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Research Article

A Note on Finite Quadrature Rules with a Kind of Freud Weight Function

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We introduce a finite class of weighted quadrature rules with the weight function $|x|^{-2a}\exp(-1/x^2)$ on $(-\infty,\infty)$ as $\int_{-\infty}^{\infty}|x|^{-2a}\exp(-1/x^2)f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + R_n[f]$, where x_i are the zeros of polynomials orthogonal with respect to the introduced weight function, w_i are the corresponding coefficients, and $R_n[f]$ is the error value. We show that the above formula is valid only for the finite values of n. In other words, the condition $a \geq \{\max n\} + 1/2$ must always be satisfied in order that one can apply the above quadrature rule. In this sense, some numerical and analytic examples are also given and compared.

1. Introduction

Recently in [1] the differential equation

$$x^2\left(px^2+q\right)\Phi_n''(x) + x\left(rx^2+s\right)\Phi_n'(x) - \left(n\left(r+(n-1)p\right)x^2 + \frac{\left(1-(-1)^n\right)s}{2}\right)\Phi_n(x) = 0 \tag{1.1}$$

is introduced, and its explicit solution is shown by

$$S_n\left(\begin{array}{cc|c} r & s & \\ p & q & x \end{array}\right) = \sum_{k=0}^{[n/2]} \binom{\left[\frac{n}{2}\right]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{\left(2i+(-1)^{n+1}+2[n/2]\right)p+r}{\left(2i+(-1)^{n+1}+2\right)q+s}\right) x^{n-2k}. \tag{1.2}$$

It is also called the generic equation of classical symmetric orthogonal polynomials [1, 2]. If this equation is written in a self-adjoint form then the first-order equation

$$x\frac{d}{dx}\left(\left(px^2+q\right)W(x)\right)=\left(rx^2+s\right)W(x) \tag{1.3}$$

is derived. The solution of (1.3) is known as an analogue of Pearson distributions family and can be indicated as

$$W\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right)=\exp\left(\int\frac{(r-2p)x^2+s}{x\left(px^2+q\right)}dx\right). \tag{1.4}$$

In general, there are four main subclasses of distributions family (1.4) (as subsolutions of (1.3)) whose explicit probability density functions are, respectively,

$$\begin{aligned} K_1W\left(\begin{matrix} -2a-2b-2, & 2a \\ -1, & 1\end{matrix}\,\middle|\, x\right) &= \frac{\Gamma(a+b+3/2)}{\Gamma(a+1/2)\Gamma(b+1)}x^{2a}\left(1-x^2\right)^b, \\ &\quad -1\le x\le 1,\quad a+\frac{1}{2}>0,\quad b+1>0, \end{aligned} \tag{1.5}$$

$$K_2W\left(\begin{matrix} -2, & 2a \\ 0, & 1\end{matrix}\,\middle|\, x\right)=\frac{1}{\Gamma(a+1/2)}x^{2a}\exp\left(-x^2\right),\quad -\infty<x<\infty,\ a+\frac{1}{2}>0, \tag{1.6}$$

$$\begin{aligned} K_3W\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1\end{matrix}\,\middle|\, x\right) &= \frac{\Gamma(b)}{\Gamma(b+a-1/2)\Gamma(-a+1/2)}\frac{x^{-2a}}{\left(1+x^2\right)^b}, \\ &\quad -\infty<x<\infty,\quad b>0,\quad a<\frac{1}{2},\quad b+a>\frac{1}{2}, \end{aligned} \tag{1.7}$$

$$K_4W\left(\begin{matrix} -2a+2, & 2 \\ 1, & 0\end{matrix}\,\middle|\, x\right)=\frac{1}{\Gamma(a-1/2)}x^{-2a}\exp\left(-\frac{1}{x^2}\right),\quad -\infty<x<\infty,\ a>\frac{1}{2}. \tag{1.8}$$

The values $K_i; i=1,2,3,4$ play the normalizing constant role in these distributions. Moreover, the value of distribution vanishes at $x=0$ in each four cases, that is, $W(0;p,q,r,s)=0$ for $s\neq 0$. Hence, (1.4) is called in [1] "The dual symmetric distributions family."

