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Research Article

Higher-Order Solutions of Coupled Systems Using the Parameter Expansion Method

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We consider periodic solution for coupled systems of mass-spring. Three practical cases of these systems are explained and introduced. An analytical technique called Parameter Expansion Method (PEM) was applied to calculate approximations to the achieved nonlinear differential oscillation equations. Comparing with exact solutions, the first approximation to the frequency of oscillation produces tolerable error 3.14% as the maximum. By the second iteration the respective error became 1/5th, as it is 0.064%. So we conclude that the first approximation of PEM is so benefit when a quick answer is required, but the higher order approximation gives a convergent precise solution when an exact solution is required.

1. Introduction

Nonlinear oscillators have been widely considered in physics and engineering. Surveys of literature with numerous references, and useful bibliographies, have been given by Mickens [1], Nayfeh and Mook [2], Agarwal et al. [3], and more recently by He [4]. To solve governing nonlinear equations and because limitation of existing exact solutions is one of the most time consuming and difficult affairs, many approaches for approximating the solutions to nonlinear oscillatory systems were excogitated. The most widely studied approximation methods are perturbation methods [5]. But these methods have a main shortcoming; there is no small parameter in the equation, and no approximation could be obtained.

Later, new analytical methods without depending on presence of small parameter in the equation were developed for solving these complicated nonlinear systems. These techniques include the Homotopy Perturbation [6–13], Modified Lindstedt-Poincaré [14], Parameter-Expanding [15–18], Parameterized Perturbation [19], Multiple Scale [20], Harmonic Balance [20, 21], Linearized Perturbation [22], Energy Balance [23–25], Variational Iteration [26, 27], Variational Approach [25, 28, 29], Iteration Perturbation [30], Variational Homotopy Perturbation [31] methods, and more [32]. Among these methods, Parameter Perturbation Method (PEM) is considered to be one powerful method that capable to handle strongly nonlinear behaviors. For this sake, we apply PEM to analysis of three practical cases [2, 33, 34] of nonlinear oscillatory system. Unlike the past investigations, here, it had assumed that the spring's property is nonlinear. The TDOF oscillation systems were consist of two coupled nonhomogeneous ordinary differential equations. So, we attempted to transform the equations of motion of a mechanical system which associated with the linear and nonlinear springs into a set of differential algebraic equations by introducing new variables. The analytical solutions of practical cases based on the cubic oscillation are presented by means of PEM for two iterations. Comparisons between analytical and exact solutions show that PEM can converge to an accurate periodic solution for nonlinear systems.

2. The Models of Nonlinear Oscillation Systems

In this section, a practical case of nonlinear oscillation system of SDOF in Case 1 and two cases of TDOF systems in Cases 2 and 3 are considered.

2.1. Single-Degree-of-Freedom

Case 1 (Model of a Bulking Column). First, we consider the system shown in Figure 1. The mass m can move in the horizontal direction only. Using this model representing a column, we demonstrate how one can study its static stability by determining the nature of the singular point at $u = 0$ of the dynamic equations. This "dynamic" approach is simpler to use, and arguments are more satisfying than the "static" approach [2]. Vito [35] analyzed the stability of vibration of a particle in a plane constrained by identical springs.

Neglecting the weight of springs and columns shows that the governing equation for the motion of m is [2]

$$m\ddot{u} + \left(k_1 - \frac{2P}{l}\right)u + \left(k_3 - \frac{2P}{l^3}\right)u^3 + \cdots = 0, \tag{2.1}$$

where $u(0) = A$, $\dot{u}(0) = 0$. The spring force is given by

$$F_{\text{spring}} = k_1 u + k_3 u^3 + \cdots . \tag{2.2}$$

2.2. Two-Degree-of-Freedom

Case 2 (Two-Mass System with Three Springs). The model of two-mass system with three springs is shown in Figure 2. In this system, two equal masses m are connected with the fixed supports using spring k_1. The connection between two masses makes a compact item which is a spring with nonlinear properties. The linear coefficient of spring elasticity is k_2 and of the

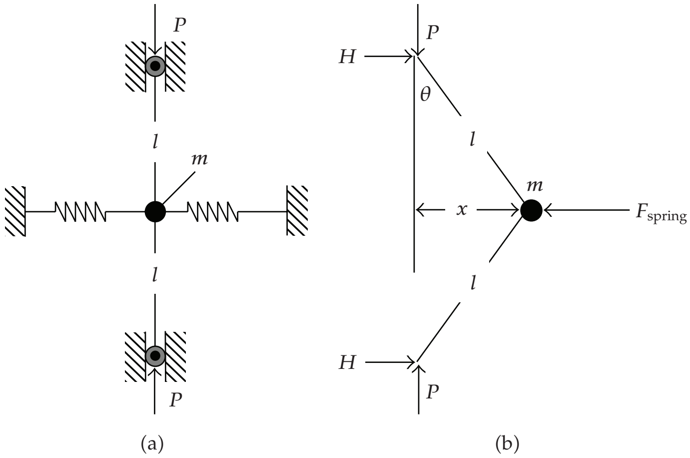


Figure 1: Model for the bulking of a column [2].

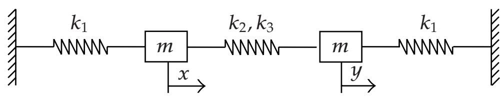


Figure 2: . Model of the two-mass system with three springs [34].

cubic nonlinearity is k_3. The system has two degrees of freedom. The generalized coordinates are x and y.

