_k K_0\left(\sqrt{s + \lambda_k^2} R'\right)}{s}, \quad \text{on } \Gamma_1, \end{gathered} \tag{3.28}$$

R' has the same meaning as in (3.8).

Under polar coordinates representation of Laplace operator and by using methods of separation of variables, we obtain a general solution [11–13]:

$$\begin{aligned} \widehat{\psi_k}(s, x, y; s, x', 0) = {} & \left[A_{0k} I_0(\zeta_k \rho) + B_{0k} K_0(\zeta_k \rho)\right]\left[a_{0k}\theta + b_{0k}\right] \\ & + \sum_{m=1}^{\infty}\left[A_{mk} I_m(\zeta_k \rho) + B_{mk} K_m(\zeta_k \rho)\right]\left[a_{mk}\cos(m\theta) + b_{mk}\sin(m\theta)\right], \end{aligned} \tag{3.29}$$

where $A_{ik}, B_{ik}, a_{ik}, b_{ik}$, $i = 0, 1, 2, \ldots$, are undetermined coefficients.

Because $\widehat{\psi_k}(s, x, y; s, x', 0)$ is continuously bounded within Ω_1, but $K_i(0) = \infty$, there holds

$$B_{ik} = 0, \quad i = 0, 1, 2, \ldots. \tag{3.30}$$

There hold [9, 10]

$$\begin{gathered} K_\upsilon(z) = \left(\frac{\pi i}{2}\right) e^{\upsilon \pi i/2} H_\upsilon^{(1)}(zi), \\ I_\upsilon(z) = e^{-\upsilon \pi i/2} J_\upsilon(zi), \end{gathered} \tag{3.31}$$

where $K_\upsilon(z)$ is modified Bessel function of second kind and order υ, $I_\upsilon(z)$ is modified Bessel function of first kind and order υ, $J_\upsilon(z)$ is Bessel function of first kind and order υ, $H_\upsilon^{(1)}(z)$ is Hankel function of first kind and order υ, and $i = \sqrt{-1}$.

And there hold (see [14, page 979])

$$H_0^{(1)}(\sigma R') = J_0(\sigma\rho_0)H_0^{(1)}(\sigma R_e) + 2\sum_{m=1}^{\infty} J_m(\sigma\rho_0)H_m^{(1)}(\sigma R_e)\cos(m\theta), \tag{3.32}$$

$$K_0(\zeta_k R') = \left(\frac{\pi i}{2}\right)H_0^{(1)}(i\zeta_k R'). \tag{3.33}$$

Let $\sigma = i\zeta_k$, (note that $i^2 = -1$), substituting (3.31) into (3.32) and using (3.33), we have the following Cosine Fourier expansions of $K_0(\zeta_k R')$ (see [14, page 952]):

$$\begin{aligned} K_0(\zeta_k R') &= \left(\frac{\pi i}{2}\right)\left[J_0(i\zeta_k\rho_0)H_0^{(1)}(i\zeta_k R_e) + 2\sum_{m=1}^{\infty} J_m(i\zeta_k\rho_0)H_m^{(1)}(i\zeta_k R_e)\cos(m\theta)\right] \\ &= J_0(i\zeta_k\rho_0)K_0(\zeta_k R_e) + 2\sum_{m=1}^{\infty} e^{-m\pi i/2} J_m(i\zeta_k\rho_0)K_m(\zeta_k R_e)\cos(m\theta) \\ &= I_0(\zeta_k\rho_0)K_0(\zeta_k R_e) + 2\sum_{m=1}^{\infty} I_m(\zeta_k\rho_0)K_m(\zeta_k R_e)\cos(m\theta). \end{aligned} \tag{3.34}$$

So, we obtain

$$\frac{\beta_k K_0(\zeta_k R')}{s} = \frac{\beta_k\left[I_0(\zeta_k\rho_0)K_0(\zeta_k R_e) + 2\sum_{m=1}^{\infty} I_m(\zeta_k\rho_0)K_m(\zeta_k R_e)\cos(m\theta)\right]}{s}. \tag{3.35}$$

Note that $\widehat{\varphi_k} = \beta_k K_0(\zeta_k R')/s$ on Γ_1, and comparing coefficients of Cosine Fourier expansions of $K_0(\zeta_k R')/s$ in (3.35) and (3.29), we obtain

$$a_{0k} = 0, \qquad b_{0k} = 1, \qquad b_{ik} = 0, \quad i = 1, 2, \ldots. \tag{3.36}$$

Define

$$\Upsilon_{mk} = a_{mk}A_{mk}, \quad k = 0, 1, 2, \ldots \tag{3.37}$$

and recall (3.29), then we have

$$\widehat{\varphi_k}(s, x, y; s, x', 0) = \sum_{m=0}^{\infty} \Upsilon_{mk} I_m(\zeta_k\rho)\cos(m\theta), \quad k = 0, 1, 2, \ldots, \tag{3.38}$$

where

$$\Upsilon_{0k} = \frac{\beta_k K_0(\zeta_k R_e)I_0(\zeta_k\rho_0)}{sI_0(\zeta_k R_e)}, \tag{3.39}$$

$$\Upsilon_{mk} = \frac{2\beta_k K_m(\zeta_k R_e)I_m(\zeta_k\rho_0)}{sI_m(\zeta_k R_e)}. \tag{3.40}$$

In the appendix, we can prove

$$\sum_{k=1}^{\infty}\left|\widehat{\varphi_k}\cos\left(\frac{k\pi z}{H}\right)\right| \approx 0. \tag{3.41}$$

Thus we only consider the case $k = 0$, in (3.38) and (3.40), let $k = 0$, we have

$$\widehat{\varphi_0}(s, x, y; s, x', 0) = \sum_{m=0}^{\infty} \Upsilon_{m0} I_m(\zeta_0\rho)\cos(m\theta), \tag{3.42}$$

where

$$\zeta_0 = \sqrt{s}, \tag{3.43}$$

$$\Upsilon_{m0} I_m(\zeta_0\rho) = \frac{\beta_0 K_m(\sqrt{s}R_e) I_m(\sqrt{s}\rho_0) I_m(\sqrt{s}\rho)}{s I_m(\sqrt{s}R_e)} = f_{1m}(s) \times f_{2m}(s), \tag{3.44}$$

where

$$f_{1m}(s) = \left(\frac{\beta_0 R_e^m}{2^{1+m}}\right)\left[\frac{2^{1+m}s^{m/2}K_m(\sqrt{s}R_e)}{R_e^m}\right], \quad m = 0, 1, 2, \ldots, \tag{3.45}$$

$$f_{2m}(s) = \frac{I_m(\sqrt{s}\rho_0) I_m(\sqrt{s}\rho)}{s^{m/2+1} I_m(\sqrt{s}R_e)}, \quad m = 0, 1, 2, \ldots. \tag{3.46}$$

And there holds

$$\mathcal{L}^{-1}\{f_{1m}(s)\} = \left(\frac{\beta_0 R_e^m}{2^{1+m}}\right)\left[\frac{\exp\left(-\left(R_e^2/4t\right)\right)}{t^{m+1}}\right], \quad m = 0, 1, 2, \ldots, \tag{3.47}$$

where $\mathcal{L}^{-1}$ is Inverse Laplace transform operator.

Since $s = 0$, $s = -\gamma_{mn}$ are simple poles of meromorphic function $f_{2m}(s)$, if using partial fraction expansion of meromorphic function, there holds [15]

$$f_{2m}(s) = \frac{B_{m0}}{s} + \sum_{n=1}^{\infty}\frac{B_{mn}}{s + \gamma_{mn}}, \tag{3.48}$$

where B_{m0}, B_{mn} are residues at poles $s = 0$, $s = -\gamma_{mn}$, respectively, and

$$\begin{aligned} B_{m0} &= \frac{(\rho_0\rho)^m}{2^m m! R_e^m}, \\ \gamma_{mn} &= \frac{\varepsilon_{mn}^2}{R_e^2}, \end{aligned} \tag{3.49}$$

ε_{mn} is the nth root of equation $J_m(x) = 0$.