As a special case of $W(x;p,q,r,s)$, let us choose the values $p=1$, $q=0$, $r=-2a+2$, and $s=2$ corresponding to distribution (1.8) here and replace them in (1.1) to get

$$x^4\Phi_n''(x)+2x\left((1-a)x^2+1\right)\Phi_n'(x)-\left(n(n+1-2a)x^2+1-(-1)^n\right)\Phi_n(x)=0. \tag{1.9}$$

If (1.9) is solved, the polynomial solution of monic type

$$\overline{S}_n\left(\begin{matrix}-2a+2 & 2\\ 1 & 0\end{matrix}\middle| x\right)=\prod_{i=0}^{[n/2]-1}\frac{2}{2i+2[n/2]+(-1)^{n+1}+2-2a}\times\sum_{k=0}^{[n/2]}\binom{\left[\frac{n}{2}\right]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{2i+2[n/2]+(-1)^{n+1}+2-2a}{2}\right)x^{n-2k} \tag{1.10}$$

is obtained. According to [1], these polynomials are finitely orthogonal with respect to a special kind of Freud weight function, that is, $x^{-2a}\exp(-1/x^2)$, on the real line $(-\infty,\infty)$ if and only if $a\geq\{\max n\}+1/2$; see also [3, 4]. In other words, we have

$$\int_{-\infty}^{\infty}|x|^{-2a}\exp\left(-\frac{1}{x^2}\right)\overline{S}_n\left(\begin{matrix}-2a+2 & 2\\ 1 & 0\end{matrix}\middle| x\right)\overline{S}_m\left(\begin{matrix}-2a+2 & 2\\ 1 & 0\end{matrix}\middle| x\right)dx=\left(\prod_{i=1}^{n}\frac{2(-1)^i(i-a)+2a}{(2i-2a+1)(2i-2a-1)}\right)\Gamma\left(a-\frac{1}{2}\right)\delta_{n,m}, \tag{1.11}$$

if and only if $m,n=0,1,2,\ldots,$ $N=\max\{m,n\}\leq a-1/2$, $(-1)^{2a}=1$ and

$$\delta_{n,m}=\begin{cases}0, & \text{if } n\neq m,\\ 1, & \text{if } n=m.\end{cases} \tag{1.12}$$

Furthermore, the polynomials (1.10) also satisfy a three-term recurrence relation as

$$\overline{S}_{n+1}(x)=x\,\overline{S}_n(x)-\frac{2(-1)^n(n-a)+2a}{(2n-2a+1)(2n-2a-1)}\,\overline{S}_{n-1}(x),\quad \overline{S}_0(x)=1,\ \overline{S}_1(x)=x,\ n\in\mathbb{N}. \tag{1.13}$$

But the polynomials $\overline{S}_n(x;1,0,-2a+2,2)$ are suitable tool to finitely approximate arbitrary functions, which satisfy the Dirichlet conditions (see, e.g., [5]). For example, suppose that $N=\max\{m,n\}=3$ and $a>7/2$ in (1.10). Then, the function $f(x)$ can finitely be approximated as

$$f(x)\cong C_0\overline{S}_0(x;1,0,-2a+2,2)+C_1\overline{S}_1(x;1,0,-2a+2,2)+C_2\overline{S}_2(x;1,0,-2a+2,2)+C_3\overline{S}_3(x;1,0,-2a+2,2), \tag{1.14}$$

where

$$C_m=\int_{-\infty}^{\infty}\frac{|x|^{-2a}\exp(-1/x^2)\overline{S}_m\left(\begin{smallmatrix}-2a+2 & 2\\ 1 & 0\end{smallmatrix}\middle| x\right)f(x)dx}{\left(\prod_{i=1}^{m}\left(\left(2\ (-1)^i(i-a)+2a\right)/(2i-2a+1)(2i-2a-1)\right)\Gamma(a-1/2)\right)}, \tag{1.15}$$

for $m=0,1,2,3$.

Clearly (1.14) is valid only when the general function $x^m|x|^{-2a}\exp(-1/x^2)\,f(x)$ in (1.15) is integrable for any $m=0,1,2,3$. This means that the finite set $\{\overline{S}_i(x;1,0,-2a+2,2)\}_{i=0}^{3}$ is a basis space for all polynomials of degree at most three. So if $f(x)=a_3x^3+a_2x^2+a_1x+a_0$, the approximation (1.14) is exact. By noting this, here is a good position to express an application of the mentioned polynomials in weighted quadrature rules [6, 7] by a straightforward example. Let us consider a two-point approximation as

