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Research Article

MRA Parseval Frame Wavelets and Their Multipliers in $L^2(R^n)$

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We characterize all generalized lowpass filters and multiresolution analysis(MRA) Parseval frame wavelets in $L^2(R^n)$ with matrix dilations of the form $(Df)(x) = \sqrt{2}f(Ax)$, where A is an arbitrary expanding $n\times n$ matrix with integer coefficients, such that $|\det A| = 2$. At first, we study the pseudo-scaling functions, generalized lowpass filters, and multiresolution analysis (MRA) Parseval frame wavelets and give some important characterizations about them. Then, we describe the multiplier classes associated with Parseval frame wavelets in $L^2(R^n)$ and give an example to prove our theory.

1. Introduction

Wavelet theory has been studied extensively in both theory and applications since 1980's (see [1–3]). One of the basic advantage of wavelets is that an even can be simultaneously described in the frequency domain as well as in the time domain. This feature permits a multiresolution analysis of data with different behaviors on different scales. The main advantage of wavelets is their time-frequency localization property. Many signals can be efficiently represented by wavelets.

The classical MRA wavelets are probably the most important class of orthonormal wavelets. Because they guarantee the existence of fast implementation algorithm, many of the famous examples often used in applications belong to this class. However, there are useful filters, such as $m(\omega) = (1/2)(1 + e^{3i\omega})$, that do not produce orthonormal basis; nevertheless, they do produce systems that have the reconstruction property as well as many other useful features. It is natural, therefore, to develop a theory involving more general filters that can produce systems having these properties.

A tight wavelet frame is a generalization of an orthonormal wavelet basis by introducing redundancy into a wavelet system [3]. By allowing redundancy in a wavelet system, one has much more freedom in the choice of wavelets. Tight wavelet frames have some desirable features, such as near translation invariant wavelet frame transforms, and it may be easier to recognize patterns in a redundant transform. For advantages and applications of tight wavelet frames, the reader is referred to [4–15] and many references therein. Recently, the theory of high dimensional wavelet is widely studied by the people, such as [16, 17].

In [18], authors discussed wavelet multipliers, scaling function multipliers, and lowpass filter multipliers in $L^2(R^n)$. In [19], authors introduced a class of generalized lowpass filter that allowed them to define and construct the MRA Parseval frame wavelets. This led them to an associated class of generalized scaling functions that were not necessarily obtained from a multiresolution analysis. Also, they generalized notions of the wavelet multipliers in [18] to the case of wavelet frame and got several properties of the multipliers of Parseval frame wavelets.

In this paper, we characterize all generalized lowpass filter and MRA Parseval frame wavelets (PFWs) in $L^2(R^n)$ with matrix dilations of the form $(Df)(x) = \sqrt{2}f(Ax)$, where A is an arbitrary expanding $n \times n$ matrix with integer coefficients, such that $|\det A| = 2$. Firstly, we study some properties of the generalized wavelets, scaling functions, and filters in $L^2(R^n)$. Our result is a generalization of the construction of PFWs from generalized lowpass filters that is introduced in [19]. Though we follow [19] as a blueprint, it is well known that the situation in higher dimension is so complex that we have to recur to some special matrices to solve problem. Thus, our ways are different from original ones. And we borrow some thoughts and technique in [16]. Then, we give some characterizations of the multiplier classes associated with Parseval frame wavelets in $L^2(R^n)$.

Let us now describe the organization of the material that follows. Section 2 is of a preliminary character: it contains various results on matrices belonging to the class $E_n^{(2)}$ and some facts about a Parseval frame wavelet. In Section 3, we study the pseudoscaling functions, the generalized lowpass filters, and the MRA PFWs and give some important characterizations about them. In Section 4, we describe the multiplier classes associated with Parseval frame wavelets in $L^2(R^n)$. At last, we give an example to prove our theory.

2. Preliminaries

Let us now establish some basic notations.