From (3.48), we have

$$\mathcal{L}^{-1}\{f_{2m}(s)\} = B_{m0} + \sum_{n=1}^{\infty} B_{mn}\exp(-\gamma_{mn}t) = \sum_{n=0}^{\infty} B_{mn}\exp(-\gamma_{mn}t), \tag{3.50}$$

where $\gamma_{m0} = 0$.

According to the convolution theorem [12], from (3.44), there holds

$$\begin{aligned}\mathcal{L}^{-1}\{f_{1m}(s)\times f_{2m}(s)\} &= \left(\frac{\beta_0 R_e^m}{2^{1+m}}\right)\left\{\sum_{n=0}^{\infty} B_{mn}\int_0^t \left[\frac{\exp(-(R_e^2/4\tau))}{\tau^{m+1}}\right]\exp\left[-\gamma_{mn}(t-\tau)\right]d\tau\right\}\\ &= C_{m0} + D_m,\end{aligned} \tag{3.51}$$

where

$$\begin{aligned}C_{m0} &= \left(\frac{\beta_0 R_e^m B_{m0}}{2^{1+m}}\right)\int_0^t \left[\frac{\exp\left(-\left(R_e^2/4\tau\right)\right)}{\tau^{m+1}}\right]d\tau\\ &= \left[\frac{\beta_0(\rho_0\rho)^m}{m!\times 2^{1+2m}}\right]\int_0^t \left[\frac{\exp\left(-\left(R_e^2/4\tau\right)\right)}{\tau^{m+1}}\right]d\tau,\\ D_m &= \left(\frac{\beta_0 R_e^m}{2^{1+m}}\right)\left\{\sum_{n=1}^{\infty} B_{mn}\int_0^t \left[\frac{\exp\left(-\left(R_e^2/4\tau\right)\right)}{\tau^{m+1}}\right]\exp\left[-\gamma_{mn}(t-\tau)\right]d\tau\right\}.\end{aligned} \tag{3.52}$$

Recall (3.38), (3.44), and (3.51), there holds

$$\begin{aligned}\psi_0(t,x,y;x',0) &= \mathcal{L}^{-1}\{\widehat{\psi_0}(s,x,y;x',0)\}\\ &= \sum_{m=0}^{\infty}\mathcal{L}^{-1}\{f_{1m}(s)\times f_{2m}(s)\}\cos(m\theta)\\ &= \sum_{m=0}^{\infty}\left(C_{m0}+D_m\right)\cos(m\theta).\end{aligned} \tag{3.53}$$

Using Laplace asymptotic integration (see [16, page 221]), when γ_{mn} is sufficiently large, then

$$\int_0^t \left[\frac{\exp\left(-\left(R_e^2/4\tau\right)\right)}{\tau^{m+1}}\right]\exp\left[-\gamma_{mn}(t-\tau)\right]d\tau \approx \frac{\exp\left(-\left(R_e^2/4\tau\right)\right)}{\gamma_{mn}t^{m+1}}, \tag{3.54}$$

therefore,

$$D_m = \left[\frac{\beta_0 R_e^m \exp\left(-\left(R_e^2/4t\right)\right)}{2^{1+m}t^{m+1}}\right]\sum_{n=1}^{\infty}\frac{B_{mn}}{\gamma_{mn}}. \tag{3.55}$$

There holds [9]

$$
\begin{aligned}
I_m(x) &= \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!}\left(\frac{x}{2}\right)^{m+2k} \\
&= \left[\left(\frac{1}{m!}\right)\left(\frac{x}{2}\right)^{m}\right]\left\{1+\frac{m!}{(m+1)!}\left(\frac{x}{2}\right)^{2}+\left[\frac{m!}{2!(m+2)!}\right]\left(\frac{x}{2}\right)^{4}+\cdots\right\}.
\end{aligned}
\tag{3.56}
$$

Using (3.48) and (3.56), and note that

$$
\frac{1}{1+x} = 1-x+x^2-x^3+O(x^3),
\tag{3.57}
$$

so (3.46) can be written as

$$
\begin{aligned}
&f_{2m}(s) \\
&\quad= \frac{\left\{(1/m!)\left(\sqrt{s}\rho_0/2\right)^{m}\left[1+\mathscr{A}\left(\sqrt{s}\rho_0/2\right)^{2}+\cdots\right]\right\}\left\{(1/m!)\left(\sqrt{s}\rho/2\right)^{m}\left[1+\mathscr{A}\left(\sqrt{s}\rho/2\right)^{2}+\cdots\right]\right\}}{s^{1+m/2}\left\{(1/m!)\left(\sqrt{s}R_e/2\right)^{m}\left[1+(1/(m+1))\left(\sqrt{s}R_e/2\right)^{2}+\cdots\right]\right\}} \\
&\quad= \frac{B_{m0}}{s}\left[1+\frac{\rho_0^2 s}{4(m+1)}+\cdots\right]\left[1+\frac{\rho^2 s}{4(m+1)}+\cdots\right]\left[1-\frac{R_e^2 s}{4(m+1)}+\cdots\right] \\
&\quad= \frac{B_{m0}}{s}+\left(\frac{1}{m!}\right)\left(\frac{\rho_0\rho}{2R_e}\right)^{m}\left[\frac{1}{4(m+1)}\left(\rho_0^2+\rho^2-R_e^2\right)+O(s)\right],
\end{aligned}
\tag{3.58}
$$

where $\mathscr{A}$ denotes $(m!/(m+1)!)$, thus from (3.48), we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{B_{mn}}{\gamma_{mn}} &= \lim_{s\to 0}\left[f_{2m}(s)-\frac{B_{m0}}{s}\right] \\
&= \left[\frac{1}{4(m+1)!}\right]\left(\frac{\rho_0\rho}{2R_e}\right)^{m}\left(\rho_0^2+\rho^2-R_e^2\right),
\end{aligned}
\tag{3.59}
$$

therefore,

$$
\begin{aligned}
D_m &= \left[\frac{\beta_0 R_e^m \exp\left(-\left(R_e^2/4t\right)\right)}{2^{1+m}t^{m+1}}\right]\left[\frac{\left(\rho_0^2+\rho^2-R_e^2\right)}{4(m+1)!}\right]\left(\frac{\rho_0\rho}{2R_e}\right)^{m} \\
&= \left[\frac{\beta_0 \exp\left(-\left(R_e^2/4t\right)\right)}{8t(m+1)!}\right]\left(\rho_0^2+\rho^2-R_e^2\right)\left(\frac{\rho_0\rho}{4t}\right)^{m}.
\end{aligned}
\tag{3.60}
$$

RP is defined as real part operator, for example, $\mathrm{RP}(e^{im\theta})$ means real part of $e^{im\theta}$,

$$
\mathrm{RP}(e^{im\theta}) = \cos(m\theta).
\tag{3.61}
$$

There holds

$$\exp\left(\frac{\rho_0\rho e^{i\theta}}{4\tau}\right)=\sum_{m=0}^{\infty}\left(\frac{1}{m!}\right)\left(\frac{\rho_0\rho e^{i\theta}}{4\tau}\right)^m, \tag{3.62}$$

and define

$$\eta=\frac{R_e^2}{4}-\left(\frac{\rho_0\rho e^{i\theta}}{4}\right), \tag{3.63}$$

η is a complex number.