$$\int_{-\infty}^{\infty}|x|^{-2a}\exp\left(-\frac{1}{x^2}\right)f(x)dx \cong w_1 f(x_1)+w_2 f(x_2), \tag{1.16}$$

provided that $a > 5/2$. According to the described themes, (1.16) must be exact for all elements of the basis $f(x)=\{x^3,x^2,x,1\}$ if and only if x_1,x_2 are two roots of $\overline{S}_2(x;1,0,-2a+2,2)$. For instance, if $a=3>5/2$ then (1.16) should be changed to

$$\int_{-\infty}^{\infty}x^{-6}\exp\left(-\frac{1}{x^2}\right)f(x)dx \cong w_1 f\left(\sqrt{\frac{2}{3}}\right)+w_2 f\left(-\sqrt{\frac{2}{3}}\right), \tag{1.17}$$

in which $\sqrt{2/3}$ and $-\sqrt{2/3}$ are zeros of $\overline{S}_2(x;1,0,-4,2)$, and w_1,w_2 are computed by solving the linear system

$$w_1+w_2=\int_{-\infty}^{\infty}x^{-6}\exp\left(-\frac{1}{x^2}\right)dx=\frac{3}{4}\sqrt{\pi},\qquad \sqrt{\frac{2}{3}}\,(w_1-w_2)=\int_{-\infty}^{\infty}x^{-5}\exp\left(-\frac{1}{x^2}\right)dx=0. \tag{1.18}$$

Hence, after solving (1.18) the final form of (1.16) is known as

$$\int_{-\infty}^{\infty}x^{-6}\exp\left(-\frac{1}{x^2}\right)f(x)dx \cong \frac{3}{8}\sqrt{\pi}\left(f\left(\sqrt{\frac{2}{3}}\right)+f\left(-\sqrt{\frac{2}{3}}\right)\right). \tag{1.19}$$

This approximation is exact for all arbitrary polynomials of degree at most 3.

2. Application of Polynomials (1.10) in Weighted Quadrature Rules: General Case

As we know, the general form of weighted quadrature rules is given by

$$\int_{\alpha}^{\beta}w(x)f(x)dx=\sum_{i=1}^{n}w_i f(x_i)+R_n[f], \tag{2.1}$$

in which the weights $\{w_i\}_{i=1}^n$ and the nodes $\{x_i\}_{i=1}^n$ are unknown values, $w(x)$ is a positive function, and $[\alpha,\beta]$ is an arbitrary interval; see, for example, [6, 7]. Moreover the residue $R_n[f]$ is determined (see, e.g., [7]) by

$$R_n[f] = \frac{f^{(2n)}(\xi)}{(2n)!}\int_\alpha^\beta w(x)\prod_{i=1}^n (x-x_i)^2 dx, \quad \alpha < \xi < \beta. \tag{2.2}$$

It can be proved in (2.1) that $R_n[f] = 0$ for any linear combination of the sequence $\{1, x, x^2, \dots, x^{2n-1}\}$ if and only if $\{x_i\}_{i=1}^n$ are the roots of orthogonal polynomials of degree n with respect to the weight function $w(x)$ on the interval $[\alpha,\beta]$. For more details, see [6]. Also, it is proved that to derive $\{w_i\}_{i=1}^n$ in (2.1), it is not required to solve the following linear system of order $n \times n$:

$$\sum_{i=1}^n w_i x_i^j = \int_\alpha^\beta w(x)x^j dx \quad \text{for } j = 0,1,\dots,2n-1, \tag{2.3}$$

rather, one can directly use the relation

$$\frac{1}{w_i} = \widehat{P}_0^2(x_i) + \widehat{P}_1^2(x_i) + \cdots + \widehat{P}_{n-1}^2(x_i) \quad \text{for } i = 1,2,\dots,n, \tag{2.4}$$

where $\widehat{P}_i(x)$ are orthonormal polynomials of $P_i(x)$ defined as

$$\widehat{P}_i(x) = \left(\int_\alpha^\beta w(x)P_i^2(x)dx\right)^{-1/2} P_i(x). \tag{2.5}$$

In this way, as it is shown in [8, 9], $\widehat{P}_i(x)$ satisfies a particular type of three-term recurrence as

$$x\widehat{P}_{n-1}(x) = \alpha_n\widehat{P}_n(x) + \beta_n\widehat{P}_{n-1}(x) + \alpha_{n-1}\widehat{P}_{n-2}(x). \tag{2.6}$$

