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Research Article

Self-Similar Solutions for Nonlinear Schrödinger Equations

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We study the self-similar solutions for nonlinear Schrödinger type equations of higher order with nonlinear term $|u|^{\alpha}u$ by a scaling technique and the contractive mapping method. For some admissible value α, we establish the global well-posedness of the Cauchy problem for nonlinear Schrödinger equations of higher order in some nonstandard function spaces which contain many homogeneous functions. we do this by establishing some nonlinear estimates in the Lorentz spaces or Besov spaces. These new global solutions to nonlinear Schrödinger equations with small data admit a class of self-similar solutions.

1. Introduction

This paper is concerned with the following Cauchy problem for the nonlinear Schrödinger type equation:

$$iu_t + (-\Delta)^m u = \mu|u|^{\alpha}u, \quad x \in R^n, \; t \in R^+,$$
$$u(x,0) = u_0(x), \quad x \in R^n, \tag{1.1}$$

where $\mu \in R$ is a constant, $m \geq 1$ is an integer, $u = u(t,x)$ is a complex-valued function defined on $R^+ \times R^n$ $(R^+ \equiv [0,+\infty))$, and the initial data $u_0(x)$ is a complex-valued function defined in R^n. Pecher and Wahl [1] have established the existence of the classical solution to the Cauchy problem for the higher-order Schrödinger equation (1.1) by making use of L^p-estimates of the associated elliptic equation in conjunction with the compactness method. Recently Sjögren and Sjölin studied the local smoothing effect of the solutions to the Cauchy problem (1.1) by means of the Strichartz estimates in nonhomogeneous spaces ([2, 3]). Moreover, there are some work ([4–6]) which is devoted to the investigation of the global well-posedness and the scattering theory of the problem (1.1). However, little attention is paid to the self-similar solutions of the Cauchy problem (1.1).

Our goal is to prove the existence of the global self-similar solutions to the Cauchy problem (1.1) for some admissible parameter α. From the scaling principle, it is easy to see that if $u(t,x)$ is a solution of the Cauchy problem (1.1), then $u_\lambda(t,x) = \lambda^{2m/\alpha}u(\lambda^{2m}t, \lambda x)$ with $\lambda > 0$ is also a solution of equation in (1.1) with the initial value $\lambda^{2m/\alpha}u_0(\lambda x)$. We thus have the following definition.

Definition 1.1. $u(t,x)$ is said to be a self-similar solution to the higher-order Schrödinger equation in (1.1) if

$$u(t,x) = u_\lambda(t,x) = \lambda^{2m/\alpha}u\left(\lambda^{2m}t, \lambda x\right), \quad \forall\lambda > 0. \tag{1.2}$$

By Definition 1.1, we know that the self-similar solution to (1.1) is of the form

$$u(t,x) = t^{-(1/\alpha)}U\left(\frac{x}{\sqrt[2m]{t}}\right), \tag{1.3}$$

where $U: R^n \rightarrow C$ is called profile of the solution, and the initial value u_0 is of the form

$$u_0(x) = \frac{\Omega(x')}{|x|^{2m/\alpha}}, \tag{1.4}$$

where $x' = x/|x|$ and Ω is defined on the unit sphere S^n of R^n. Therefore the problem (1.1) can be studied through a nonlinear higher-order elliptic equation on U. However, this is usually very complicated. By virtue of this method, Kavian and Weissler [7] have dealt with the radially symmetric solutions of (1.1) in the case $m = 1, u_0(x) = |x|^{-(2/\alpha)}$.

Another important way of looking for self-similar solutions for the nonlinear Schrödinger equation in (1.1) is to study the small global well-posedness of associated Cauchy problem (1.1) in some suitable function spaces. These global solutions admit a class of self-similar solutions. As a consequence, if $u(t,x)$ is the unique solution of the Cauchy problem (1.1) with the initial data u_0 given by (1.4), then $u(t,x)$ is a self-similar solution of the problem.

On the other hand, if $u(t,x)$ is a self-similar solution to the problem (1.1), then the initial function is $u_0(x) = \lambda^{2m/\alpha}u_0(\lambda x)$. So $u_0(x)$ is homogeneous of degree $-(2m/\alpha)$. In general, such homogeneous functions do not belong to the usual Lebesgue spaces and Sobolev spaces.

To do our work, several definitions and notations are required. Denote by $S(R^n)$ and $S'(R^n)$ the Schwartz space and the space of Schwartz distribution functions, respectively. $L^r(R^n)$ denotes the usual Lebesgue space on R^n with the norm $\|\cdot\|_r$ for $1 \le r \le \infty$. For $s \in R$ and $1 < r < \infty$, let $H^s_r(R^n) = (1-\Delta)^{-(s/2)}L^r(R^n)$, the inhomogeneous Sobolev space in terms of Bessel potentials; let $\dot{H}^s_r(R^n) = (-\Delta)^{-(s/2)}L^r(R^n)$, the homogeneous Sobolev space in terms of Riesz potentials, and write $H^s(R^n) = H^s_2(R^n)$ and $\dot{H}^s(R^n) = \dot{H}^s_2(R^n)$. We will omit R^n from spaces and norms. For any interval $I \subset R^+$ (or $I = R^+$) and for any Banach space X, we denote by $C(I;X)$ the space of strongly continuous functions from I to X and by $L^q(I;X)$ the space of strongly measurable functions from I to X with $\|u(\cdot)\|_X \in L^q(I)$. Finally, let $q > 0$, q' stands for the dual to q, that is, $(1/q) + (1/q') = 1$; $[a]$ denotes the largest integer less or equal to a.