Note that $\beta_0=-4/H$, recall (3.53), define

$$\begin{aligned}
\Lambda_1&=\sum_{m=0}^{\infty}C_{m0}\cos(m\theta)\\
&=\left(\frac{\beta_0}{2}\right)\left\{\int_0^t\left[\frac{\exp(-(R_e^2/4\tau))}{\tau}\right]\sum_{m=0}^{\infty}\left[\frac{\cos(m\theta)}{m!}\left(\frac{\rho_0\rho}{4\tau}\right)^m\right]d\tau\right\}\\
&=\left(\frac{\beta_0}{2}\right)\times RP\left\{\int_0^t\left[\frac{\exp(-(R_e^2/4\tau))}{\tau}\right]\exp\left(\frac{\rho_0\rho e^{i\theta}}{4\tau}\right)d\tau\right\}\\
&=\left(\frac{\beta_0}{2}\right)\times RP\left\{\int_0^t\left[\frac{\exp(-(\eta/\tau))}{\tau}\right]d\tau\right\}\\
&=-\left(\frac{\beta_0}{2}\right)\times RP\left\{Ei\left(-\frac{\eta}{t}\right)\right\}\\
&=\left(\frac{2}{H}\right)\times RP\left\{Ei\left(-\frac{\eta}{t}\right)\right\}.
\end{aligned} \tag{3.64}$$

In (3.60), let

$$\chi=\frac{\rho_0\rho}{4t}, \tag{3.65}$$

and define

$$\begin{aligned}
\Lambda_2&=\sum_{m=0}^{\infty}D_m\cos(m\theta)\\
&=\left[\frac{\beta_0\exp(-(R_e^2/4t))}{8t}\right]\left(\rho_0^2+\rho^2-R_e^2\right)\sum_{m=0}^{\infty}\left[\frac{\cos(m\theta)}{(m+1)!}\left(\frac{\rho_0\rho}{4t}\right)^m\right]\\
&=\left[\frac{\beta_0\exp(-(R_e^2/4t))}{8t}\right]\left(\rho_0^2+\rho^2-R_e^2\right)\sum_{m=0}^{\infty}RP\left\{\frac{1}{(m+1)!}\left(\frac{\rho_0\rho e^{i\theta}}{4t}\right)^m\right\}\\
&=-\left[\frac{\exp(-(R_e^2/4t))}{2tH}\right]\left(\rho_0^2+\rho^2-R_e^2\right)\times RP\left\{\left(\frac{1}{\chi e^{i\theta}}\right)\left[\exp\left(\chi e^{i\theta}\right)-1\right]\right\},
\end{aligned} \tag{3.66}$$

thus we obtain

$$\varphi_0 = \mathcal{L}^{-1}\{\widehat{\varphi_0}\} = \Lambda_1 + \Lambda_2, \tag{3.67}$$

and there holds [9, 10]

$$\mathcal{L}^{-1}\left\{\frac{K_0(a\sqrt{s})}{s}\right\} = -\frac{1}{2}Ei\left(-\frac{a^2}{4t}\right), \tag{3.68}$$

thus if we recall (3.26) and define

$$\Lambda_3 = -\mu_0 = -\mathcal{L}^{-1}\{\widehat{\mu_0}\}, \tag{3.69}$$

then

$$\Lambda_3 = -\mathcal{L}^{-1}\left[\frac{\beta_0 K_0(\sqrt{s}R')}{s}\right] = -\left(\frac{2}{H}\right)Ei\left(-\frac{R'^2}{4t}\right) \tag{3.70}$$

and R' has the same meaning as in (3.7).

In the above equations, $Ei(-x)$ is exponential integral function,

$$Ei(-x) = \int_{-\infty}^{-x}\frac{\exp(u)}{u}du, \quad (0 < x < \infty). \tag{3.71}$$

Recall (3.27), there holds

$$\varphi_0(t,x,y;x',0) = \varphi_0 - \mu_0 = \Lambda_1 + \Lambda_2 + \Lambda_3. \tag{3.72}$$

Combining (3.12), (3.26), (3.27), and (3.41), we obtain

$$\begin{aligned} P(t;x,y,z;x',y',z') &= \varphi_0 + \mathcal{L}^{-1}\left\{\sum_{k=1}^{\infty}\widehat{\varphi_k}\cos\left(\frac{k\pi z}{H}\right)\right\} \\ &= \varphi_0 + \mathcal{L}^{-1}\left\{\sum_{k=1}^{\infty}(\widehat{\varphi_k} - \widehat{\mu_k})\cos\left(\frac{k\pi z}{H}\right)\right\} \\ &= \varphi_0 - \sum_{k=1}^{\infty}\cos\left(\frac{k\pi z}{H}\right)\mathcal{L}^{-1}\left\{\frac{\beta_k K_0\left(\sqrt{s+\lambda_k^2}R'\right)}{s}\right\}. \end{aligned} \tag{3.73}$$

Equation (3.73) is the pressure distribution equation of an off-center point sink in the cylindrical body. If the point sink $\mathbf{r}_0$ and the observation point $\mathbf{r}$ are not on a radius of the drainage circle, $\theta \neq 0$, recall (3.7), R' cannot be simplified, we cannot obtain exact inverse Laplace transform of (3.73), but if necessary, we may obtain numerical inverse Laplace transform results.

If the point sink is at the center of the drainage circle, then

$$\rho_0 = 0, \quad \eta = \frac{R_e^2}{4},$$
$$\varphi_0 = \left(\frac{2}{H}\right)\left[Ei\left(-\frac{R_e^2}{4t}\right) - Ei\left(-\frac{\rho^2}{4t}\right)\right] + \left[\frac{\exp(-(R_e^2/4t))}{2tH}\right](R_e^2 - \rho^2). \tag{3.74}$$

In Figure 2, if the point sink $\mathbf{r}_0$ and the observation point $\mathbf{r}$ are on a radius, then

$$\theta = 0, \quad \eta = \frac{R_e^2}{4} - \left(\frac{\rho_0\rho}{4}\right), \tag{3.75}$$

$$\varphi_0 = \left(\frac{2}{H}\right)\left\{Ei\left[-\left(\frac{R_e^2 - \rho\rho_0}{4t}\right)\right] - Ei\left[-\frac{(\rho-\rho_0)^2}{4t}\right]\right.$$
$$\left. - \left(\frac{\rho_0^2 + \rho^2 - R_e^2}{\rho\rho_0}\right)\left[\exp\left(\frac{\rho\rho_0 - R_e^2}{4t}\right) - \exp\left(-\frac{R_e^2}{4t}\right)\right]\right\}. \tag{3.76}$$

4. Uniform Line Sink Solution

Although the off-center partially penetrating vertical well is represented in the model by a line sink, we only concern in the pressures at the wellbore face.

For convenience, in the following reference, every variable below is dimensionless but we drop the subscript D.

The well line sink is located along the line $\{(x', 0, z) : L_1 \le z \le L_2\}$. If the observation point $\mathbf{r}$ is on the wellbore, $R' = R_w$, note that $R_0 \gg R_w$, and there hold

$$\theta = 0, \quad \rho = \rho_0 + R_w = R_0 + R_w,$$
$$\rho_0 = R_0, \qquad \rho - \rho_0 = R_w, \tag{4.1}$$
$$\rho_0\rho \approx R_0^2, \qquad \rho + \rho_0 \approx 2\rho_0 = 2R_0,$$

then

$$\eta = \frac{R_e^2}{4} - \frac{R_0^2}{4},$$
$$\chi = \frac{\rho\rho_0}{4t} \approx \frac{R_0^2}{4t}, \tag{4.2}$$

and recall (3.64), then

$$\Lambda_1(t; R_0, 0) = -\left(\frac{\beta_0}{2}\right)Ei\left[-\left(\frac{R_e^2 - R_0^2}{4t}\right)\right]. \tag{4.3}$$

Recall (3.66), then

$$\Lambda_2(t;R_0,0)=-\left[\frac{2\exp\left(-\left(R_e^2/4t\right)\right)}{HR_0^2}\right]\left(2R_0^2-R_e^2\right)\left[\exp\left(\frac{R_0^2}{4t}\right)-1\right], \tag{4.4}$$

and recall (3.70), then

$$\Lambda_3(t;R_0,0)=\left(\frac{\beta_0}{2}\right)Ei\left[-\frac{(\rho-\rho_0)^2}{4t}\right]=\left(\frac{\beta_0}{2}\right)Ei\left(-\frac{R_w^2}{4t}\right). \tag{4.5}$$