We denote by $\mathbf{T}^n$ the n-dimensional torus. By $L^p(\mathbf{T}^n)$ we denote the space of all Z^n-periodic functions f (i.e., f is 1-periodic in each variable) such that $\int_{\mathbf{T}^n}|f(x)|^p dx < +\infty$. The standard unit cube $[-(1/2), 1/2)^n$ will be denoted by C. The subsets of R^n invariant under Z^n translations and the subsets of $\mathbf{T}^n$ are often identified.

We use the Fourier transform in the form

$$\widehat{f}(\omega) = \int_{R^n} f(x) e^{-2\pi i \langle x,\omega \rangle} dx, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in R^n.

For $f, g \in L^2(R^n)$ we denote the function $[f, g](\omega)$ as follows:

$$[f, g](\omega) = \sum_{k \in Z^n} f(\omega + k)\overline{g(\omega + k)}. \tag{2.2}$$

In particular, for $f \in L^2(R^n)$, we will write $\sigma_f(\omega) := \sum_{k\in Z^n} |\widehat{f}(\omega+k)|^2$, which is named as the bracket function of f. For $\sigma_f(\omega) = \sum_{k\in Z^n} |\widehat{f}(\omega+k)|^2$, we let Ω_f be the Z^n-translation invariant subset of R^n defined, modulo a null set, by $\Omega_f = \operatorname{supp} \sigma_f = \{\omega \in R^n : \widehat{f}(\omega+k) \neq 0, \text{ for some } k \in Z^n\}$.

The Lebesgue measure of a set $S \subseteq R^n$ will be denoted by $|S|$. When measurable sets X and Y are equal up to a set of measure zero, we write $X \doteq Y$.

Then we introduce some notations and the existing results about expanding matrices.

Let $E_n^{(2)}$ denote the set of all expanding matrices A such that $|\det A| = 2$. The expanding matrices mean that all eigenvalues have magnitude greater than 1. For $A \in E_n^{(2)}$, we denote by B the transpose of $A : B = A^t$. It is obvious that $B \in E_n^{(2)}$.

The following elementary lemma [16, Lemma 2.2] provides us with a convenient description of BZ^n for an arbitrary $A \in E_n^{(2)}$, and it will be used in Section 3.

Lemma 2.1. *Let $B \in E_n^{(2)}$ be any integer matrix such that $|\det B| = 2$. Then the group Z^n/BZ^n is isomorphic to $B^{-1}Z^n/Z^n$and the order of Z^n/BZ^n is equal to 2. In particular, if $\alpha \in Z^n/BZ^n$ and $\beta = B^{-1}\alpha$, then $Z^n = BZ^n \cup (BZ^n + \alpha)$ and $B^{-1}Z^n = Z^n \cup (Z^n + \beta)$.*

Our standard example that will be frequently used is the quincunx matrix $Q = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right) \in E_n^{(2)}$. Observe that Q acts on R^2 as rotation by $\pi/4$ composed with dilation by $\sqrt{2}$. In the quincunx case, our standard choice will be $\alpha = (1,0), \beta = (1/2, 1/2)$.

In this paper, we will work with two families of unitary operators on $L^2(R^n)$. The first one consists of all translation operators $T_k : L^2(R)^n \to L^2(R^n), k \in Z^n$, defined by $(T_k f)(x) = f(x-k)$. The second one consists of all integer powers of the dilation operator $D_A : L^2(R^n) \to L^2(R^n)$ defined by $(Df)(x) = \sqrt{2} f(Ax)$ with $A \in E_n^{(2)}$.

Let us now fix an arbitrary matrix $A \in E_n^{(2)}$. For a function $\psi \in L^2(R^n)$, we will consider the affine system Ψ defined by

$$\Psi = \left\{ \psi_{j,\,k}(x) \mid \psi_{j,\,k}(x) = 2^{j/2} \psi\left(A^j x - k\right),\ j \in Z,\ k \in Z^n \right\}. \tag{2.3}$$

Let us recall the definition of a Parseval frame and a Parseval frame wavelet.