When $m = 1$, the equation in (1.1) becomes the classical Schrödinger equation

$$iu_t - \Delta u = \mu |u|^{\alpha} u, \quad x \in R^n,\ t \in R^+, \qquad u(x,0) = u_0(x), \quad x \in R^n, \tag{1.5}$$

which describes many physical phenomena, and the well-posedness as well as the scattering theory for the Cauchy problem (1.5) has been extensively studied by many authors ([8–11]). Cazenave and Weissler [12, 13] (also Ribaud and Youssfi [14]) have studied the self-similar solutions of the equation in (1.5) with initial value $u_0(x)$ as (1.4). Their common ideas are to introduce the new function space $E_{s,p} = E_{s,p}(R^+ \times R^n)$ which consists of all Bochner measurable functions $u : (0,\infty) \to \dot{H}^s_p(R^n)$ such that $\|u\|_{E_{s,p}} = \sup_{t>0} t^{\sigma} \|u(t,x)\|_{\dot{H}^s_p} < \infty$, where $2 \le p < \infty, 0 \le s < n/p$ and $\sigma = \sigma(s,p) = (1/2)((2/\alpha) - (n/p) + s)$. They then established the existence of global self-similar solutions in $E_{s,p}$ for the problem (1.5) under the condition that $\|u_0\|_{E_{s,p}} < \varepsilon$.

This paper is organized as follows. In the next section, we will recall the definition and basic properties of function spaces that we require. Then in Section 3 we state the main results and the related propositions. The last section is devoted to the proof of main results.

2. Function Spaces

2.1. Lorentz Spaces $L^{p,q}(R^n)$

Definition 2.1. Let $f^*(t)$, $t \in (0,\infty)$, *be the nonincreasing rearrangement of a measurable function* $f(x)$, $x \in R^n$, *then* $f \in S'(R^n)$ *is said to be in* $L^{p,q}(R^n)$ *if and only if*

$$\|f\|_{p,q} = \left\{ \int_0^{\infty} \left(t^{1/p} f^*(t) \right)^q \frac{dt}{t} \right\}^{1/q}, \tag{2.1}$$

when $1 \le p, q < \infty$, and

$$\|f\|_{p,\infty} = \sup_{t \ge 0} t^{1/p} f^*(t) < +\infty, \tag{2.2}$$

when $1 \le p < \infty$, where $\|u\|_{p,q}$ is the quasinorm of space $L^{p,q}(R^n)$.

We refer the reader to [15, 16] for the definitions and detailed properties of the nonincreasing rearrangement functions and Lorentz spaces. In fact, Lorentz space $L^{p,q}(R^n)$ is a generalization of Lebesgue space $L^p(R^n)$. We have $L^{p,q}(R^n) = L^p(R^n)$ as $p = q$, and $L^p(R^n) \subset L^{p,q}(R^n) \subset L^{p,\infty}(R^n)$ as $q > p$. Meanwhile, a lot of properties of Lebesgue spaces are still valid in Lorentz spaces.

We may prove the following results according to Definition 2.1.

Proposition 2.2. *Suppose that* $1 \le p < \infty,\ 1 \le q \le \infty$, *then*

$$\left|\int_{R^n} f(x)g(x)dx\right| \le C\|f\|_{p,q} \cdot \|g\|_{p',q'}, \tag{2.3}$$

$$\left\|\int_{R^n} f(\cdot,y)dy\right\|_{p,q} \le C\int_{R^n} \|f(\cdot,y)\|_{p,q}dy, \tag{2.4}$$

$$\||f|^{\alpha}\|_{p,q} = \|f\|^{\alpha}_{p\alpha,q\alpha}. \tag{2.5}$$

The inequalities (2.3) and (2.4) are essentially the Hölder and Minkowski inequality in Lorentz spaces, respectively, and they can be proved by using Definition 2.1. Furthermore, noting that $L^{p,q}(R^n)$ is a real interpolation of Lebesgue space, we immediately obtain the following proposition.

Proposition 2.3. *Let* $0 < \alpha < n,\ 1 \le p < r < \infty,\ 1 \le q \le \infty$ *and* $1/r = (1/p) - (\alpha/n)$, *then*

$$\left\|\int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy\right\|_{r,q} \le C\|f\|_{p,q}. \tag{2.6}$$

2.2. Besov Spaces

We first recall briefly the definition of Besov spaces. For detailed properties and embedding theorems, we are referred to [15, 17].

Let $\varphi_0 \in S(R^n)$ satisfy $\widehat{\varphi}_0(\xi) = 1$ as $|\xi| \le 1$ and $\widehat{\varphi}_0(\xi) = 0$ as $|\xi| \ge 2$,

$$\widehat{\varphi}_j(\xi) = \widehat{\varphi}_0\left(2^{-j}\xi\right), \quad \widehat{\psi}_j(\xi) = \widehat{\varphi}_0\left(2^{-j}\xi\right) - \widehat{\varphi}_0\left(2^{-j+1}\xi\right), \quad j \in Z, \tag{2.7}$$

then we have the Littlewood-Paley decomposition

$$\begin{gathered}\widehat{\varphi}_0(\xi) + \sum_{j=0}^{\infty}\widehat{\psi}_j(\xi) = 1, \quad \xi \in R^n, \\ \sum_{j\in Z}\widehat{\psi}_j(\xi) = 1, \quad \xi \in R^n \setminus \{0\}, \\ \lim_{j\to+\infty}\widehat{\varphi}_j(\xi) = 1, \quad \xi \in R^n.\end{gathered} \tag{2.8}$$

For convenience, we introduce the following notions:

$$\Delta_j f = \mathcal{F}^{-1}\widehat{\psi}_j\mathcal{F}f = \psi_j * f, \quad S_j f = \mathcal{F}^{-1}\widehat{\varphi}_j\mathcal{F}f = \varphi_j * f, \quad j \in Z, \tag{2.9}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for Fourier and inverse Fourier transforms, respectively.