Define

$$\begin{aligned}
\Gamma_1&=\int_{L_1}^{L_2}\Lambda_1(t;R_0,0)dz'\\
&=\int_{L_1}^{L_2}\left[-\left(\frac{\beta_0}{2}\right)\right]Ei\left[-\left(\frac{R_e^2-R_0^2}{4t}\right)\right]dz'\\
&=-\left(\frac{\beta_0}{2}\right)(L_2-L_1)Ei\left[-\left(\frac{R_e^2-R_0^2}{4t}\right)\right]\\
&=\left(\frac{2L_{pr}}{H}\right)Ei\left[-\left(\frac{R_e^2-R_0^2}{4t}\right)\right],\\
\Gamma_2&=\int_{L_1}^{L_2}\Lambda_2(t;R_0,0)dz'\\
&=\int_{L_1}^{L_2}-\left[\frac{2\exp(-(R_e^2/4t))}{HR_0^2}\right]\left(2R_0^2-R_e^2\right)\left[\exp\left(\frac{R_0^2}{4t}\right)-1\right]dz'\\
&=-\left[\frac{2L_p\exp\left(-\left(R_e^2/4t\right)\right)}{HR_0^2}\right]\left(2R_0^2-R_e^2\right)\left[\exp\left(\frac{R_0^2}{4t}\right)-1\right]\\
&=-\left(\frac{2L_{pr}}{HR_0^2}\right)\left(2R_0^2-R_e^2\right)\left[\exp\left(\frac{R_0^2-R_e^2}{4t}\right)-\exp\left(-\frac{R_e^2}{4t}\right)\right],\\
\Gamma_3&=\int_{L_1}^{L_2}\Lambda_3(t;R_0,0)dz'\\
&=\left(\frac{\beta_0}{2}\right)\int_{L_1}^{L_2}Ei\left(-\frac{R_w^2}{4t}\right)dz'\\
&=-\left(\frac{2L_{pr}}{H}\right)Ei\left(-\frac{R_w^2}{4t}\right).
\end{aligned} \tag{4.6}$$

In order to calculate the pressure at the wellbore, using principle of potential superposition, integrating z' at both sides of (3.72) from L_1 to L_2, then

$$\begin{aligned}
\Psi_0(t) &= \int_{L_1}^{L_2} \varphi_0(t; R_0, 0)\,dz' \\
&= \Gamma_1 + \Gamma_2 + \Gamma_3 \\
&= \left(\frac{2L_{pr}}{H}\right) Ei\left[-\left(\frac{R_e^2 - R_0^2}{4t}\right)\right] \\
&\quad - \left(\frac{2L_{pr}}{HR_0^2}\right)\left(2R_0^2 - R_e^2\right)\left[\exp\left(\frac{R_0^2 - R_e^2}{4t}\right) - \exp\left(-\frac{R_e^2}{4t}\right)\right] \\
&\quad - \left(\frac{2L_{pr}}{H}\right) Ei\left(-\frac{R_w^2}{4t}\right) \\
&= \left(\frac{2L_{pr}}{H}\right)\left\{ Ei\left[-\left(\frac{R_e^2 - R_0^2}{4t}\right)\right] - Ei\left(-\frac{R_w^2}{4t}\right)\right. \\
&\qquad\qquad \left. - \left(\frac{1}{R_0^2}\right)\left(2R_0^2 - R_e^2\right)\left[\exp\left(\frac{R_0^2 - R_e^2}{4t}\right) - \exp\left(-\frac{R_e^2}{4t}\right)\right]\right\}.
\end{aligned} \tag{4.7}$$

Recall (3.26), and note that $\rho - \rho_0 = R_w$, we have

$$\widehat{\mu}_k = \frac{\beta_k K_0\left(R_w\sqrt{s + \lambda_k^2}\right)}{s}, \tag{4.8}$$

and define

$$\begin{aligned}
\widehat{\sigma}_k &= \int_{L_1}^{L_2} \widehat{\mu}_k\,dz' \\
&= \beta_k\left(\frac{1}{s}\right)\int_{L_1}^{L_2} K_0\left(R_w\sqrt{s + \lambda_k^2}\right)dz' \\
&= -\left(\frac{8}{Hs}\right) K_0\left(R_w\sqrt{s + \lambda_k^2}\right)\int_{L_1}^{L_2} \cos\left(\frac{k\pi z'}{H}\right)dz' \\
&= -\left(\frac{8}{k\pi s}\right) K_0\left(R_w\sqrt{s + \lambda_k^2}\right)\left[\sin\left(\frac{k\pi L_2}{H}\right) - \sin\left(\frac{k\pi L_1}{H}\right)\right],
\end{aligned} \tag{4.9}$$

because when s is very small, (time t is sufficiently long), there holds

$$K_0\left(R_w\sqrt{s + \lambda_k^2}\right) = K_0\left(R_w\lambda_k\sqrt{1 + s/\lambda_k^2}\right) \approx K_0(R_w\lambda_k), \tag{4.10}$$

so when time is sufficiently long,

$$\sigma_k = \mathcal{L}^{-1}\{\widehat{\sigma}_k\} = -\left(\frac{8}{k\pi}\right)K_0(R_w\lambda_k)\left[\sin\left(\frac{k\pi L_2}{H}\right) - \sin\left(\frac{k\pi L_1}{H}\right)\right]. \tag{4.11}$$

Recall (3.12), (3.24), (3.27), and (3.41), when time t is sufficiently long, define

$$\begin{aligned} U &= \sum_{k=1}^{\infty}\sigma_k \cos\left(\frac{k\pi z}{H}\right) \\ &= \sum_{k=1}^{\infty} -\left(\frac{8}{k\pi}\right)K_0(R_w\lambda_k)\left[\sin\left(\frac{k\pi L_2}{H}\right) - \sin\left(\frac{k\pi L_1}{H}\right)\right]\cos\left(\frac{k\pi z}{H}\right) \\ &= -\left(\frac{8}{\pi}\right)\sum_{n=1}^{\infty}\frac{K_0(n\pi R_w/H)}{n}\left[\sin\left(\frac{n\pi L_2}{H}\right) - \sin\left(\frac{n\pi L_1}{H}\right)\right]\cos\left(\frac{n\pi z}{H}\right). \end{aligned} \tag{4.12}$$

Therefore, the wellbore pressure at point $(R_0 + R_w, z)$ is

$$\begin{aligned} P(R_w, z) &= \int_{L_1}^{L_2} P(R_0 + R_w, 0, z, t; R_0, 0, z')dz' \\ &\approx \Psi_0(t) - U. \end{aligned} \tag{4.13}$$

Considering the bottom point of the well line sink, then $z = L_{pr}$, $L_1 = 0$, thus $L_2 = L_{pr}$, in this case, (4.12) reduces to

$$\begin{aligned} U &= -\left(\frac{8}{\pi}\right)\sum_{n=1}^{\infty}\frac{K_0(n\pi R_w/H)}{n}\sin\left(\frac{n\pi L_{pr}}{H}\right)\cos\left(\frac{n\pi z}{H}\right) \\ &= -\left(\frac{4}{\pi}\right)\sum_{n=1}^{\infty}\frac{K_0(n\pi R_w/H)}{n}\sin\left(\frac{2n\pi L_{pr}}{H}\right) \\ &= I_1 + I_2, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} I_1 &= -\left(\frac{4}{\pi}\right)\sum_{n=1}^{N}\frac{K_0(n\pi R_w/H)}{n}\sin\left(\frac{2n\pi L_{pr}}{H}\right), \\ I_2 &= -\left(\frac{4}{\pi}\right)\sum_{n=N+1}^{\infty}\frac{K_0(n\pi R_w/H)}{n}\sin\left(\frac{2n\pi L_{pr}}{H}\right), \\ N &= 4\left[\frac{H}{\pi R_w}\right], \end{aligned} \tag{4.15}$$

where $[H/\pi R_w]$ is the integer part of $H/\pi R_w$.