Definition 2.2. We say that a countable family $\{f_j\}, j \in J$, in a separable Hilbert space H, is a Parseval frame (PF) for H if the equality $\|f\|^2 = \sum_{j\in J} |\langle f, f_j\rangle|^2$ is satisfied for all $f \in H$.

Definition 2.3. We say that $\psi \in L^2(R^n)$ is a Parseval frame wavelet (briefly: PFW) if the system (2.3) is a Parseval frame for $L^2(R^n)$.

Then we recall a result from [20] that characterizes Parseval frame wavelets associated with more general matrix dilations. We state the special case of that theorem appropriate to the discussion in this paper.

Lemma 2.4 ([20, Theorem 6.12]). *Let A be an arbitrary matrix in $E_n^{(2)}$, $B = A^t$, and $\psi \in L^2(R^n)$. Then the system* (2.3) *is a PFW if and only if both the equality*

$$\sum_{j\in Z} \left|\widehat{\psi}(B^j\omega)\right|^2 = 1, \quad a.e. \tag{2.4}$$

and the equality

$$\sum_{j=0}^{+\infty}\widehat{\varphi}\left(B^j\omega\right)\overline{\widehat{\varphi}\left(B^j(\omega+Bk+\alpha)\right)}=0,\quad a.e.,\ \forall\ k\in Z^n,\ \alpha\in Z^n/BZ^n \tag{2.5}$$

are satisfied.

In the following, we will give some definitions which will be used in this paper. In fact, they are some generalizations of the notations in [19].

Definition 2.5. A measurable Z^n-periodic function m on R^n is called a generalized filter if it satisfies

$$|m(\omega)|^2+|m(\omega+\beta)|^2=1\quad \text{a.e.}\,\omega, \tag{2.6}$$

where β is defined in Lemma 2.1.

We will denote by $\widetilde{F}$ the set of generalized filters and put $\widetilde{F}^+=\{m\in\widetilde{F}:m\geq 0\}$. Observe that $m\in\widetilde{F}\Rightarrow|m|\in\widetilde{F}^+$.

Definition 2.6. A function $\varphi\in L^2(R^n)$ is called a pseudoscaling function if there exists a filter $m\in\widetilde{F}$ such that

$$\widehat{\varphi}(B\omega)=m(\omega)\widehat{\varphi}(\omega)\quad \text{a.e. }\omega. \tag{2.7}$$

Notice that m is not uniquely determined by the pseudo-scaling function φ. Therefore, we shall denote by $\widetilde{F}_\varphi$ the set of all $m\in\widetilde{F}$ such that m satisfies (2.7) for φ. For example, if $\varphi=0$, then, $\widetilde{F}_\varphi=\widetilde{F}$. If φ is a scaling function of an orthonormal MRA wavelet, then $\widetilde{F}_\varphi$ is a singleton; its only element is the lowpass filter m associated with φ. Notice that for a pseudo-scaling function φ, the function $|\widehat{\varphi}|$ is also a pseudo-scaling function, and if $m\in\widetilde{F}$, then $|m|\in\widetilde{F}_{|\widehat{\varphi}|}$.

Suppose that $m\in\widetilde{F}^+$. Since $0\leq m(\omega)\leq 1$, a.e. ω, the function

$$\widehat{\varphi_m}(\omega)=:\prod_{j=1}^{+\infty}m\left(B^{-j}\omega\right) \tag{2.8}$$

is well defined a.e.; moreover, we have

$$\widehat{\varphi_m}(B\omega)=m(\omega)\widehat{\varphi_m}(\omega),\quad \text{a.e. }\ \omega. \tag{2.9}$$

Following [16], the function $\widehat{\varphi_m}$ defined by (2.9) belongs to $L^2(R^n)$, and the function $\widehat{\varphi_m}$ is a pseudo-scaling function such that $m\in\widetilde{F}_{\varphi_m}$.