Definition 2.4. Assume that $s \in R$, $1 \le q \le \infty$, then

$$B_p^{s,q} = \left\{ f \in S'(R^n) \mid \|f\|_{B_p^{s,q}} = \|S_0 f\|_p + \left(\sum_{j=1}^{\infty} 2^{jsq} \|\Delta_j f\|_p^q \right)^{1/q} \right.$$
$$\left. = \|\varphi_0 * f\|_p + \left(\sum_{j=1}^{\infty} 2^{jsq} \|\varphi_j * f\|_p^q \right)^{1/q} < \infty \right\} \tag{2.10}$$

is called Besov space, and

$$\dot{B}_p^{s,q} = \left\{ f \in S'(R^n) \mid \|f\|_{\dot{B}_p^{s,q}} = \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\Delta_j f\|_p^q \right)^{1/q} = \left(\sum_{j \in Z} 2^{jsq} \|\varphi_j * f\|_p^q \right)^{1/q} < \infty \right\} \tag{2.11}$$

is homogeneous Besov space.

In particular, we have

$$\dot{B}_p^{s,\infty} = \left\{ f \in S'(R^n) \mid \|f\|_{\dot{B}_p^{s,\infty}} = \sup_{j \in Z} 2^{js} \|\Delta_j f\|_p = \sup_{j \in Z} 2^{js} \|\varphi_j * f\|_p < \infty \right\}. \tag{2.12}$$

Besides the classical Besov spaces, we also need the so-called generalized Besov spaces.

Definition 2.5. Let E be a Banach space, then, for $s \in R$ and $1 \le q \le \infty$, defines $\dot{B}_E^{s,q}$as

$$\dot{B}_E^{s,q} = \left\{ f \in E \mid \|f\|_{\dot{B}_E^{s,q}} = \left(\sum_{j \in Z} 2^{jsq} \|\Delta_j f\|_E^q \right)^{1/q} < \infty \right\}, \tag{2.13}$$

where Δ_j is the Littlewood-Paley operator on R^n defined as above.

Remark 2.6. If E is the Lorentz space $L^{p,r}(R^n)$, then

$$\dot{B}_{L^{p,r}}^{s,q} = \left\{ f \in L^{p,r} \mid \|f\|_{\dot{B}_{L^{p,r}}^{s,q}} = \left(\sum_{j \in Z} 2^{jsq} \|\Delta_j f\|_{L^{p,r}}^q \right)^{1/q} < \infty \right\}. \tag{2.14}$$

This space is useful in the study of self-similar solutions.

Remark 2.7. Let $E = L^q(I; L^r)$ with $I = R^+$ or $I \subset R^+$ being an interval, then we have

$$\dot{B}^{s,p}_{L^q(I;L^r)} = \left\{ f \in L^q(I; L^r) \mid \|f\|_{\dot{B}^{s,p}_{L^q(I;L^r)}} = \left(\sum_{j\in Z} 2^{jsp} \|\Delta_j f\|^p_{L^q(I;L^r)} \right)^{1/p} < \infty \right\},$$
$$\dot{B}^{s,\infty}_{L^q(I;L^r)} = \left\{ f \in L^q(I; L^r) \mid \|f\|_{\dot{B}^{s,\infty}_{L^q(I;L^r)}} = \sup_{j\in Z} 2^{js} \|\Delta_j f\|_{L^q(I;L^r)} < \infty \right\}, \tag{2.15}$$

where $1 \le q \le \infty,\ 1 \le r \le \infty,\ 1 \le p < \infty$.

Remark 2.8. In addition to the Besov spaces norm in Definition 2.4, we usually use the following equivalent norms for the Besov spaces $\dot{B}^{s,q}_p$ and $B^{s,q}_p$:

$$\|v\|_{\dot{B}^{s,q}_p} = \sum_{|\alpha|=N} \left(\int_0^\infty t^{-q\sigma} \sup_{|y|\le t} \left\| \Delta_y^2 \partial^\alpha v \right\|_p^q \frac{dt}{t} \right)^{1/q},$$
$$\|v\|_{B^{s,q}_p} = \|v\|_p + \|v\|_{\dot{B}^{s,q}_p}, \tag{2.16}$$

where $\Delta_y^2 v = \tau_y v + \tau_{-y} v - 2v, \tau_{\pm y} v(\cdot) = v(\cdot \pm y); \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n},\ \partial_i = \partial/\partial x_i,\ i = 1, 2, \ldots, n:$ $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $s = N + \sigma$ with a nonnegative integer N and $0 < \sigma < 2$. When s is not an integer, (2.16) is also equivalent to the following norm:

$$\|v\|_{\dot{B}^{s,q}_p} = \sum_{|\alpha|=[s]} \left(\int_0^\infty t^{-q(s-[s])} \sup_{|y|\le t} \left\| \Delta_y \partial^\alpha v \right\|_p^q \frac{dt}{t} \right)^{1/q}, \tag{2.17}$$

where $\Delta_{\pm y} v(\cdot) = \tau_{\pm y} v - v$. In the case when $q = \infty$, the above norm should be modified as follows:

$$\|v\|_{\dot{B}^{s,\infty}_p} = \sum_{|\alpha|=N} \sup_{t>0} \sup_{|y|\le t} t^{-\sigma} \left\| \Delta_y^2 \partial^\alpha v \right\|_p, \quad s \in R,$$
$$\|v\|_{\dot{B}^{s,\infty}_p} = \sum_{|\alpha|=[s]} \sup_{t>0} \sup_{|y|\le t} t^{-s+[s]} \left\| \Delta_y \partial^\alpha v \right\|_p, \quad s \notin Z. \tag{2.18}$$