For I_2 it holds the following estimate:

$$
\begin{aligned}
|I_2| &= \left| \sum_{n=N+1}^{\infty} \left(\frac{4}{\pi}\right) \frac{K_0(n\pi R_w/H)}{n} \sin\left(\frac{2n\pi L_{pr}}{H}\right) \right| \\
&\leq \left| \sum_{n=N+1}^{\infty} \left(\frac{4}{\pi}\right) \frac{K_0(n\pi R_w/H)}{n} \right| \\
&\leq \int_N^{\infty} \left(\frac{4}{\pi}\right) \frac{K_0(4x/N)}{x} dx \\
&= \left(\frac{4}{\pi}\right) \int_4^{\infty} \frac{K_0(y)}{y} dy \\
&= 2.7 \times 10^{-3} \\
&\approx 0.
\end{aligned}
\tag{4.16}
$$

So, (4.14) reduces to

$$
U \approx I_1 = -\left(\frac{4}{\pi}\right) \sum_{n=1}^{N} \frac{K_0(n\pi R_w/H)}{n} \sin\left(\frac{2n\pi L_{pr}}{H}\right). \tag{4.17}
$$

Combining (4.7), (4.13), and (4.17), pressure at the bottom point of the producing portion is

$$
P(R_w, L_{pr}) = \Psi_0(t) + \left(\frac{4}{\pi}\right) \sum_{n=1}^{N} \frac{K_0(n\pi R_w/H)}{n} \sin\left(\frac{2n\pi L_{pr}}{H}\right). \tag{4.18}
$$

In order to obtain average wellbore pressure, recall (4.12) and (4.17), integrate both sides of (4.13) with respect to z from L_1 to L_2, then divided by L_{pr}, average wellbore pressure is obtained:

$$
\begin{aligned}
P_{a,w} &= \frac{1}{L_{pr}} \int_{L_1}^{L_2} P(R_w, z) dz \\
&\approx \Psi_0(t) + \left(\frac{8}{\pi}\right) \sum_{n=1}^{N} \frac{K_0(n\pi R_w/H)}{n} \left[\sin\left(\frac{n\pi L_2}{H}\right) - \sin\left(\frac{n\pi L_1}{H}\right) \right] \left[\frac{1}{L_{pr}} \int_{L_1}^{L_2} \cos\left(\frac{n\pi z}{H}\right) dz \right] \\
&= \Psi_0(t) + \left(\frac{8H}{\pi^2 L_{pr}}\right) \sum_{n=1}^{N} \frac{K_0(n\pi R_w/H)}{n^2} \left[\sin\left(\frac{n\pi L_2}{H}\right) - \sin\left(\frac{n\pi L_1}{H}\right) \right]^2 \\
&= \Psi_0(t) + \left(\frac{32H}{\pi^2 L_{pr}}\right) \sum_{n=1}^{N} \frac{K_0(n\pi R_w/H)}{n^2} \sin^2\left(\frac{n\pi L_{pr}}{2H}\right) \cos^2\left[\frac{n\pi (L_2 + L_1)}{2H}\right],
\end{aligned}
\tag{4.19}
$$

where we use

$$\frac{1}{L_{pr}}\int_{L_1}^{L_2}\cos\left(\frac{n\pi z}{H}\right)dz=\left(\frac{H}{n\pi L_{pr}}\right)\left[\sin\left(\frac{n\pi L_2}{H}\right)-\sin\left(\frac{n\pi L_1}{H}\right)\right]. \tag{4.20}$$

5. Dimensionless Wellbore Pressure Equations

Combining (4.7) and (4.19), the dimensionless average wellbore pressure of an off-center partially penetrating vertical well in a circular cylinder drainage volume is

$$\begin{aligned}P_{wD}=\left(\frac{2L_{prD}}{H_D}\right)\Bigg\{&Ei\left[-\left(\frac{R_{eD}^2-R_{0D}^2}{4t_D}\right)\right]-Ei\left(-\frac{R_{wD}^2}{4t_D}\right)\\&-\left(\frac{1}{R_{0D}^2}\right)\left(2R_{0D}^2-R_{eD}^2\right)\left[\exp\left(\frac{R_{0D}^2-R_{eD}^2}{4t_D}\right)-\exp\left(-\frac{R_{eD}^2}{4t_D}\right)\right]\Bigg\}+S_p,\end{aligned} \tag{5.1}$$

where

$$S_p=\left(\frac{32H_D}{\pi^2L_{prD}}\right)\sum_{n=1}^{N}\frac{K_0(n\pi R_{wD}/H_D)}{n^2}\sin^2\left(\frac{n\pi L_{prD}}{2H_D}\right)\cos^2\left[\frac{n\pi(L_{2D}+L_{1D})}{2H_D}\right], \tag{5.2}$$

$$N=4\left[\frac{H_D}{\pi R_{wD}}\right], \tag{5.3}$$

$[H_D/\pi R_{wD}]$ is the integer part of $H_D/\pi R_{wD}$.

Equation (5.1) is applicable to impermeable upper and lower boundaries and long after the time when pressure transient reaches the upper and lower boundaries. And S_p denotes pseudo-skin factor due to partial penetration.

If $L_{pr}=L=H$, the drilled well length is equal to formation thickness, for a fully penetrating well, $S_p=0$, (5.1) reduces to

$$\begin{aligned}P_{wD}=2\Bigg\{&Ei\left[-\left(\frac{R_{eD}^2-R_{0D}^2}{4t_D}\right)\right]-Ei\left(-\frac{R_{wD}^2}{4t_D}\right)\\&-\left(\frac{1}{R_{0D}^2}\right)\left(2R_{0D}^2-R_{eD}^2\right)\left[\exp\left(\frac{R_{0D}^2-R_{eD}^2}{4t_D}\right)-\exp\left(-\frac{R_{eD}^2}{4t_D}\right)\right]\Bigg\}.\end{aligned} \tag{5.4}$$

If the well is located at the center of the cylindrical body, then $x'=R_0=0$, there holds

$$\lim_{R_{0D}\to 0}\left[\frac{\left(2R_{0D}^2-R_{eD}^2\right)}{R_{0D}^2}\right]\left[\exp\left(\frac{R_{0D}^2-R_{eD}^2}{4t_D}\right)-\exp\left(-\frac{R_{eD}^2}{4t_D}\right)\right]=-\left(\frac{R_{eD}^2}{4t_D}\right)\exp\left(\frac{-R_{eD}^2}{4t_D}\right). \tag{5.5}$$

Thus, (5.1) reduces to

$$P_{wD} = \left(\frac{2L_{prD}}{H_D}\right)\left\{Ei\left[-\left(\frac{R_{eD}^2}{4t_D}\right)\right] - Ei\left(-\frac{R_{wD}^2}{4t_D}\right) + \left(\frac{R_{eD}^2}{4t_D}\right)\exp\left[-\left(\frac{R_{eD}^2}{4t_D}\right)\right]\right\} + S_p, \tag{5.6}$$

where S_p has the same meaning as in (5.2).

If the well is a fully penetrating well in an infinite reservoir, $R_e = \infty$, there holds

$$Ei\left(-\frac{R_{eD}^2}{4t_D}\right) = 0, \qquad \left(\frac{R_{eD}^2}{4t_D}\right)\exp\left(-\frac{R_{eD}^2}{4t_D}\right) = 0. \tag{5.7}$$

Thus, (5.6) reduces to

$$P_{wD} = -2Ei\left(-\frac{R_{wD}^2}{4t_D}\right). \tag{5.8}$$

Substitute (2.12) and (2.15) into (2.17), then simplify and rearrange the resulting equation, we obtain

$$P_i - P_w = \left(\frac{\mu QB}{8\pi K_h L_{pr}}\right)P_{wD}, \tag{5.9}$$

where Q is total flow rate of the well, and P_{wD} can be calculated by (5.1), (5.4), (5.6), and (5.8) for different cases.

During production, the unsteady state pressure drop of an off-center partially penetrating vertical well in a circular cylinder drainage volume can be calculated by (5.9).

6. Examples and Discussions

Recall (5.2), pseudo-skin factor due to partial penetration S_p is a function of L_1, L_2 and H is not a function of well off-center distance R_0 or drainage radius R_e.

For an isotropic reservoir, (5.2) reduces to

$$S_p = \left(\frac{32H}{\pi^2 L_{pr}}\right)\sum_{n=1}^{N}\frac{K_0(n\pi R_w/H)}{n^2}\sin^2\left(\frac{n\pi L_{pr}}{2H}\right)\cos^2\left[\frac{n\pi(L_1+L_2)}{2H}\right], \tag{6.1}$$

and (5.3) reduces to

$$N = 4\left[\frac{H}{\pi R_w}\right], \tag{6.2}$$

$[H/\pi R_w]$ is the integer part of $H/\pi R_w$.