The mathematical model of the system is [34]

$$\begin{aligned} m\ddot{x} + k_1 x + k_2(x-y) + k_3(x-y)^3 &= \varepsilon f_1(x,\dot{x},y,\dot{y}), \quad x(0) = X_0,\ \dot{x}(0) = 0, \\ m\ddot{y} + k_1 y + k_2(y-x) + k_3(y-x)^3 &= \varepsilon f_2(x,\dot{x},y,\dot{y}), \quad y(0) = Y_0,\ \dot{y}(0) = 0, \end{aligned} \tag{2.3}$$

where εf_i is small nonlinearity $(i = 1,2)$. Dividing (2.3) by mass m yields

$$\begin{aligned} \ddot{x} + \frac{k_1}{m}x + \frac{k_2}{m}(x-y) + \frac{k_3}{m}(x-y)^3 &= \frac{\varepsilon}{m} f_1(x,\dot{x},y,\dot{y}), \\ \ddot{y} + \frac{k_1}{m}y + \frac{k_2}{m}(y-x) + \frac{k_3}{m}(y-x)^3 &= \frac{\varepsilon}{m} f_2(x,\dot{x},y,\dot{y}). \end{aligned} \tag{2.4}$$

Introducing the new variables

$$\begin{aligned} x &= u, \\ y - x &= v. \end{aligned} \tag{2.5}$$

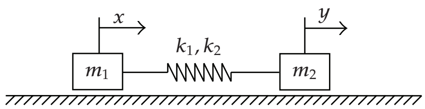


Figure 3: Model of the two-mass system with spring [33].

Transforming (2.4) yields

$$\ddot{u} + \frac{k_1}{m}u - \frac{k_2}{m}v - \frac{k_3}{m}v^3 = \frac{\varepsilon}{m}f_1(u, \dot{u}, v + u, \dot{v} + \dot{u}), \tag{2.6}$$

$$\ddot{v} + \ddot{u} + \frac{k_1}{m}(v + u) + \frac{k_2}{m}v + \frac{k_3}{m}v^3 = \frac{\varepsilon}{m}f_2(u, \dot{u}, v + u, \dot{v} + \dot{u}). \tag{2.7}$$

From (2.6), we have

$$\ddot{u} + \frac{k_1}{m}u = \frac{\varepsilon}{m}f_1(u, \dot{u}, v + u, \dot{v} + \dot{u}) + \frac{k_2}{m}v + \frac{k_3}{m}v^3, \tag{2.8}$$

Substituting (2.8) into (2.7) gives

$$\begin{gathered}\ddot{v} + \left[\frac{k_1 + 2k_2}{m}\right]v + \left[\frac{2k_3}{m}\right]v^3 = \zeta\left(f_2(u, \dot{u}, v + u, \dot{v} + \dot{u}) - f_1(u, \dot{u}, v + u, \dot{v} + \dot{u})\right), \\ v(0) = y(0) - x(0) = Y_0 - X_0 = A, \qquad \dot{v}(0) = 0.\end{gathered} \tag{2.9}$$

Setting $\varepsilon = 0$, (2.9) can be written as

$$\ddot{v} + \left[\frac{k_1 + 2k_2}{m}\right]v + \left[\frac{2k_3}{m}\right]v^3 = 0, \qquad v(0) = y(0) - x(0) = Y_0 - X_0 = A, \qquad \dot{v}(0) = 0. \tag{2.10}$$

Note that the case of $k_3 > 0$ corresponds to a hardening spring while $k_3 < 0$ indicates a softening one.

Case 3 (Two-Mass System with a Connection Spring). Similarly, the model of system with one spring is shown in Figure 3. Two masses, m_1 and m_2, are connected with a spring in which linear coefficient of rigidity is k_1, and the nonlinear coefficient is k_3. The system has two degrees of freedom.

The generalized coordinates of the system are x and y. The equation of motion of the system is described by [33]:

$$\begin{gathered}m_1\ddot{x} + k_1(x - y) + k_3(x - y)^3 = 0, \qquad x(0) = X_0, \qquad \dot{x}(0) = 0, \\ m_2\ddot{y} + k_1(y - x) + k_3(y - x)^3 = 0, \qquad y(0) = Y_0, \qquad \dot{y}(0) = 0.\end{gathered} \tag{2.11}$$

Similar to the previous section, to simplify these equations, we apply the variables that was introduced in (2.5). Using these variables, (2.11) transformed to

$$m_1 \ddot{u} - k_1 v - k_2 v^3 = 0, \tag{2.12}$$

$$m_2(\ddot{v} + \ddot{u}) + k_1 v + k_2 v^3 = 0. \tag{2.13}$$

Solving (2.12) for u yields

$$\ddot{u} = \frac{k_1}{m_1} v + \frac{k_2}{m_1} v^3, \tag{2.14}$$

Substituting (2.14) into (2.13) gives

$$\begin{gathered} \ddot{v} + \left[\frac{k_1(m_1 + m_2)}{m_1 m_2}\right] v + \left[\frac{k_2(m_1 + m_2)}{m_1 m_2}\right] v^3 = 0, \\ v(0) = y(0) - x(0) = Y_0 - X_0 = A, \qquad \dot{v}(0) = 0. \end{gathered} \tag{2.15}$$

As mentioned, these models can be transformed to a cubic nonlinear differential equation in general form with different values α and β. The general form of cubic nonlinear differential is as follows:

$$\ddot{v} + \alpha v + \beta v^3 = 0, \qquad v(0) = A, \qquad \dot{v}(0) = 0. \tag{2.16}$$

3. Basic Idea of PEM

In order to use the PEM, we rewrite the general form of Duffing equation in the following form [7]:

$$\ddot{v} + \alpha v + 1 \cdot N(v, t) = 0, \tag{3.1}$$

where $N(v, t)$ includes the nonlinear term. Expanding the solution v, α as a coefficient of v, and 1 as a coefficient of $N(v, t)$, the series of p can be introduced as follows:

$$v = v_0 + p v_1 + p^2 v_2 + \cdots, \tag{3.2}$$

$$\alpha = \omega^2 + p\gamma_1 + p^2\gamma_2 + \cdots, \tag{3.3}$$

$$1 = p\delta_1 + p^2\delta_2 + \cdots. \tag{3.4}$$

Substituting (3.2)–(3.4) into (3.1) and equating the terms with the identical powers of p, we have

$$p^0 : \ddot{v}_0 + \omega^2 v_0 = 0, \tag{3.5}$$

$$\begin{aligned} p^1 &: \ddot{v}_1 + \omega^2 v_1 + \gamma_1 v_0 + \delta_1 N(v_0, t) = 0 \\ &\vdots \end{aligned} \tag{3.6}$$

Considering the initial conditions $v_0(0) = A$ and $\dot{v}_0(0) = 0$, the solution of (3.5) is $v_0 = A\cos(\omega t)$. Substituting v_0 into (3.6), we obtain

$$p^1 : \ddot{v}_1 + \omega^2 v_1 + \gamma_1 A\cos(\omega t) + \delta_1 N(A\cos(\omega t), t) = 0. \tag{3.7}$$