3. Main Results

To solve our problems, we may rewrite (1.1) in the equivalent integral equation of the form

$$u(t) = S(t)u_0(x) - i\mu \int_0^t S(t-\tau) \left(|u(\tau)|^\alpha u(\tau) \right) d\tau, \tag{3.1}$$

where $S(t) = e^{i(-\Delta)^m t} = \mathcal{F}^{-1}(e^{i|\xi|^{2m}t} \mathcal{F} \cdot)$ is the free group generated by the free equation of Schrödinger type $iv_t + (-\Delta)^m v = 0$.

Definition 3.1. One calls (q, r) a classical admissible pair with respect to the $2m$-order Schrödinger operator if

$$\frac{2}{q} = \frac{n}{m}\left(\frac{1}{2} - \frac{1}{r}\right), \tag{3.2}$$

where $2 \le r < \infty$ for $n \le 2m$; $2 \le r \le 2n/(n-2m)$ for $n > 2m$.

To prove Theorem 3.3 we need the following generalized Strichartz estimates which follow directly from the stationary phase method, the Strichartz estimates, and interpolation theorems (see [5, 15, 18] for details).

Proposition 3.2. *Let $S(t) = e^{i(-\Delta)^m t}$, $2 \le p,\ l \le \infty$ and (q, r) satisfy (3.2); then*

$$\|S(t)\varphi(x)\|_{p,l} \le C|t|^{-(n/m)((1/2)-(1/p))} \|\varphi(x)\|_{p',l'}, \tag{3.3}$$

$$\|S(t)\varphi(x)\|_{L^{q,2}(I;L^{r,2})} \le C\|\varphi(x)\|_2, \tag{3.4}$$

$$\left\|\int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L^\infty(I;L^2)} \le C\|f\|_{L^{q',2}(I;L^{r',2})}, \tag{3.5}$$

$$\left\|\int_0^t S(t-\tau)f(x,\tau)d\tau\right\|_{L^{q,2}(I;L^{r,2})} \le C\|f\|_{L^{q',2}(I;L^{r',2})}. \tag{3.6}$$

Moreover, if $\alpha > 4m/n, 2/\beta = (n/m)((1/2) - (s_c/n) - (1/(\alpha+2)))$, then

$$\|S(t)\varphi(x)\|_{L^{\beta,\infty}(I;L^{\alpha+2,\infty})} \le C\|\varphi(x)\|_{\dot{B}_2^{s_c,\infty}}, \tag{3.7}$$

where $s_c = (n/2) - (2m/\alpha)$.

Our main results state as follows.

Theorem 3.3. (i) *Let $\beta = 2m\alpha(\alpha+2)/(4m-(n-2m)\alpha), 4m/n < \alpha < \infty$ for $n \le 2m$; $4m/n < \alpha < 4m/(n-2m)$ for $n > 2m$. There exists an $\varepsilon > 0$ such that if $u_0 \in \dot{B}_2^{s_c,\infty}$ with $\|u_0\|_{\dot{B}_2^{s_c,\infty}} \le \varepsilon$, then the Cauchy problem* (1.1) *(or* (3.1)*) has a unique global solution $u(t,x)$ with*

$$\begin{gathered} u(t,x) \in L^\infty\left(R^+; \dot{B}_2^{s_c,\infty}\right) \cap L^{\beta,\infty}\left(R^+; L^{\alpha+2,\infty}\right), \quad n \le 2m \\ u(t,x) \in L^\infty\left(R^+; \dot{B}_2^{s_c,\infty}\right) \cap \dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n-2m),2})} \cap L^{\beta,\infty}\left(R^+; L^{\alpha+2,\infty}\right), \quad n > 2m. \end{gathered} \tag{3.8}$$

(ii) *Let $\alpha \in 2N, n > 2m$, and $\alpha \ge 4m/(n-2m)$. There exists an $\varepsilon > 0$ such that if $u_0 \in \dot{B}_2^{s_c,\infty}$ with $\|u_0\|_{\dot{B}_2^{s_c,\infty}} \le \varepsilon$, then* (1.1) *has a unique global solution*

$$u(t,x) \in L^\infty\left(R^+; \dot{B}_2^{s_c,\infty}\right) \cap \dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n-2m),2})}. \tag{3.9}$$

(iii) *Let* $\alpha \notin 2N$, *and let the condition (a)* $2m < n < 4\sqrt{2m}$ *for* $1 \le m < 8$, $\alpha \ge 4m/(n-2m)$; *or (b)* $n > 2m$ *for* $m \ge 8$, $\alpha \in [4m/(n-2m), \alpha_-) \cup (\alpha_+, \infty)$ *be satisfied, where* α_- *and* α_+ *are two positive roots of equation* $2x^2 - nx + 4m = 0$ *and* $\alpha_- < \alpha_+$. *There exists an* $\varepsilon > 0$ *such that if* $u_0 \in \dot{B}_2^{s_c,\infty}$ *with* $\|u_0\|_{\dot{B}_2^{s_c,\infty}} \le \varepsilon$, *then the problem* (1.1) *has a unique global solution:*

$$u(t,x) \in L^{\infty}\left(R^+; \dot{B}_2^{s_c,\infty}\right) \cap \dot{B}^{s_c,\infty}_{L^2\left(R^+; L^{2n/(n-2m),2}\right)}. \tag{3.10}$$

Corollary 3.4 (see [19]). *Let* $u_0(x) = \varepsilon_0|x|^{-(2m/\alpha)}$, *where* ε_0 *is a positive constant,* α *satisfies the assumptions in Theorem 3.3; then there exists a unique global self-similar solution for the Cauchy problem* (1.1) *with the initial value* $u_0(x)$.