If we define

$$f_1 = \frac{L_1}{H}, \qquad f_2 = \frac{L_2}{H}, \qquad f_3 = \frac{R_w}{H}, \tag{6.3}$$

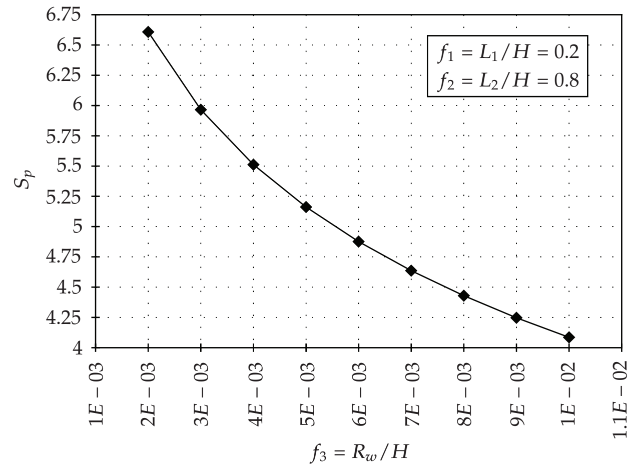

Figure 3: Pseudo-skin factor versus R_w/H plot.

then (6.1) can be written as

$$S_p = \left[\frac{32}{\pi^2(f_2 - f_1)}\right]\sum_{n=1}^{N}\frac{K_0(n\pi f_3)}{n^2}\sin^2\left[\frac{n\pi}{2}(f_2 - f_1)\right]\cos^2\left[\frac{n\pi}{2}(f_2 + f_1)\right]. \tag{6.4}$$

Example 6.1. Equation (6.4) shows that pseudo-skin factor S_p is a function of the three parameters f_1, f_2, f_3, fix two parameters, and generate plots that show the trend of S_p with the third parameter.

Solution

Case 1. Figure 3 shows the trend of S_p with f_3 when $f_1 = 0.2,\ f_2 = 0.8$, it can be found that S_p is a weak decreasing function of f_3.

Case 2. Figure 4 shows the trend of S_p with f_1 when $f_2 = 0.9,\ f_3 = 0.002$, it can be found that S_p is an increasing function of f_1. When f_2 is a constant, we may assume H is a constant, then L_2 is also a constant; when f_1 increases, L_1 also increases, thus the well producing length $L_{pr} = L_2 - L_1$ decreases, and pseudo-skin factor due to partial penetration increases.

Case 3. Figure 5 shows the trend of S_p with f_2 when $f_1 = 0.1,\ f_3 = 0.002$, it can be found that S_p is a decreasing function of f_2. When f_1 is a constant, we may assume H is a constant, then L_1 is also a constant; when f_2 increases, L_2 also increases, thus the well producing length $L_{pr} = L_2 - L_1$ increases, and pseudo-skin factor due to partial penetration decreases.

Example 6.2. A fully penetrating off-center vertical well, if

$$R_{eD} = 20, \qquad R_{wD} = 0.01, \tag{6.5}$$

compare the wellbore pressure responses when $R_{oD} = 5, 10, 15$.

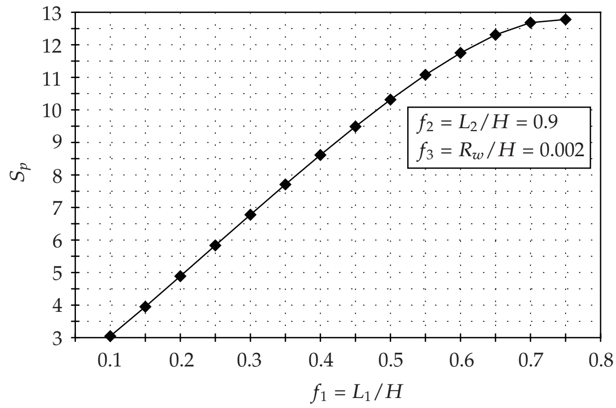

Figure 4: Pseudo-skin factor versus L_1/H plot.

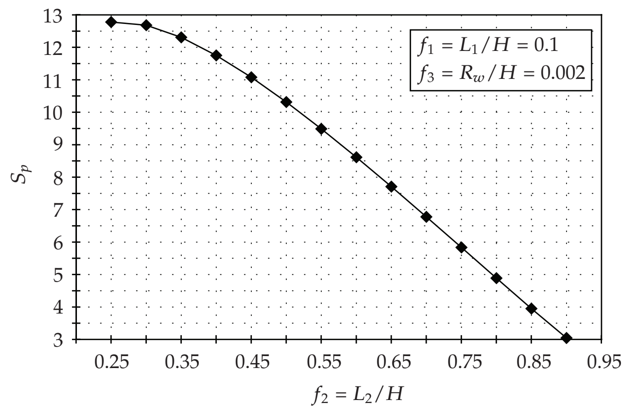

Figure 5: Pseudo-skin factor versus L_2/H plot.

Solution

Equation (5.4) is used to calculate P_{wD}, the results are shown in Figure 6.

Figure 6 shows that at early times, the well is in infinite acting period. When producing time is long, the influence from outer boundary appears. Because the outer boundary is at constant pressure, when the producing time is sufficiently long, steady state will be reached, P_{wD} becomes a constant.

At a given time t_D, if drainage radius R_{eD} is a constant, when well off-center distance R_{oD} increases, P_{wD} decreases, which indicates the effect from constant pressure outer boundary is more pronounced.

Example 6.3. A fully penetrating off-center vertical well, if

$$R_{oD} = 10, \qquad R_{wD} = 0.01, \tag{6.6}$$

compare the wellbore pressure responses when $R_{eD} = 20, 30, 40$.

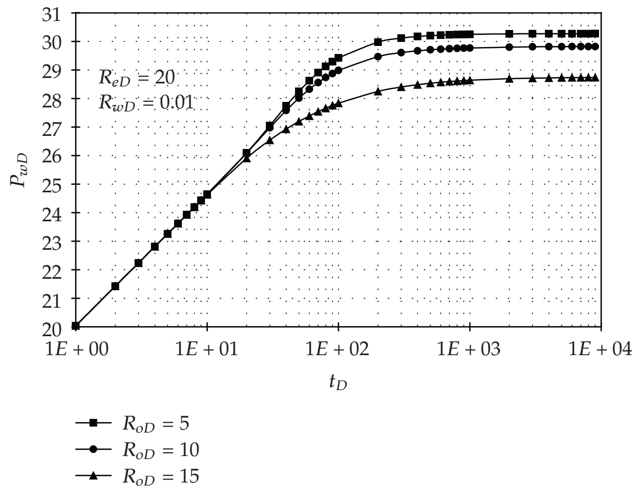

Figure 6: The effect of well off-center distance on wellbore pressure.

Solution

Equation (5.4) is used to calculate P_{wD}, the results are shown in Figure 7.

Figure 7 shows that at a given time t_D, if well off-center distance R_{oD} is a constant, when drainage radius R_{eD} increases, P_{wD} also increases, which indicates the effect from constant pressure outer boundary is more pronounced.

7. Conclusions

The following conclusions are reached.

(1) The proposed equations provide fast analytical tools to evaluate the performance of a vertical well which is located arbitrarily in a circular drainage volume with constant pressure outer boundary.

(2) The well off-center distance has significant effect on well pressure drop behavior, but it does not have any effect on pseudo-skin factor due to partial penetration.

(3) Because the outer boundary is at constant pressure, when producing time is sufficiently long, steady-state is definitely reached.

(4) At a given time in a given drainage volume, if the well off-center distance increases, the pressure drop at wellbore decreases.

(5) When well producing length is equal to payzone thickness, the pressure drop equations for a fully penetrating well are obtained.

Appendix

In this appendix, we want to prove (3.41).

For convenience, in the following reference, every variable below is dimensionless but we drop the subscript D.

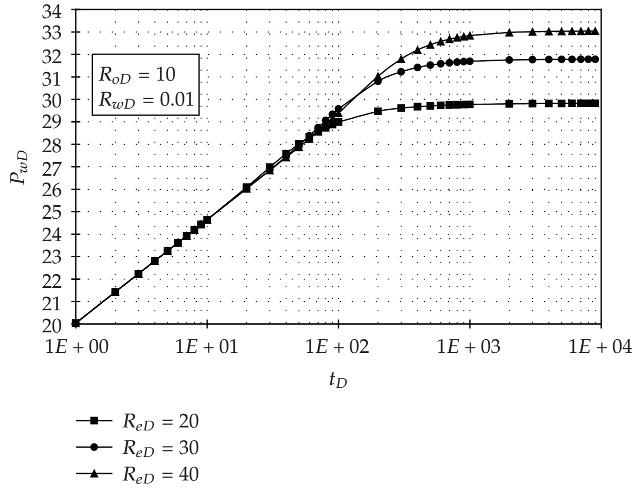

Figure 7: The effect of drainage radius on wellbore pressure.