For achieving the secular term, we use Fourier expansion series as follows:

$$N(A\cos(\omega t), t) = \sum_{n=0}^{\infty} b_{2n+1} \cos[(2n+1)\omega t]. \tag{3.8}$$

Substituting (3.8) into (3.7) yields

$$p^1 : \ddot{v}_1 + \omega^2 v_1 + (\gamma_1 A + \delta_1 b_1)\cos(\omega t) = 0. \tag{3.9}$$

For avoiding secular term, we have

$$(\gamma_1 A + \delta_1 b_1) = 0. \tag{3.10}$$

Setting $p = 1$ in (3.3) and (3.4), we have:

$$\gamma_1 = \alpha - \omega^2, \tag{3.11}$$

$$\delta_1 = 1. \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we will achieve the first-order approximation frequency (2.10). Note that, from (3.4) and (3.12), we can find that $\delta_i = 0$, for all $i = 2, 3, 4, \ldots$. In the following section we will describe the second-order of modified PEM solution in details for solving the cubic nonlinear differential equation.

4. Application of PEM to Cubic Equation

In order to use the PEM, we rewrite (2.16) as follows:

$$\ddot{v} + \alpha v + 1 \cdot \left(\beta v^3\right) = 0, \qquad v(0) = A, \qquad \dot{v}(0) = 0. \tag{4.1}$$

Substituting (3.2) and (3.4) into (4.1) and equating the terms with the identical powers of p, yields

$$p^0 : \ddot{v}_0 + \omega^2 v_0 = 0, \tag{4.2}$$

$$p^1 : \ddot{v}_1 + \omega^2 v_1 + (\alpha\delta_1 + \gamma_1)v_0 + \beta v_0^3 = 0, \tag{4.3}$$

$$p^2 : \ddot{v}_1 + \omega^2 v_1 + (\delta_2\alpha + \gamma_2)v_0 + (\gamma_1 + \delta_1\alpha)v_1 + \delta_2\beta v_0^3 + 3\delta_1\beta v_0^2 v_1 = 0. \tag{4.4}$$

$$\vdots$$

Considering the initial conditions $v(0) = A$ and $\dot{v}(0) = 0$, the solution of (4.2) is $v_0 = A\cos(\omega t)$. Substituting u_0 into (4.3), we obtain

$$p^1 : \ddot{v}_1 + \omega^2 v_1 + \gamma_1 A\cos(\omega t) + \delta_1\beta A^3\cos^3(\omega t) = 0. \tag{4.5}$$

It is possible to perform the following Fourier series expansion:

$$\begin{aligned}\beta A^3\cos^3(\omega t) &= \sum_{n=0}^{\infty} b_{2n+1}\cos[(2n+1)\omega t] = b_1\cos(\omega t) + \sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t]\\ &= \left(\frac{4}{\pi}\beta A^3\int_0^{\pi/2}\left(\cos^4(\varphi)\right)d\varphi\right)\cos(\omega t) + \sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t]\\ &= \frac{3A^3\beta}{4}\cos(\omega t) + \sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t].\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5) gives

$$\ddot{v}_1 + \omega^2 v_1 + \left(\gamma_1 A + \frac{3\delta_1 A^3\beta}{4}\right)\cos(\omega t) + \delta_1\sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] = 0. \tag{4.7}$$

No secular term in v_1 requires that

$$\gamma_1 = -\frac{3\delta_1 A^2\beta}{4}. \tag{4.8}$$

Setting $p = 1$ in (3.3) and (3.4) gives

$$\gamma_1 = \alpha - \omega^2, \tag{4.9}$$

$$\delta_1 = 1. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8), we obtain

$$\omega_1 = \sqrt{\alpha + \frac{3}{4}\beta A^2}, \tag{4.11}$$

$$T_1 = \frac{4\pi}{\sqrt{4\alpha + 3\beta A^2}}. \tag{4.12}$$

From (4.7) and (4.8), then (4.9) can be rewritten in the following form:

$$\ddot{v}_1 + \omega^2 v_1 = -\sum_{n=1}^{\infty} \zeta_{2n+1} \cos[(2n+1)\omega t], \qquad v_1(0) = 0, \qquad \dot{v}_1(0) = 0. \tag{4.13}$$

The periodic solution of (4.13) can be written [19]

$$v_1 = \sum_{n=0}^{\infty} \lambda_{2n+1} \cos[(2n+1)\omega t]. \tag{4.14}$$

Substituting (4.14) into (4.13) gives

$$-\omega^2 \sum_{n=0}^{\infty} 4n(n+1)\lambda_{2n+1} \cos[(2n+1)\omega t] = -\sum_{n=1}^{\infty} \zeta_{2n+1} \cos[(2n+1)\omega t]. \tag{4.15}$$

From (4.15), the coefficients λ_{2n+1} (for $n \geq 1$) can be written as follows:

$$\lambda_{2n+1} = \frac{\zeta_{2n+1}}{4n(n+1)}, \tag{4.16}$$

Taking into account that $v_1(0) = 0$, (4.14) yields

$$\lambda_1 = -\sum_{n=1}^{\infty} \lambda_{2n+1}. \tag{4.17}$$

To determine the second-order approximate solution, it is necessary to substitute (4.14) into (4.4). Then secular term is eliminated, and parameter γ_2 can be calculated. $v_1(t)$ has an infinite series; however, to simplify the solution procedure, we can truncate the series expansion of (4.14) and (4.17) and write an approximate equation $v_1(t)$ in the following form:

$$v_1(t) = \lambda_3(\cos 3\omega t - \cos \omega t). \tag{4.18}$$

Substituting $\delta_2 = 0$ and (4.8) and (4.18) into (4.4) gives

$$\ddot{v}_2 + \omega^2 v_2 - 3\lambda_3 \beta A^2 \cos^3(\omega t) + \beta A^2 \lambda_3 \cos(3\omega t)\left(3\cos^2(\omega t) - \frac{3}{4}\right) + \left(\frac{3}{4}\lambda_3 \beta A^2 + A\gamma_2\right)\cos(\omega t) = 0. \tag{4.19}$$