Theorem 3.5. *Let* $u_0(x) \in \dot{H}^{s_c}$ *satisfy the conditions of Theorem 3.3; then the global solution* $u(t,x)$ *obtained in Theorem 3.3 satisfies* $u(t,x) \in C(R^+; \dot{H}^{s_c})$.

4. The Proof of Main Results

To prove the main results, we need the following lemmas.

Lemma 4.1 (see [20]). *Let* $\delta_p = n \cdot \max(0, (1/p) - 1)$ *and* $m \in N$ *with* $m \ge 2$. *Suppose that*

$$\min_{k=1,2,\ldots,m} \sum_{k \ne j} \frac{1}{r_k} < 1, \quad \frac{1}{p} = \frac{1}{p_j} + \sum_{k \ne j} \frac{1}{r_k}, \quad j = 1,2,\ldots,m. \tag{4.1}$$

If $s > \delta_p$, *then there exists a constant* $C > 0$ *such that*

$$\left\| \prod_{i=1}^{m} f_i \right\|_{\dot{B}_p^{s,q}} \le C \sum_{j=1}^{m} \left(\|f_j\|_{\dot{B}_{p_j}^{s,q}} \right) \prod_{k \ne j} \|f_k\|_{L^{r_k}} \tag{4.2}$$

for all $(f_1, f_2, \ldots, f_m) \in \prod_{j=1}^{m} (\dot{B}^s_{p_j,q} \cap L^{r_j})$.

Lemma 4.2. *Let* $F = L^{\beta,\infty}\ (R^+; L^{\alpha+2,\infty})$, *where* $\beta = 2m\alpha(\alpha+2)/(4m-(n-2m)\alpha)$, $0 < \alpha < \infty$ *for* $n \le 2m$; $0 < \alpha < 4m/(n-2m)$ *for* $n > 2m$, *then*

$$\left\| \int_0^t S(t-\tau)\left(|u(\tau)|^{\alpha} u(\tau)\right) d\tau \right\|_F \le C\|u\|_F^{\alpha+1}. \tag{4.3}$$

Proof. By (2.4) in Proposition 2.2, we have

$$\left\| \int_0^t S(t-\tau)\left(|u(\tau)|^{\alpha} u(\tau)\right) d\tau \right\|_F \le C \left\| \int_0^t \left\| S(t-\tau)\left(|u(\tau)|^{\alpha} u(\tau)\right) \right\|_{L^{\alpha+2,\infty}} dt \right\|_{L^{\beta,\infty}}. \tag{4.4}$$

We get from (3.3) in Proposition 3.2

$$\begin{aligned}\left\|S(t-\tau)\left(|u(\tau)|^{\alpha}u(\tau)\right)\right\|_{L^{\alpha+2,\infty}} &\leq C|t-\tau|^{-n\alpha/(2m(\alpha+2))}\left\||u(\tau)|^{\alpha}u(\tau)\right\|_{L^{(\alpha+2)/(\alpha+1),\infty}} \\ &\leq C|t-\tau|^{-n\alpha/(2m(\alpha+2))}\|u(\tau)\|_{L^{\alpha+2,\infty}}^{\alpha+1}.\end{aligned} \tag{4.5}$$

Therefore, we obtain from Proposition 2.3 and (2.5)

$$\begin{aligned}\left\|\int_0^t S(t-\tau)\left(|u(\tau)|^{\alpha}u(\tau)\right)d\tau\right\|_F &\leq C\left\|\int_0^t C|t-\tau|^{-n\alpha/(2m(\alpha+2))}\|u(\tau)\|_{L^{\alpha+2,\infty}}^{\alpha+1}d\tau\right\|_{L^{\beta,\infty}} \\ &\leq C\left\|\|u\|_{L^{\alpha+2,\infty}}^{\alpha+1}\right\|_{L^{2m\alpha(\alpha+2)/[4m-(n-2m)\alpha](\alpha+1),\infty}} \leq C\|u\|_F^{\alpha+1}.\end{aligned} \tag{4.6}$$

□

Lemma 4.3 (see [21]). *Suppose that* $E = \dot{B}^{s_c,\infty}_{L^{4m(\alpha+2)/n\alpha}(R^+;L^{\alpha+2,2})}$; $F = L^{\beta,\infty}(R^+;L^{\alpha+2,\infty})$, *then one has*

$$\left\||u|^{\alpha}u\right\|_{\dot{B}^{s_c,\infty}_{L^{4m(\alpha+2)/(8m-(n-4m)\alpha),2}(R^+;L^{(\alpha+2)/(\alpha+1),2})}} \leq C\|u\|_F^{\alpha}\|u\|_E \tag{4.7}$$

for $n \leq 2m$.

Lemma 4.4 (see [22]). *Let* $f(u) = |u|^{\alpha}u$, $s_c = (n/2) - (2m/\alpha)$ *and* $1 \leq s_c < \alpha$, *then*

$$\left\|f(u)\right\|_{\dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n+2m),2})}} \leq C\|u\|_{\dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n+2m),2})}}\|u\|^{\alpha}_{L^{\infty}(R^+;\dot{B}_2^{s_c,\infty})}, \tag{4.8}$$

$$\left\|f'(u)\right\|_{\dot{B}^{s_c,\infty}_{L^2(R^+;L^{l,2})}} \leq C\|u\|_{\dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n+2m),2})}}\|u\|^{\alpha-1}_{L^{\infty}(R^+;\dot{B}_2^{s_c,\infty})}, \tag{4.9}$$

where $l = 2n\alpha/((n+2m)\alpha - 4m)$.