There hold [14]

$$I_m(x) \simeq \frac{\exp(x)}{(2\pi x)^{1/2}}, \quad K_m(x) \simeq \frac{[\pi/(2x)]^{1/2}}{\exp(x)}, \quad x \gg 1,\ \forall m \geq 0. \tag{A.1}$$

Since

$$\zeta_k = \sqrt{\lambda_k^2 + s} > \lambda_k = \frac{k\pi}{H} > 0, \quad \forall k \geq 1, \tag{A.2}$$

and note that H is in dimensionless form in the above equation, recall (2.11), (2.13) and (2.21), for dimensionless H, R_e, ρ_0, ρ, there hold

$$\zeta_k R_e \gg 1, \qquad \zeta_k \rho_0 \gg 1, \qquad \zeta_k \rho \gg 1, \tag{A.3}$$

thus, we obtain

$$\frac{K_m(\zeta_k R_e)}{I_m(\zeta_k R_e)} \approx \pi \exp(-2\zeta_k R_e), \tag{A.4}$$

$$\begin{aligned} I_m(\zeta_k\rho_0)I_m(\zeta_k\rho) &\approx \left[\frac{\exp(\zeta_k\rho_0)}{(2\pi\zeta_k\rho_0)^{1/2}}\right]\left[\frac{\exp(\zeta_k\rho)}{(2\pi\zeta_k\rho)^{1/2}}\right] \\ &= \frac{\exp[\zeta_k(\rho+\rho_0)]}{(2\pi\zeta_k)(\rho\rho_0)^{1/2}}, \end{aligned} \tag{A.5}$$

$$\begin{aligned}
Y_{mk} I_m(\zeta_k \rho) &= \frac{2\beta_k K_m(\zeta_k R_e) I_m(\zeta_k \rho_0) I_m(\zeta_k \rho)}{s I_m(\zeta_k R_e)} \\
&\approx \left(\frac{2\beta_k}{s}\right)\left[\pi \exp(-2\zeta_k R_e)\right]\left\{\frac{\exp[\zeta_k(\rho+\rho_0)]}{(2\pi\zeta_k)(\rho\rho_0)^{1/2}}\right\} \\
&= \left(\frac{2\beta_k}{s}\right)\left[\frac{\pi}{(2\pi\zeta_k)(\rho\rho_0)^{1/2}}\right] \exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right] \\
&= \left[\frac{\beta_k}{s\zeta_k(\rho\rho_0)^{1/2}}\right] \exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right].
\end{aligned} \tag{A.6}$$

There holds

$$\begin{aligned}
|\widehat{\varphi_k}| &= \sum_{m=0}^{\infty}\left|Y_{mk} I_m(\zeta_k \rho)\cos(m\theta)\right| \\
&< \sum_{m=0}^{\infty}\left|Y_{mk} I_m(\zeta_k \rho)\right| \\
&= \sum_{m=0}^{\infty}\left|\frac{2\beta_k K_m(\zeta_k R_e) I_m(\zeta_k \rho_0) I_m(\zeta_k \rho)}{s I_m(\zeta_k R_e)}\right|.
\end{aligned} \tag{A.7}$$

Combining (2.21), (3.20), (A.6), and (A.7), we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty}\left|\widehat{\varphi_k}\cos\left(\frac{k\pi z}{H}\right)\right| &\le \sum_{k=1}^{\infty}|\widehat{\varphi_k}| \\
&= \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\left|\frac{2\beta_k K_m(\zeta_k R_e) I_m(\zeta_k \rho_0) I_m(\zeta_k \rho)}{s I_m(\zeta_k R_e)}\right| \\
&= \sum_{k=1}^{\infty}\left[\left|\frac{2\beta_k K_0(\zeta_k R_e) I_0(\zeta_k \rho_0) I_0(\zeta_k \rho)}{s I_0(\zeta_k R_e)}\right| \right.\\
&\qquad \left. + \sum_{m=1}^{\infty}\left|\frac{2\beta_k K_m(\zeta_k R_e) I_m(\zeta_k \rho_0) I_m(\zeta_k \rho)}{s I_m(\zeta_k R_e)}\right|\right] \\
&= \sum_{k=1}^{\infty}\left|\frac{2\beta_k K_0(\zeta_k R_e) I_0(\zeta_k \rho_0) I_0(\zeta_k \rho)}{s I_0(\zeta_k R_e)}\right| \\
&\quad + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left|\frac{2\beta_k K_m(\zeta_k R_e) I_m(\zeta_k \rho_0) I_m(\zeta_k \rho)}{s I_m(\zeta_k R_e)}\right| \\
&= \Xi_1 + \Xi_2,
\end{aligned} \tag{A.8}$$

where

$$\Xi_1 = \sum_{k=1}^{\infty}\left|\frac{2\beta_k K_0(\zeta_k R_e) I_0(\zeta_k \rho_0) I_0(\zeta_k \rho)}{s I_0(\zeta_k R_e)}\right|, \tag{A.9}$$

$$\Xi_2 = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left|\frac{2\beta_k K_m(\zeta_k R_e) I_m(\zeta_k \rho_0) I_m(\zeta_k \rho)}{s I_m(\zeta_k R_e)}\right|. \tag{A.10}$$

It is easy to prove if

$$x > y > 0, \quad a > 0, \tag{A.11}$$

then

$$\frac{\exp(-ax)}{x} < \frac{\exp(-ay)}{y}, \tag{A.12}$$

since $\zeta_k > \lambda_k$, thus

$$\left(\frac{1}{\zeta_k}\right)\exp\left[-\zeta_k(2R_e - \rho_0 - \rho)\right] < \left(\frac{1}{\lambda_k}\right)\exp\left[-\lambda_k(2R_e - \rho_0 - \rho)\right]. \tag{A.13}$$

Thus, there holds

$$\begin{aligned}
\Xi_1 &= \sum_{k=1}^{\infty}\left|\frac{2\beta_k K_0(\zeta_k R_e) I_0(\zeta_k \rho_0) I_0(\zeta_k \rho)}{s I_0(\zeta_k R_e)}\right| \\
&\approx \sum_{k=1}^{\infty}\left[\frac{|\beta_k|}{s\zeta_k(\rho\rho_0)^{1/2}}\right]\exp\left[-\zeta_k(2R_e - \rho_0 - \rho)\right] \\
&< \sum_{k=1}^{\infty}\left[\frac{|\beta_k|}{s\lambda_k(\rho\rho_0)^{1/2}}\right]\exp\left[-\lambda_k(2R_e - \rho_0 - \rho)\right] \\
&= \sum_{k=1}^{\infty}\left[\frac{8}{s\pi k(\rho\rho_0)^{1/2}}\right]\exp\left[-\left(\frac{k\pi}{H}\right)(2R_e - \rho_0 - \rho)\right] \\
&< \sum_{k=1}^{\infty}\left[\frac{8}{s\pi(\rho\rho_0)^{1/2}}\right]\exp\left[-\left(\frac{k\pi}{H}\right)(2R_e - \rho_0 - \rho)\right] \\
&\approx \left[\frac{8}{s\pi(\rho\rho_0)^{1/2}}\right]\left\{\frac{\exp[-(\pi/H)(2R_e - \rho_0 - \rho)]}{1 - \exp[-(\pi/H)(2R_e - \rho_0 - \rho)]}\right\}, \\
&\approx 0,
\end{aligned} \tag{A.14}$$

where we use (2.30),

$$\exp\left[-\left(\frac{\pi}{H}\right)(2R_e-\rho_0-\rho)\right]\approx 0, \tag{A.15}$$

$$x+x^2+x^3+x^4+x^5+\cdots=\frac{x}{1-x},\quad 0<x<1. \tag{A.16}$$

If $m>-1/2$, there holds [14]