It is possible to do the following Fourier series expansion:

$$\begin{aligned}
3\lambda_3 \beta A^2 \cos^2(\omega t)(\cos(3\omega t) - \cos(\omega t)) &= \sum_{n=0}^{\infty} \eta_{2n+1} \cos[(2n+1)\omega t] \\
&= \eta_1 \cos(\omega t) + \sum_{n=1}^{\infty} \eta_{2n+1} \cos[(2n+1)\omega t] \\
&\cong \left(\frac{12\beta^2 \lambda_3 A^3}{\pi} \int_0^{\pi/2} (\cos(3\varphi) - \cos(\varphi))\cos^3\varphi \, d\varphi\right) \\
&\quad \times \cos(\omega t) + \sum_{n=1}^{\infty} \eta_{2n+1} \cos[(2n+1)\omega t] \\
&= -\frac{3}{4}\beta A^2 \lambda_3 \cos(\omega t) + \sum_{n=1}^{\infty} \eta_{2n+1} \cos[(2n+1)\omega t].
\end{aligned} \tag{4.20}$$

Substituting (4.20) into (4.19) and collecting, we have

$$\ddot{v}_2 + \omega^2 v_2 - \left(\frac{3}{4}\beta A^2 \lambda_3 - A\gamma_2\right)\cos(\omega t) - \frac{3}{4}\lambda_3 \beta A^2 \cos(3\omega t) + \sum_{n=1}^{\infty} \eta_{2n+1} \cos[(2n+1)\omega t] = 0. \tag{4.21}$$

The secular term in the solution for $v_2(t)$ can be eliminated if

$$\frac{3}{4}\beta A^2 \lambda_3 - A\gamma_2 = 0. \tag{4.22}$$

Solving (4.22) gives

$$\gamma_2 = \frac{3}{4}\beta A \lambda_3. \tag{4.23}$$

On the other hand, From (4.16), the following expression for the coefficient λ_3 is obtained:

$$\lambda_3 = \frac{\zeta_3}{8\omega^2} = \frac{\left((4/\pi)\beta A^3 \int_0^{\pi/2} \cos^3\varphi \ \cos 3\varphi \, d\varphi\right)}{8\omega^2} = \frac{\beta A^3}{32\omega^2}. \tag{4.24}$$

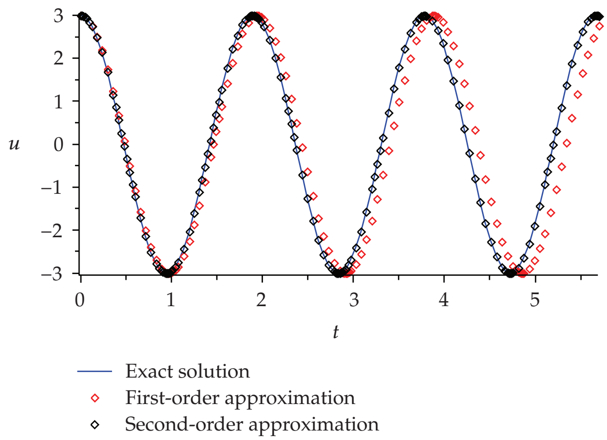


Figure 4: Comparison of approximate periodic solutions of Bucking of a Column equation (Case 1) with the exact one for $m = 1.0$, $l = 1.5$, $P = 5.0$, $k_1 = 5.0$, and $k_3 = 6.0$ with $u(0) = 3.0$.

Then, we can obtain

$$\gamma_2 = \frac{3\beta^2 A^4}{128\omega^2}. \tag{4.25}$$

From (3.3), (3.4), and (4.8), and taking $p = 1$ and considering $\delta_1 = 1$, we have

$$\gamma_2 = \alpha - \omega^2 - \frac{3A^2\beta}{4}. \tag{4.26}$$

Comparing the right hands of (4.25) and (4.26), one can easily obtain the following expression for the second-order approximate frequency and period

$$\omega_2 = \frac{\sqrt{8\alpha + 6\beta A^2 + \sqrt{64\alpha^2 + 96\alpha\beta A^2 + 30\beta^2 A^4}}}{4}, \tag{4.27}$$

$$T_2 = \frac{8\pi}{\sqrt{8\alpha + 6\beta A^2 + \sqrt{64\alpha^2 + 96\alpha\beta A^2 + 30\beta^2 A^4}}}. \tag{4.28}$$

5. Analytical Solution of Practical Cases

In this section, we present the first and second approximate frequency and period values of (2.16) for different values of α and β. Substituting $\alpha = (k_1 + 2P/l)/m$ and $\beta = (k_3 + 2P/l^3)/m$

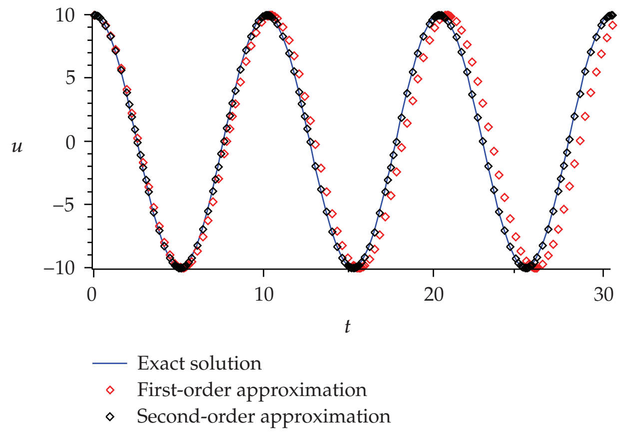


Figure 5: Comparison of approximate periodic solutions of Bucking of a Column equation (Case 1) with the exact one for $m = l = P = k_1 = 10.0$, and $k_3 = 50.0$ with $u(0) = 10.0$.