4.1. The Proof of Theorem 3.3

We first prove (i). Defining the following map by (3.1),

$$\Phi(u)(t) = S(t)u_0(x) - i\mu\int_0^t S(t-\tau)\left(|u(\tau)|^{\alpha}u(\tau)\right)d\tau. \tag{4.10}$$

For $n \leq 2m$, we have from Lemma 4.2 and (3.7) in Proposition 3.2,

$$\|\Phi(u)\|_F \leq C\left(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_F^{\alpha+1}\right), \tag{4.11}$$

$$\|\Phi(u) - \Phi(v)\|_F \leq C\left(\|u\|_F^{\alpha} + \|v\|_F^{\alpha}\right)\|u - v\|_F. \tag{4.12}$$

Let $F_\varepsilon = \{u \mid u \in F, \|u\|_F \le 2C\varepsilon\} \subset F$ and choose

$$\varepsilon \le \left(\frac{1}{2C}\right)^{1+(1/\alpha)}, \tag{4.13}$$

then we get by (4.11) and (4.12)

$$\begin{gathered} \|\Phi(u)\|_{F_\varepsilon} \le 2C\varepsilon \\ \|\Phi(u) - \Phi(v)\|_{F_\varepsilon} \le \frac{1}{2}\|u - v\|_{F_\varepsilon} \end{gathered} \tag{4.14}$$

for all $u, v \in F_\varepsilon$.

This implies that Φ is a contraction map from F_ε into F_ε. Thus, there exists a unique solution $u \in F$ of (1.1) with $\|u\|_F \le 2C\varepsilon$.

Let $E = \dot{B}_\Pi^{s_c,\infty}$, where $\Pi = L^{4m(\alpha+2)/n\alpha,2}(R^+; L^{\alpha+2,2})$. Then we derive from (3.4) and (3.6)

$$\begin{aligned} \|u\|_E &\le \|S(t)u_0\|_E + |\mu| \left\| \int_0^t S(t-\tau)(|u|^\alpha u)d\tau \right\|_E \\ &\le C\left(\sup_j 2^{js_c} \|\Delta_j S(t)u_0\|_\Pi + \sup_j 2^{js_c} \left\| \int_0^t S(t-\tau)\Delta_j(|u|^\alpha u)d\tau \right\|_\Pi \right) \\ &\le C\left(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \||u|^\alpha u\|_{\dot{B}_{\Pi'}^{s_c,\infty}} \right) \end{aligned} \tag{4.15}$$

for $u \in F_\varepsilon$, where $\Pi' = L^{4m(\alpha+2)/(8m-(n-4m)\alpha),2}(R^+; L^{(\alpha+2)/(\alpha+1),2})$. As a consequence, we get by Lemma 4.3 that

$$\|u\|_E \le C\left(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_E \|u\|_F^\alpha \right). \tag{4.16}$$

It follows that from (4.13), $C\|u\|_F^\alpha \le 1/2$. So, (4.16) implies that

$$\|u\|_E \le 2C\|u_0\|_{\dot{B}_2^{s_c,\infty}} < \infty. \tag{4.17}$$

Taking the $L^\infty(R^+; \dot{B}_2^{s_c,\infty})$ norm in both sides of (3.1), we obtain from the definition of generalized Besov spaces, Lemma 4.3 and (3.4) and (3.5)

$$\begin{aligned} \|u\|_{L^\infty(R^+;\dot{B}_2^{s_c,\infty})} &\le \|S(t)u_0\|_{L^\infty(R^+;\dot{B}_2^{s_c,\infty})} + |\mu| \left\| \int_0^t S(t-\tau)(|u|^\alpha u)(\tau)d\tau \right\|_{L^\infty(R^+;\dot{B}_2^{s_c,\infty})} \\ &\le C\left(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \||u|^\alpha u\|_{\dot{B}_{\Pi'}^{s_c,\infty}} \right) \\ &\le C\left(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_E \|u\|_F^\alpha \right) < \infty, \end{aligned} \tag{4.18}$$

which implies $u \in L^\infty(R^+; \dot{B}_2^{s_c,\infty})$. Therefore, in the case of $n \leq 2m$, we have

$$u(t,x) \in L^\infty(R^+; \dot{B}_2^{s_c,\infty}) \cap L^{\beta,\infty}\left(R^+; L^{\alpha+2,\infty}\right). \tag{4.19}$$

For $n > 2m$, let $G = L^\infty(R^+; \dot{B}_2^{s_c,\infty})$, $H = \dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n-2m),2})}$ and $X = F \cap G \cap H$, then we obtain from the assumption in (i) $0 < s_c < m$. In the case of $0 < s_c < 1$, according to the equivalent norm of Besov spaces and Hölder inequality it follows that

$$\begin{aligned} \left\| |u|^\alpha u \right\|_{\dot{B}^{s_c,\infty}_{L^2(R^+;L^{(2n/(n+2m)),2})}} &= \sup_{|y|\leq\tau} \tau^{-s_c} \left\| \Delta_y (|u|^\alpha u) \right\|_{L^2\left(R^+;L^{(2n/(n-2m)),2}\right)} \\ &\leq C \sup_{|y|\leq\tau} \tau^{-s_c} \left\| \Delta_y u \right\|_{L^2\left(R^+;L^{(2n/(n-2m)),2}\right)} \left\| \left|\tau_y u\right|^\alpha + |u|^\alpha \right\|_{L^\infty\left(R^+;L^{n/2m,\infty}\right)} \\ &\leq C \sup_{|y|\leq\tau} \tau^{-s_c} \left\| \Delta_y u \right\|_{L^2\left(R^+;L^{2n/(n-2m),2}\right)} \|u\|^\alpha_{L^\infty\left(R^+;L^{n\alpha/2m,\infty}\right)}. \end{aligned} \tag{4.20}$$