Now, by noting these comments and the fact that the symmetric polynomials $\overline{S}_n(x;1,0,-2a+2,2)$ are finitely orthogonal with respect to the weight function $W(x,a) = |x|^{-2a}\exp(-1/x^2)$ on the real line, we can define a finite class of quadrature rules as

$$\int_{-\infty}^{\infty} |x|^{-2a}\exp\left(-\frac{1}{x^2}\right)f(x)dx = \sum_{j=1}^n w_j f(x_j) + R_n[f], \tag{2.7}$$

in which x_j are the roots of $\overline{S}_n(x;1,0,-2a+2,2)$ and w_j are computed by

$$\frac{1}{w_j} = \sum_{i=0}^{n-1}\left(\overline{S}_i^*(1,0,-2a+2,2;x_j)\right)^2, \quad \text{for } j = 0,1,2,\dots,n. \tag{2.8}$$

Moreover, for the residue value we have

$$R_n[f] = \frac{f^{(2n)}(\xi)}{(2n)!}\int_{-\infty}^{\infty}|x|^{-2a}\exp\left(-\frac{1}{x^2}\right)\prod_{j=1}^{n}(x-x_j)^2dx, \quad \xi\in\mathbb{R}. \tag{2.9}$$

2.1. *An Important Remark*

It is important to note that by applying the change of variable $1/x^2 = t$ in the left-hand side of (2.7) the orthogonality interval $(-\infty,\infty)$ changes to $[0,\infty)$ and subsequently

$$\int_{-\infty}^{\infty}|x|^{-2a}\exp\left(-\frac{1}{x^2}\right)f(x)dx = \int_{0}^{\infty}t^{a-3/2}e^{-t}f\left(\frac{1}{\sqrt{t}}\right)dt. \tag{2.10}$$

As it is observed, the right-hand integral of (2.10) contains the well-known Laguerre weight function $x^u e^{-x}$ for $u = a - 3/2$. Hence, one can use Gauss-Laguerre quadrature rules [8, 9] with the special parameter $u = a - 3/2$. This process changes (2.7) in the form

$$\int_{-\infty}^{\infty}|x|^{-2a}\exp\left(-\frac{1}{x^2}\right)f(x)dx = \sum_{j=1}^{n}w_j^{(a-3/2)}f\left(\frac{1}{\sqrt{x_j^{(a-3/2)}}}\right) + R_n\left[f\left(\frac{1}{\sqrt{x}}\right)\right], \tag{2.11}$$

in which $x_j^{(a-3/2)}$ are the zeros of Laguerre polynomials $L_n^{(a-3/2)}(x)$. But, there is a large disadvantage for formula (2.11). According to (2.2) or (2.9), the residue of integration rules generally depends on $f^{(2n)}(\xi)$; $\alpha < \xi < \beta$. Thus, by noting (2.11) we should have

$$\frac{d^{2n}f(1/\sqrt{x})}{dx^{2n}} = \sum_{i=0}^{2n}\phi_i(x)f^{(i)}\left(\frac{1}{\sqrt{x}}\right), \tag{2.12}$$

where $\phi_i(x)$ are real functions to be computed and $f^{(i)}$, $i = 0,1,2,\ldots,2n$, are the successive derivatives of function $f(x)$.

As we observe in (2.12), $f(x)$ cannot be in the form of an arbitrary polynomial function in order that the right-hand side of (2.12) is equal to zero. In other words, (2.11) is not exact for the basis space $f(x) = x^j$, $j = 0,1,2,\ldots,2n-1$. This is the main disadvantage of using (2.11), as the examples of next section support this claim.

3. Examples

Example 3.1. Since a 2-point formula was presented in (1.19), in this example we consider a 3-point integration formula. For this purpose, we should first note that according to (1.11) the condition $a > 7/2$ is necessary. Hence, let us, for instance, assume that $a = 4$. After some computations the related quadrature rule would take the form

$$\int_{-\infty}^{\infty}x^{-8}\exp\left(-\frac{1}{x^2}\right)f(x)dx = \frac{3}{16}\sqrt{\pi}\left(3f\left(\sqrt{\frac{2}{3}}\right)+4f(0)+3f\left(-\sqrt{\frac{2}{3}}\right)\right)+R_3[f], \tag{3.1}$$