in (4.10), (4.11), (4.26), and (4.27) gives the following results for first- and second-order approximations of the model of nonlinear SDOF Bucking Column system in Case 1:

$$\begin{aligned}
\omega_1 &= \frac{1}{2}\sqrt{\frac{4k_1l^3 - 8Pl^2 + 3k_3A^2l^3 - 6PA^2}{ml^3}}, \\
T_1 &= \frac{4\pi\sqrt{ml^3}}{\sqrt{4k_1l^3 - 8Pl^2 + 3k_3A^2l^3 - 6PA^2}}, \\
\omega_2 &= \frac{1}{4ml^3}\Big(8l^3k_1 - 16l^2P + 6A^2k_3l^3 - 12A^2P \\
&\qquad + \big(64l^6k_1^2 - 256l^5k_1P + 256l^4P^2 + 96A^2l^6k_1k_3 \\
&\qquad - 192A^2l^3k_1P - 192A^2l^5k_3P + 384A^2l^2P^2 \\
&\qquad + 30A^4k_3^2l^6 - 120A^4Pk_3l^3 + 120A^4P^2\big)^{1/2}\Big)^{1/2}, \\
T_2 &= 8\pi ml^3 \Big/ \Big(8l^3k_1 - 16l^2P + 6A^2k_3l^3 - 12A^2P \\
&\qquad + \big(64l^6k_1^2 - 256l^5k_1P + 256l^4P^2 + 96A^2l^6k_1k_3 \\
&\qquad - 192A^2l^3k_1P - 192A^2l^5k_3P + 384A^2l^2P^2 + 30A^4k_3^2l^6 \\
&\qquad - 120A^4Pk_3l^3 + 120A^4P^2\big)^{1/2}\Big)^{1/2},
\end{aligned} \tag{5.1}$$

Table 1: Comparison of approximate and exact periods for Case 1.

Constant parameters						Approximate solutions		Exact solution	$\lvert T - T_{ex}\rvert / T_{ex}$	
m	l	p	k_1	k_3	A	T_1	T_2	T_e	$T = T_1$	$T = T_2$
1	1	1	10	5	1	1.96254	1.96451	1.96451	0.101%	0.000%
5	1.5	5	5	6	3	3.23743	3.32518	3.32368	2.664%	0.045%
10	10	10	10	50	10	0.32418	0.33145	0.33143	2.210%	0.031%
50	25	40	30	100	20	0.25640	0.26216	0.26208	2.216%	0.031%
70	20	−30	50	100	10	0.60486	0.61827	0.61809	2.187%	0.030%
100	50	150	70	20	100	0.16221	0.16586	0.16580	2.218%	0.031%
500	150	220	120	500	0.5	9.67637	9.71682	9.71672	0.417%	0.001%
1000	500	1000	500	500	1	6.73241	6.75877	6.75871	0.391%	0.001%

Also, we can obtain the first and second-order approximations solutions for Case 2, by substituting $\alpha = (k_1 + 2k_2)/m$ and $\beta = 2k_3/m$ into (4.11), (4.12), (4.27), and (4.28):

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{k_1 + 2k_2 + 1.5k_3A^2}{m}}, \\
T_1 &= \frac{\pi\sqrt{8m}}{\sqrt{2k_1 + 4k_2 + 3A^2k_3}}, \\
\omega_2 &= \frac{1}{4}\sqrt{\frac{8(k_1 + 2k_2) + 12k_3A^2 + \sqrt{64(k_1 + 2k_2)^2 + 192(k_1 + 2k_2)k_3A^2 + 120k_3^2A^4}}{m}}, \\
T_2 &= \frac{4\pi\sqrt{m}}{\sqrt{8(k_1 + 2k_2) + 12k_3A^2 + \sqrt{64(k_1 + 2k_2)^2 + 192(k_1 + 2k_2)k_3A^2 + 120k_3^2A^4}}}.
\end{aligned} \tag{5.2}$$

Similarly, for $\alpha = k_1(m_1 + m_2)/m_1m_2$ and $\beta = k_2(m_1 + m_2)/m_1m_2$, we obtain the following frequency and period values for Case 3:

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{(m_1 + m_2)}{m_1m_2}\left(k_1 + \frac{3}{4}A^2k_2\right)}, \\
T_1 &= \frac{2\pi\sqrt{m_1m_2}}{\sqrt{(m_1 + m_2)(k_1 + (3/4)A^2k_2)}}, \\
\omega_2 &= \frac{1}{4}\sqrt{\frac{(m_1 + m_2)}{m_1m_2}\left(8k_1 + 6A^2k_2 + \sqrt{(64k_1^2 + 96A^2k_1k_2 + 30A^4k_2^2)}\right)}, \\
T_2 &= \frac{8\pi}{(1/4)\sqrt{((m_1 + m_2)/m_1m_2)\left(8k_1 + 6A^2k_2 + \sqrt{(64k_1^2 + 96A^2k_1k_2 + 30A^4k_2^2)}\right)}}.
\end{aligned} \tag{5.3}$$

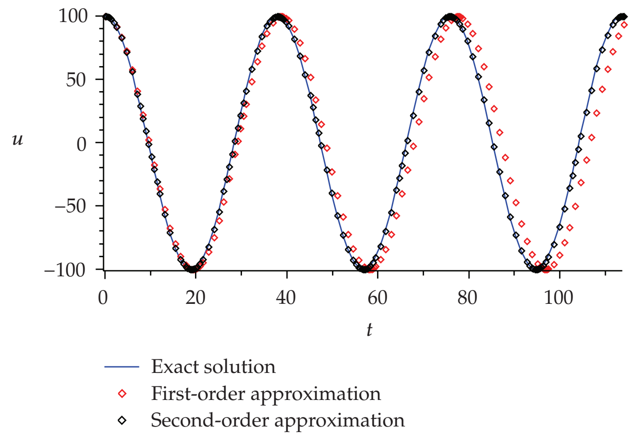


Figure 6: Comparison of approximate periodic solutions of Bucking of a Column equation (Case 1) with the exact one for $m = 100.0$, $l = 50.0$, $P = 150.0$, $k_1 = 70.0$, and $k_3 = 20.0$ with $u(0) = 100.0$.