Using the Sobolev embedding theorem $\dot{B}_2^{s_c,\infty} \hookrightarrow L^{n\alpha/2m,\infty}$, we get that

$$\left\| |u|^\alpha u \right\|_{\dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n+2m)})}} \leq C \|u\|_H \cdot \|u\|_G^\alpha. \tag{4.21}$$

Consequently, from Remark 2.7, (3.4), (3.5), and (4.21), it follows that

$$\begin{aligned} \|\Phi(u)\|_H &\leq \|S(t)u_0\|_H + |\mu| \left\| \int_0^t S(t-\tau)(|u|^\alpha u)\, d\tau \right\|_H \\ &\leq C\left(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \left\| |u|^\alpha u \right\|_{\dot{B}^{s_c,\infty}} \right) \\ &\leq C\left(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_H \cdot \|u\|_G^\alpha \right) \\ &\leq C\left(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_X^{\alpha+1} \right). \end{aligned} \tag{4.22}$$

Similarly

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_H &\leq |\mu| \left\| \int_0^t S(t-\tau)(|u|^\alpha u - |v|^\alpha v)\, d\tau \right\|_H \\ &\leq C\left(\|u\|_G^\alpha + \|v\|_G^\alpha \right) \|u - v\|_H \\ &\leq C\left(\|u\|_X^\alpha + \|v\|_X^\alpha \right) \|u - v\|_X. \end{aligned} \tag{4.23}$$

By using (3.4), (3.5), and (4.21), and arguing similarly as in deriving (4.18) one obtain that

$$\begin{aligned} \|\Phi(u)\|_G &\le \|S(t)u_0\|_G + |\mu| \left\| \int_0^t S(t-\tau)(|u|^\alpha u)d\tau \right\|_G \\ &\le C\Big(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \big\| |u|^\alpha u \big\|_{L^2(R^+;L^{2n/(n+2m)})} \Big) \\ &\le C\Big(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_H \cdot \|u\|_G^\alpha \Big) \\ &\le C\Big(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_X^{\alpha+1} \Big), \end{aligned} \tag{4.24}$$

$$\|\Phi(u) - \Phi(v)\|_G \le C\left(\|u\|_X^\alpha + \|v\|_X^\alpha\right)\|u-v\|_X. \tag{4.25}$$

From (4.11) and (4.12), it follows that

$$\|\Phi(u)\|_F \le C\Big(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_X^{\alpha+1} \Big) \tag{4.26}$$

$$\|\Phi(u) - \Phi(v)\|_F \le C\left(\|u\|_X^\alpha + \|v\|_X^\alpha\right)\|u-v\|_X. \tag{4.27}$$

Thus, by (4.22)–(4.27) we have

$$\|\Phi(u)\|_X \le C\Big(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_X^{\alpha+1} \Big), \tag{4.28}$$

$$\|\Phi(u) - \Phi(v)\|_X \le C\left(\|u\|_X^\alpha + \|v\|_X^\alpha\right)\|u-v\|_X. \tag{4.29}$$

Letting $X_\varepsilon = \{u \mid u \in X, \|u\|_X \le 2C\varepsilon\}$, and choosing $\varepsilon \le \left(1/(2C)^{\alpha+1}\right)^{1/\alpha}$, then (4.28) and (4.29) imply that Φ is a contraction map from X_ε into X_ε. By the Banach contraction mapping principle we conclude that there is a unique solution $u(t,x) \in X_\varepsilon \subset X$ such that

$$u(t,x) \in L^\infty\left(R^+; \dot{B}_2^{s_c,\infty}\right) \cap L^{\beta,\infty}\Big(R^+; L^{\alpha+2,\infty}\Big) \cap \dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n-2m),2})}. \tag{4.30}$$

In the case of $1 < s_c < m$, the proof above can see that of (iii) below.

For a proof of (ii) see [18].

We now prove (iii). Note that $s_c = (n/2) - (2m/\alpha) \ge m > 1$ and $s_c = (n/2) - (2m/\alpha) \le \alpha$ under the assumption in (iii).

Let $Y = L^\infty(R^+; \dot{B}_2^{s_c,\infty}) \cap \dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n-2m),2})} = G \cap H$, then by using (4.8) in Lemma 4.4 and arguing similarly as in deriving (4.24) we have

$$\|\Phi(u)\|_H \le C\Big(\|u_0\|_{\dot{B}_2^{s_c,\infty}} + \|u\|_Y^{\alpha+1} \Big). \tag{4.31}$$

On the other hand, since $f(u)-f(v)=\int_0^1(u-v)\cdot f'(u+\theta(v-u))d\theta$, where $f(u)=|u|^\alpha u$, it follows from Proposition 3.2, Lemma 4.1, and (4.9) in Lemma 4.4 that

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_H &= |\mu|\left\|\int_0^t S(t-\tau)\big(f(u)-f(v)\big)d\tau\right\|_H \\
&\le C\|f(u)-f(v)\|_{\dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n+2m),2})}} \\
&= C\left\|(u-v)\int_0^1 f'(u+\theta(v-u))d\theta\right\|_{\dot{B}^{s_c,\infty}_{L^2(R^+;L^{(2n/(n+2m)),2})}} \\
&\le C\|v-u\|_{\dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n-2m),2})}}\cdot\left\|\int_0^1 f'(u+\theta(v-u))d\theta\right\|_{L^\infty(R^+;L^{n/2m})} \\
&\quad+\|u-v\|_{L^\infty(R^+;L^{n\alpha/2m})}\cdot\left\|\int_0^1 f'(u+\theta(v-u))d\theta\right\|_{\dot{B}^{s_c,\infty}_{L^2(R^+,L^{l,2})}},
\end{aligned}\tag{4.32}$$

where $l=2n\alpha/((n+2m)\alpha-4m)$.