where

$$R_3[f] = \frac{f^{(6)}(\xi)}{6!}\int_{-\infty}^{\infty} x^{-8}\exp\left(-\frac{1}{x^2}\right)\left(\overline{S}_3\left(\begin{matrix}-6 & 2\\ 1 & 0\end{matrix}\middle| x\right)\right)^2 dx$$
$$= \frac{\sqrt{\pi}}{1080} f^{(6)}(\xi), \quad \xi \in \mathbf{R}, \tag{3.2}$$

and $x_1 = \sqrt{2/3}$, $x_2 = 0$, and $x_3 = -\sqrt{2/3}$ are the roots of $\overline{S}_3(x;1,0,-6,2) = x^3 - (2/3)x$. Moreover, w_1, w_2, w_3 can be computed by

$$\frac{1}{w_j} = \sum_{i=0}^{2}\left(\overline{S}_i^*(x_j;1,0,-6,2)\right)^2, \quad j = 1,2,3, \tag{3.3}$$

in which

$$\overline{S}_i^*(x_j;1,0,-6,2) = \frac{\overline{S}_i(x_j;1,0,-6,2)}{\left\langle \overline{S}_i(x_j;1,0,-6,2), \overline{S}_i(x_j;1,0,-6,2)\right\rangle^{1/2}}. \tag{3.4}$$

Example 3.2. To have a 4-point formula, we should again note that $a > 9/2$ is a necessary condition. In this sense, if, for example, $a = 5$ then we eventually get

$$\int_{-\infty}^{\infty} x^{-10}\exp\left(-\frac{1}{x^2}\right) f(x)dx$$
$$= \frac{15}{64}\sqrt{\pi}\left(7-2\sqrt{10}\right)\left(f\left(\sqrt{\frac{10+2\sqrt{10}}{15}}\right) + f\left(-\sqrt{\frac{10+2\sqrt{10}}{15}}\right)\right)$$
$$+ \frac{15}{64}\sqrt{\pi}\left(7+2\sqrt{10}\right)\left(f\left(\sqrt{\frac{10-2\sqrt{10}}{15}}\right) + f\left(-\sqrt{\frac{10-2\sqrt{10}}{15}}\right)\right) + R_4[f], \tag{3.5}$$

where

$$R_4[f] = \frac{f^{(8)}(\xi)}{8!}\int_{-\infty}^{\infty} x^{-10}\exp\left(-\frac{1}{x^2}\right)\left(\overline{S}_4\left(\begin{matrix}-8 & 2\\ 1 & 0\end{matrix}\middle| x\right)\right)^2 dx = \frac{\sqrt{\pi}}{75600} f^{(8)}(\xi), \quad \xi \in \mathbf{R}. \tag{3.6}$$

Clearly this formula is exact for the basis elements $f(x) = x^j$, $j = 0,1,2,\ldots,7$, and the nodes of quadrature (3.5) are the roots of $\overline{S}_4(x;1,0,-8,2) = x^4 - (4/3)x^2 + 4/15$.

4. Numerical results

In this section, some numerical examples are given and compared. The numerical results related to the 2-point formula (1.19) are presented in Table 1, the results related to 3-point

Table 1: $\int_{-\infty}^{+\infty} x^{-6}\exp(-1/x^2) f(x)dx$.

$f(x)$	Approx. value (2-point)	Exact value	Error
$\cos x$	0.9103037512	0.9382539141	0.0279501629
$\exp(-2/x^2)$	0.0661839608	0.0852772257	0.0190932649
$\exp(-\cos x)$	0.6702559297	0.6812645398	0.0110086101

Table 2: $\int_{-\infty}^{+\infty} x^{-8}\exp(-1/x^2) f(x)dx$.

$f(x)$	Approx. value (3-point)	Exact value	Error
$\exp(-\cos x)$	1.494420894	1.492841821	0.001579073
$\sqrt{1+\sin x^2}$	3.866024228	3.866700560	0.000676332
$\sqrt{1+\cos x^2}$	4.544708979	4.561266761	0.016557782

Table 3: $\int_{-\infty}^{+\infty} x^{-10}\exp(-1/x^2) f(x)dx$.

$f(x)$	Approx. value (4-point)	Exact value	Error
$\sqrt{1+\cos x^2}$	16.21776936	16.21978539	0.002016030
$(1+x^2)^{-1/2}$	10.30987753	10.31704740	0.007116987
$\exp(-x^2-2)$	1.198219038	1.199125136	0.000906098

formula (3.1) are given in Table 2, and finally the results related to 4-point formula (3.5) are presented in Table 3.

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