6. Results and Discussions

To illustrate and verify accuracy of PEM, comparisons with the exact solution are given in Tables 1, 2, and 3. According to the appendix, the exact frequency, ω_{ex}, of nonlinear differential equation in the cubic form is

$$\omega_{ex}(A) = \frac{\pi\sqrt{\alpha+\beta A^2}}{2}\left(\int_0^{\pi/2}\frac{dt}{1-\delta\sin^2 t}\right)^{-1}, \qquad \delta = \frac{\beta A^2}{2(\alpha+\beta A^2)}. \tag{6.1}$$

Substituting $\alpha = (k_1 + 2P/l)/m$ and $\beta = (k_3 + 2P/l^3)/m$ into (6.1) gives the exact frequency for Case 1:

$$\omega_{ex}(A) = \frac{\pi}{2}\sqrt{\frac{k_1 l^3 - 2Pl^2 + A^2 k_3 l^3 - 2A^2 P}{ml^3}}\left(\int_0^{\pi/2}\frac{dt}{1-\delta\sin^2 t}\right)^{-1},$$
$$\delta = \frac{(l^3 k_3 - 2P)A^2}{2(k_1 l^3 - 2Pl^2 + A^2 k_3 l^3 - 2A^2 P)}. \tag{6.2}$$

Substituting $\alpha = k_1(m_1 + m_2)/m_1 m_2$ and $\beta = k_2(m_1 + m_2)/m_1 m_2$ into (6.1), the exact solution of Case 2 is

$$\omega_{ex}(A) = \frac{\pi}{2}\sqrt{\frac{(k_1 + 2k_2) + 2A^2 k_3}{m}}\left(\int_0^{\pi/2}\frac{dt}{1-\delta\sin^2 t}\right)^{-1},$$
$$\delta = \frac{2k_3 A^2}{2(k_1 + 2k_2) + 2k_3 A^2}. \tag{6.3}$$

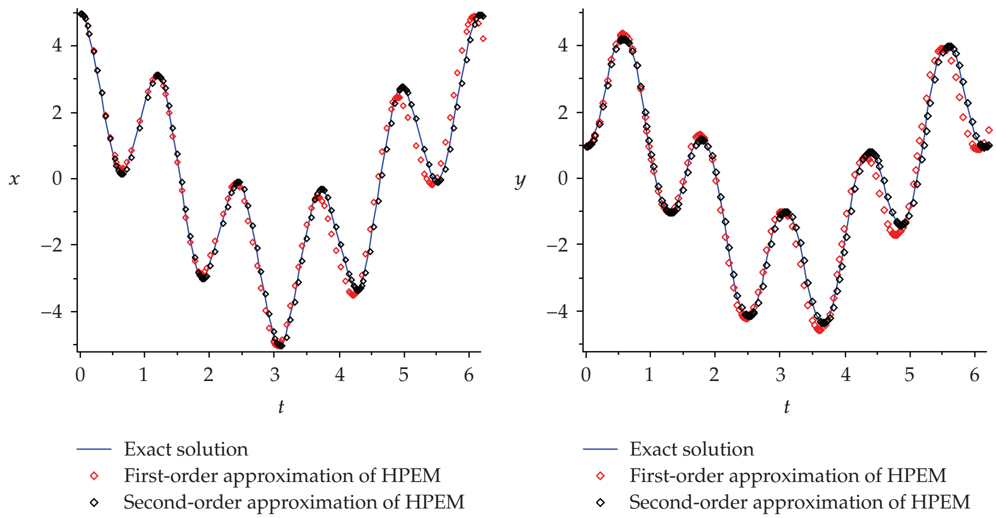


Figure 7: Comparison of the first- and second-order analytical approximate solutions with the exact solution for $m = k_1 = k_2 = k_3 = 1.0$ with $x(0) = 5.0$ and $y(0) = 1.0$ (Case 2).

Table 2: Comparison of approximate and "Exact" frequencies for case 2.

Constant parameters						Approximate solutions		Exact solution	$\lvert\omega - \omega_{ex}\rvert/\omega_{ex}$	
m	k_1	k_2	k_3	X_0	Y_0	ω_1	ω_2	ω_{ex}	$\omega = \omega_1$	$\omega = \omega_2$
1	1	1	1	5	1	5.1962	5.1068	5.1078	1.73%	0.0185%
2	1	3	5	8	10	4.3012	4.2401	4.2406	1.43%	0.0185%
5	10	20	30	−10	10	60.08328	58.7677	58.7856	2.21%	0.0305%
10	50	70	90	20	−40	220.4972	215.6448	215.7113	2.22%	0.0308%
10	25	20	0.5	−10	10	6.0415	5.9533	5.9541	1.47%	0.0132%
100	200	300	400	−50	50	244.9653	239.5715	239.6455	2.22%	0.0309%

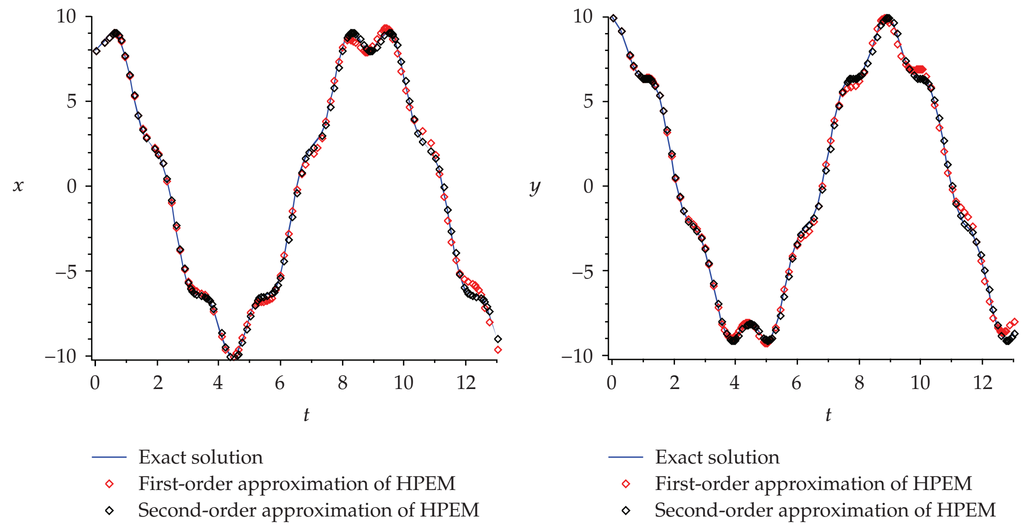
Using (2.8) and $\varepsilon = 0$, we can obtain

$$\ddot{u} + \frac{k_1}{m}u = \frac{k_2}{m}v + \frac{k_3}{m}v^3. \tag{6.4}$$

The first- and second-order analytical approximation for $u(t)$ is obtained using (6.4) and therefore, the first and second-order analytically approximates displacements $x(t)$ and $y(t)$ obtained using (2.5).