$$I_m(z)=\left[\frac{(z/2)^m}{\Gamma(m+1/2)\Gamma(1/2)}\right]\int_{-1}^{1}(1-t^2)^{m-1/2}\cosh(zt)dt, \tag{A.17}$$

thus for $m\geq 1$,

$$\begin{aligned}I_m(\zeta_k\rho)&\leq\left[\frac{(\zeta_k\rho/2)^m}{\Gamma(m+1/2)\Gamma(1/2)}\right]\int_{-1}^{1}\cosh(\zeta_k\rho t)dt\\&=\left[\frac{2(\zeta_k\rho/2)^m}{(\zeta_k\rho)\Gamma(m+1/2)\Gamma(1/2)}\right]\sinh(\zeta_k\rho)\\&=\left[\frac{(\zeta_k\rho/2)^{m-1}}{\Gamma(m+1/2)\Gamma(1/2)}\right]\sinh(\zeta_k\rho)\\&<\left[\frac{(\zeta_k\rho/2)^{m-1}}{2\Gamma(m+1/2)\Gamma(1/2)}\right]\exp(\zeta_k\rho),\end{aligned} \tag{A.18}$$

where we use

$$\begin{gathered}\int_{-1}^{1}\cosh(\zeta_k\rho t)dt=\frac{2\sinh(\zeta_k\rho)}{\zeta_k\rho},\\ \sinh(\zeta_k\rho)<\frac{\exp(\zeta_k\rho)}{2},\end{gathered} \tag{A.19}$$

and if $-1<t<1,\ m\geq 1$, then

$$(1-t^2)^{m-1/2}\leq 1. \tag{A.20}$$

Substituting (A.18) into (A.10), we obtain

$$\begin{aligned}
\Xi_2 &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left|\frac{2\beta_k K_m(\zeta_k R_e) I_m(\zeta_k\rho_0) I_m(\zeta_k\rho)}{s I_m(\zeta_k R_e)}\right| \\
&< \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{2\pi|\beta_k|}{s}\right)\exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right]\left[\frac{(\zeta_k\rho/2)^{m-1}}{2\Gamma(m+1/2)\Gamma(1/2)}\right]\left[\frac{(\zeta_k\rho_0/2)^{m-1}}{2\Gamma(m+1/2)\Gamma(1/2)}\right] \\
&= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{16\pi}{sH}\right)\frac{\left(\zeta_k^2\rho\rho_0/4\right)^{m-1}}{\left[2\Gamma(m+1/2)\Gamma(1/2)\right]^2}\exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right] \\
&= \sum_{k=1}^{\infty}\left(\frac{16\pi}{sH}\right)\exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right]\sum_{m=1}^{\infty}\frac{\left(\zeta_k^2\rho\rho_0/4\right)^{m-1}}{\left[2\Gamma(m+1/2)\Gamma(1/2)\right]^2}.
\end{aligned}\tag{A.21}$$

Note that [14]

$$\begin{aligned}
\Gamma(m+1/2) &= \frac{1\times 3\times 5\times\cdots\times(2m-1)\sqrt{\pi}}{2^m} \\
&> \frac{1\times 2\times 6\times\cdots\times(2m-2)\sqrt{\pi}}{2^m} \\
&= \frac{2^{m-1}(m-1)!\sqrt{\pi}}{2^m} \\
&= \frac{(m-1)!\sqrt{\pi}}{2}.
\end{aligned}\tag{A.22}$$

Then we obtain

$$\begin{aligned}
\Xi_2 &< \sum_{k=1}^{\infty}\left(\frac{16\pi}{sH}\right)\exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right]\sum_{m=1}^{\infty}\frac{\left(\zeta_k^2\rho\rho_0/4\right)^{m-1}}{\left[2\Gamma(m+1/2)\Gamma(1/2)\right]^2} \\
&< \sum_{k=1}^{\infty}\left(\frac{16\pi}{sH}\right)\exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right]\sum_{m=1}^{\infty}\frac{\left(\zeta_k^2\rho\rho_0/4\right)^{m-1}}{\left[(m-1)!\pi\right]^2} \\
&= \sum_{k=1}^{\infty}\left(\frac{16\pi}{sH}\right)\exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right]\sum_{n=0}^{\infty}\frac{\left(\zeta_k\sqrt{\rho\rho_0}/2\right)^{2n}}{(n!\pi)^2} \\
&= \sum_{k=1}^{\infty}\left(\frac{16\pi}{\pi^2 sH}\right)\exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right]I_0\left(\zeta_k\sqrt{\rho\rho_0}\right) \\
&= \sum_{k=1}^{\infty}\left(\frac{16}{s\pi H}\right)\exp\left[-\zeta_k(2R_e-\rho_0-\rho)\right]I_0\left(\zeta_k\sqrt{\rho\rho_0}\right),
\end{aligned}\tag{A.23}$$

where we use [14]

$$I_0(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{(n!)^2}, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{A.24}$$

It is easy to prove if $a > 0$, $x > y > 0$, there holds

$$\frac{e^{-ax}}{\sqrt{x}} < \frac{e^{-ay}}{\sqrt{y}}, \tag{A.25}$$

since

$$\zeta_k > \lambda_k, \qquad 2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0} > 0, \tag{A.26}$$

then

$$\frac{\exp\left[-\zeta_k\left(2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0}\right)\right]}{\zeta_k^{1/2}} < \frac{\exp\left[-\lambda_k\left(2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0}\right)\right]}{\lambda_k^{1/2}}. \tag{A.27}$$

Note that $\zeta_k\sqrt{\rho\rho_0} \gg 1$, and we have

$$I_0\left(\zeta_k\sqrt{\rho\rho_0}\right) \approx \frac{\exp\left(\zeta_k\sqrt{\rho\rho_0}\right)}{\left(2\pi\zeta_k\sqrt{\rho\rho_0}\right)^{1/2}} \tag{A.28}$$

thus (A.23) can be simplified as follows:

$$\begin{aligned}
\Xi_2 &< \sum_{k=1}^{\infty}\left(\frac{16}{s\pi H}\right)\exp\left[-\zeta_k(2R_e - \rho_0 - \rho)\right]\left[\frac{\exp\left(\zeta_k\sqrt{\rho\rho_0}\right)}{\left(2\pi\zeta_k\sqrt{\rho\rho_0}\right)^{1/2}}\right] \\
&= \sum_{k=1}^{\infty}\left[\frac{16}{s\pi H(2\pi)^{1/2}(\rho\rho_0)^{1/4}}\right]\frac{\exp\left[-\zeta_k\left(2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0}\right)\right]}{\zeta_k^{1/2}} \\
&< \sum_{k=1}^{\infty}\left[\frac{16}{s\pi H(2\pi)^{1/2}(\rho\rho_0)^{1/4}}\right]\frac{\exp\left[-\lambda_k\left(2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0}\right)\right]}{\lambda_k^{1/2}} \\
&= \sum_{k=1}^{\infty}\left[\frac{16}{s\pi^2(2kH)^{1/2}(\rho\rho_0)^{1/4}}\right]\exp\left[-\left(\frac{k\pi}{H}\right)\left(2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0}\right)\right] \\
&< \sum_{k=1}^{\infty}\left[\frac{16}{s\pi^2(2H)^{1/2}(\rho\rho_0)^{1/4}}\right]\exp\left[-\left(\frac{k\pi}{H}\right)\left(2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0}\right)\right] \\
&= \left[\frac{16}{s\pi^2(2H)^{1/2}(\rho\rho_0)^{1/4}}\right]\left\{\frac{\exp\left[-(\pi/H)\left(2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0}\right)\right]}{1 - \exp\left[-(\pi/H)\left(2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0}\right)\right]}\right\} \\
&\approx 0,
\end{aligned} \tag{A.29}$$

where we use (A.16) and (2.29)

$$\exp\left[-\left(\frac{\pi}{H}\right)\left(2R_e - \rho_0 - \rho - \sqrt{\rho\rho_0}\right)\right] \approx 0. \tag{A.30}$$

Combining (A.8), (A.14), and (A.29), we prove (3.41).

References

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