Because $f'(u+\theta(u-v))=(\alpha+1)|(1+\theta)u-\theta v|^\alpha$, So we derive from (4.9) and the Sobolev embedding theorem $L^\infty(R^+;\dot{B}^{s_c,\infty}_2)\hookrightarrow L^\infty(R^+;L^{n\alpha/2m})$ that

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_H &\le \|u-v\|_H\cdot\big\||u|^\alpha+|v|^\alpha\big\|_{L^\infty(R^+;L^{n/2m})}+C\|u-v\|_H\cdot\|u-v\|_G^\alpha \\
&\le C\|u-v\|_H\big(\|u\|_G^\alpha+\|v\|_G^\alpha\big)\le C\|u-v\|_Y\big(\|u\|_Y^\alpha+\|v\|_Y^\alpha\big).
\end{aligned}\tag{4.33}$$

By arguing similarly as in deriving (4.24) and (4.25) we get

$$\|\Phi(u)\|_G\le C\Big(\|u_0\|_{\dot{B}^{s_c,\infty}_2}+\|u\|_Y^{\alpha+1}\Big),\tag{4.34}$$

$$\|\Phi(u)-\Phi(v)\|_G\le C\big(\|u\|_Y^\alpha+\|v\|_Y^\alpha\big)\|u-v\|_Y.\tag{4.35}$$

it follows from (4.31)–(4.35) that

$$\|\Phi(u)\|_Y\le C\Big(\|u_0\|_{\dot{B}^{s_c,\infty}_2}+\|u\|_Y^{\alpha+1}\Big)\tag{4.36}$$

$$\|\Phi(u)-\Phi(v)\|_Y\le C\big(\|u\|_Y^\alpha+\|v\|_Y^\alpha\big)\|u-v\|_Y.\tag{4.37}$$

Let $Y_M=\{u\mid u\in Y,\ \|u\|_Y\le M\}$ with $M=2C\|u_0\|_{\dot{B}^{s_c,\infty}_2}$ and choose $\varepsilon<(1/2C)^{(\alpha+1)/\alpha}$, then (4.36) and (4.37) imply that Φ is a contraction map from Y_M into Y_M. By the Banach contraction mapping principle we obtain that there is a unique solution $u(t,x)\in Y_M\subset Y$ such that

$$u(t,x)\in L^\infty\big(R^+;\dot{B}^{s_c,\infty}_2\big)\cap\dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n-2m),2})}.\tag{4.38}$$

This complete the proof of Theorem 3.3.

4.2. The Proof of Theorem 3.5

Without loss of generality we only consider the case $n > 2m$. From Theorem 3.3 it follows that the Cauchy problem (1.1) has a unique solution $u(t,x) \in L^\infty(R^+;\dot{B}_2^{s_c,\infty}) \cap \dot{B}^{s_c,\infty}_{L^2(R^+;L^{2n/(n-2m),2})}$ provided that $\|u_0\|_{\dot{B}_2^{s_c,\infty}}$ is suitably small. If, in addition, $u_0 \in \dot{H}^{s_c} = \dot{B}_2^{s_c,2}$, then we have by letting $I = L^\infty(R^+;\dot{B}_2^{s_c,2}) \cap \dot{B}^{s_c,2}_{L^2(R^+;L^{2n/(n-2m),2})}$ that

$$\begin{aligned}
\|u\|_I &\le C\left(\|u_0\|_{\dot{H}^{s_c}} + \left\|\int_0^t S(t-\tau)(|u|^\alpha u)\,d\tau\right\|_I\right)\\
&\le C\left(\|u_0\|_{\dot{H}^{s_c}} + \||u|^\alpha u\|_{\dot{B}^{s_c,2}_{L^2(R^+;L^{(2n/(n-2m)),2})}}\right)\\
&\le C\left(\|u_0\|_{\dot{H}^{s_c}} + \|u\|^\alpha_{L^\infty(R^+;L^{n\alpha/2m,\infty})}\|u\|_{\dot{B}_{L^2(R^+;L^{(2n/(n-2m)),2})}}\right)\\
&\le C\left(\|u_0\|_{\dot{H}^{s_c}} + \|u\|^\alpha_{L^\infty(R^+;\dot{B}_2^{s_c,\infty})}\|u\|_I\right).
\end{aligned} \tag{4.39}$$

From Theorem 3.3 it follows that the problem (1.1)–(1.4) has a unique solution $u(t,x)$ such that $\|u\|^\alpha_{L^\infty(R^+;\dot{B}_2^{s_c,\infty})} \le 1/2$ provided that $\|u_0\|_{\dot{B}_2^{s_c,\infty}} \le \delta$ with enough small δ. Then we have that from (4.39)

$$\|u\|_I \le 2C\|u_0\|_{\dot{H}^{s_c}} < \infty. \tag{4.40}$$

The continuity with respect to t of $u(t,x)$ is obvious; so $u(t,x) \in C(R^+;\dot{H}^{s_c})$.

The proof of Theorem 3.5 is thus completed.

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