Similarly, substituting $\alpha = k_1(m_1 + m_2)/m_1m_2$ and $\beta = k_2(m_1 + m_2)/m_1m_2$ into (6.1), the exact solution of Case 3 is:

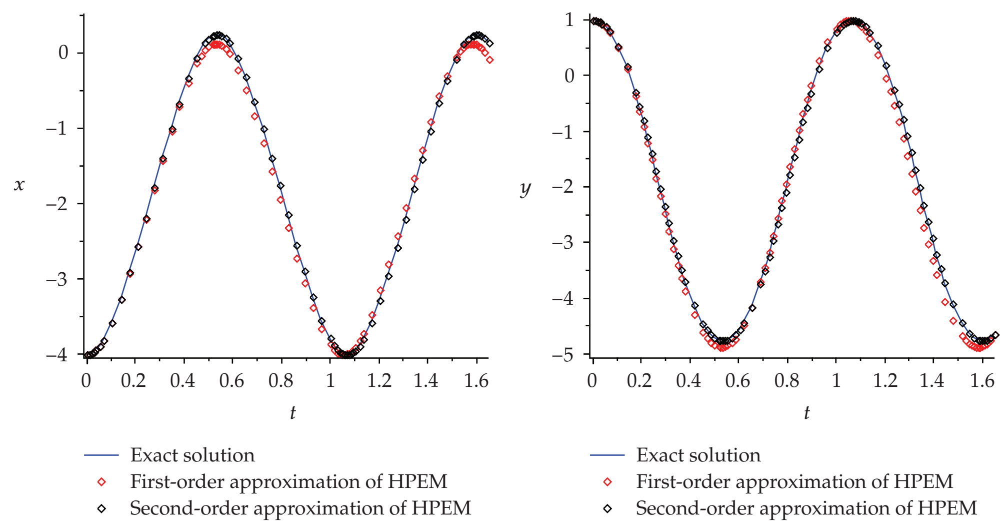
$$\omega_{ex}(A) = \frac{\pi}{2}\sqrt{\frac{(m_1 + m_2)}{m_1m_2}(k_1 + k_2A^2)}\left(\int_0^{\pi/2}\frac{dt}{1 - \delta\sin^2 t}\right)^{-1},$$
$$\delta = \frac{k_2(m_1 + m_2)A^2}{2(k_1(m_1 + m_2) + k_2(m_1 + m_2)A^2)}. \tag{6.5}$$



(a) (x-t) diagram

(b) (y-t) diagram

Figure 8: Comparison of the first- and second-order analytical approximate solutions with the exact solution for $m = 2.0$, $k_1 = 1.0$, $k_2 = 3.0$, $k_3 = 5.0$ with $x(0) = 8.0$ and $y(0) = 10.0$ (Case 2).



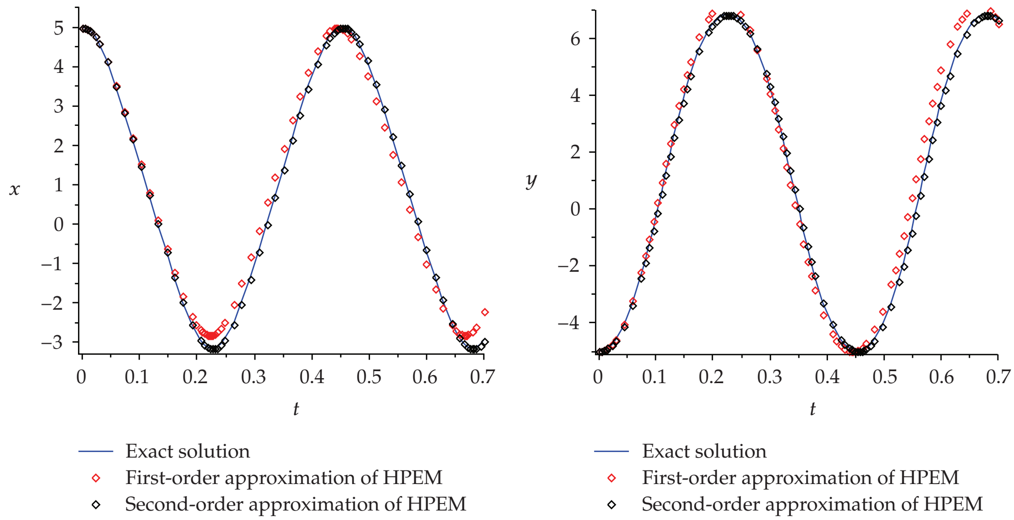
(a) (x-t) diagram

(b) (y-t) diagram

Figure 9: Comparison of the first- and second-order analytical approximate solutions with the exact solution for $m_1 = 1.0$, $m_2 = 2.0$, $k_1 = 5.0$, $k_2 = 1.0$ with $x(0) = -4.0$ and $y(0) = 1.0$ (Case 3).

After obtaining $u(t)$from (2.14), the first- and second-order analytically approximates displacements $x(t)$ and $y(t)$ obtained using (2.5).

It should be noted that ω_{ex} contains an integral which could only be solved numerically in general. The limitation of amplitude, A, in the cubic oscillation equation satisfies $\beta A^2+\alpha > 0$; the Duffing equation has a heteroclinic orbit with period $+\infty$ [36]. Hence, in order to avoid the heteroclinic orbit with period $+\infty$ for the Duffing equation in (2.16), the



(a) (x-t) diagram

(b) (y-t) diagram

Figure 10: Comparison of the first- and second-order analytical approximate solutions with the exact solution for $m_1 = 3.0$, $m_2 = 5.0$, $k_1 = 2.0$, $k_2 = 5.0$ with $x(0) = 5.0$ and $y(0) = -5.0$ (Case 3).

value of k_3 in the first two cases and k_2 for the third case should, respectively, satisfy in (6.4), (6.5), and (6.6)

$$k_3 > \frac{-k_1}{A^2} + \frac{2P}{l}\left(\frac{1}{A^2} + \frac{1}{l^2}\right), \tag{6.6}$$

$$k_3 > -\frac{k_1 + 2k_2}{2A^2}, \tag{6.7}$$

$$k_2 > -\frac{k_1}{A^2}, \tag{6.8}$$

where $k_1, k_2, k_3, l \in \mathrm{R}^+$ and $A, P \in R$.

To illustrate and verify accuracy of this analytical approach, comparisons of analytical and exact results for the practical cases are presented in Tables 1–3 and Figures 4–10. For this reason, we use the following specific parameter and initial values: Case 1: m, P, l, k_1, k_3, A, Case 2: m, k_1, k_2, k_3, X_0, Y_0, and Case 3: m_1, m_2, k_1, k_2, X_0, Y_0.

Figures 4–6, which are correspond to Case 1, indicate the comparison of this analytical method for different parameter with initial values $m = 1.0$, $l = 1.5$, $P = 5.0$, $k_1 = 5.0$ and $k_3 = 6.0$, $A = 3.0$ and $m = l = P = k_1 = 10.0$, and $k_3 = 50.0$, $A = 10.0$ and $m = 100.0$, $l = 50.0$, $P = 150.0$, $k_1 = 70.0$, $k_3 = 20.0$ and $A = 100.0$ which are in an excellent agreement with exact solutions.

Figures 7 and 8 represent the x-t and y-t diagrams which obtained analytically and exactly solving of Case 2 with different parameter and initial values $m = k_1 = k_2 = k_3 = 1.0$ with $X_0 = 5.0$, $Y_0 = 1.0$ and $m = 2.0$, $k_1 = 1.0$, $k_2 = 3.0$, $k_3 = 5.0$, $X_0 = 8.0$, $Y_0 = 10.0$. Also, the corresponding diagrams (x-t, y-t) of Case 3 are plotted in Figures 9 and 10. The different parameters and initial values that used in plotting diagrams of Case 3 are: $m_1 = 1.0$, $m_2 = 2.0$,

Table 3: Comparison of approximate and "Exact" frequencies for Case 3.

Constant parameters						Approximate solutions		Exact solution	$\lvert\omega-\omega_{ex}\rvert/\omega_{ex}$	
m_1	m_2	k_1	k_2	X_0	Y_0	ω_1	ω_2	ω_{ex}	$\omega=\omega_1$	$\omega=\omega_2$
1	2	5	1	−4	1	5.9687	5.8885	5.8892	1.35%	0.011%
3	5	2	5	5	−5	14.1798	13.8710	13.8752	2.20%	0.011%
1	5	5	1	5	−5	9.7980	9.6096	9.6119	1.94%	0.023%
10	5	10	10	20	30	15.0997	14.7763	14.7806	2.16%	0.029%
5	10	50	−0.01	−20	40	2.6268	2.5452	2.5468	3.14%	0.064%
100	1	10	5	20	25	10.2366	10.0545	10.0564	1.79%	0.020%
50	100	50	100	100	25	112.5067	110.0293	110.0633	2.22%	0.031%
1000	100	200	300	400	200	314.6461	307.7164	307.8115	2.17%	0.031%

$k_1 = 5.0$, $k_2 = 1.0$, $X_0 = -4.0$, $Y_0 = 1.0$ and $m_1 = 3.0$, $m_2 = 5.0$, $k_1 = 2.0$, $k_2 = 5.0$, $X_0 = 5.0$, $Y_0 = -5.0$.

According to these tables and figures, the difference between analytical and exact solutions is negligible. In other words, the first-order approximate results of PEM are accurate, but we significantly improve the percentage error from lower-order to second-order analytical approximations. We did it using modified PEM in second iteration for different parameters and initial amplitudes. Hence, it is concluded and provides an excellent agreement with the exact solutions.

7. Conclusions

The parameter expansion method (PEM) has been used to obtain the first- and second-order approximate frequencies and periods for Single- and Two-Degrees-Of-Freedom (SDOF and TDOF) systems. Excellent agreements between approximate frequencies and the exact one have been demonstrated and discussed, and the discrepancy of the second-order approximate frequency ω_2 with respect to the exact one is as low as 0.064%. In general, we conclude that this method is efficient for calculating periodic solutions for nonlinear oscillatory systems, and we think that the method has a great potential and could be applied to other strongly nonlinear oscillators.

Appendix

The exact solution of cubic nonlinear differential equation can be obtained by integrating the governing differential equation as follows:

$$\frac{1}{2}\dot{u}^2 + \frac{\alpha}{2}u^2 + \frac{\beta}{2}u^4 = C, \quad \forall t, \tag{A.1}$$

where C is a constant. Imposing initial conditions $u(0) = A$, $\dot{u}(0) = 0$ yields

$$C = \frac{\alpha}{2}A^2 + \frac{\beta}{4}A^4, \tag{A.2}$$

Equating (A.1) and (A.2) yields

$$\frac{1}{2}\dot{v}^2 + \frac{\alpha}{2}v^2 + \frac{\beta}{4}v^4 = \frac{\alpha}{2}A^2 + \frac{\beta}{4}A^4, \tag{A.3}$$

or equivalently

$$dt = \frac{dv}{\sqrt{\alpha(A^2 - v^2) + (\beta/2)(A^4 - v^4)}}. \tag{A.4}$$

Integrating (A.4), the period of oscillation T_e is

$$T(A) = 4\int_0^A \frac{dv}{\sqrt{\alpha(A^2 - v^2) + (\beta/2)\ (A^4 - v^4)}}. \tag{A.5}$$

Substituting $v = A\cos t$ into (A.5) and integrating

$$T(A) = \frac{4}{\sqrt{\alpha + \beta A^2}}\int_0^{\pi/2} \frac{dt}{\sqrt{1 - \delta \sin^2 t}}, \tag{A.6}$$

where

$$\delta = \frac{\beta A^2}{2(\alpha + \beta A^2)}. \tag{A.7}$$

The exact frequency ω_{ex} is also a function of A and can be obtained from the period of the oscillation as

$$\omega_{\text{ex}}(A) = \frac{\pi\sqrt{\alpha + \beta A^2}}{2}\left(\int_0^{\pi/2} \frac{dt}{1 - \delta \sin^2 t}\right)^{-1}. \tag{A.8}$$

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