Consequently, if $m\in\widetilde{F}$ is an arbitrary generalized filter, then the function $\widehat{\varphi_{|m|}}$ is a pseudo-scaling function and $|m|\in\widetilde{F}_{\varphi_{|m|}}$.

Definition 2.7. For $m \in \widetilde{F}^{+}$, define

$$N_0(m) = \left\{ \omega \in R^n : \lim_{j \to +\infty} \widehat{\varphi_m}\left(B^{-j}\omega\right) = 0 \right\}. \tag{2.10}$$

We say that $m \in \widetilde{F}$ is a generalized low-pass filter if $|N_0(|m|)| = 0$. The set of all generalized low-pass filters is denoted by $\widetilde{F}_0$.

Then, we will give the definition of MRA PFW.

Definition 2.8. A PFW ψ is an MRA PFW if there exists a pseudo-scaling function φ and $m \in \widetilde{F}_{\varphi}$ and a unimodular function $s \in L^2(T^n)$ such that

$$\widehat{\psi}(B\omega) = e^{2\pi\omega i} s(B\omega) \overline{m(\omega + \beta)} \widehat{\varphi}(\omega), \quad \text{a.e.} \quad \omega. \tag{2.11}$$

Let us conclude this introductory section by noting that many of the results that follow can be proved for dilations by expanding integer matrices with arbitrary determinant. Some of these extensions are obtained easily with essentially the same proofs; others require subtler and more involved arguments. But, for the sake of simplicity, we restrict ourselves to the class $E_n^{(2)}$.

3. MRA Parseval Frame Wavelets

The main purpose of this section is to study the pseudo-scaling functions, the generalized filters, and the MRA PFWs in $L^2(R^n)$. We give some important characterizations about them.

In the following we firstly give several lemmas in order to prove our main results.

Lemma 3.1. *Suppose that φ is a pseudo-scaling function and $m \in \widetilde{F}_{\varphi}$. If*

$$\lim_{j \to +\infty} \left| \widehat{\varphi}\left(B^{-j}\omega\right) \right| = 1, \quad \text{a.e. } \omega, \tag{3.1}$$

then,

$$|\widehat{\varphi}(\omega)| = \left| \prod_{j=1}^{+\infty} m\left(B^{-j}\omega\right) \right|, \quad \text{a.e. } \omega, \tag{3.2}$$

and $|N_0(|m|)| = 0$.

Proof. By (2.7), we have

$$|\widehat{\varphi}(\omega)| = \left| \prod_{j=1}^{n} m\left(B^{-j}\omega\right) \right| \left| \widehat{\varphi}\left(B^{-j}\omega\right) \right|, \quad \text{a.e. } \omega. \tag{3.3}$$

Using (2.9), we obtain that $|\widehat{\varphi}(\omega)| = \widehat{\varphi_{|m|}}$ and $|N_0(|m|)| = 0$ is clearly satisfied. Thus, the function $m \in \widetilde{F}$ is a generalized low-pass filter. □

Lemma 3.2. *If $f \in L^1(R^n)$, then, for a.e. $\omega \in R^n$, $\lim_{j\to+\infty}|f(B^j\omega)| = 0$.*

Proof. Assuming that $f \in L^1(R^n)$ and applying the monotone convergence theorem we obtain

$$\begin{aligned}\int_{R^n}\sum_{j\in N}\left|f\left(B^j\omega\right)\right|d\omega &= \sum_{j\in N}\int_{R^n}\left|f\left(B^j\omega\right)\right|d\omega \\ &= \sum_{j\in N}2^{-j}\int_{R^n}\left|f(\xi)d\xi\right| = \|f\|_1 < +\infty.\end{aligned} \tag{3.4}$$

It follows that for a.e. $\omega \in R^n$, $\sum_{j\in N}|f(B^j\omega)|$ is finite. Therefore, for a.e. $\omega \in R^n$, $\lim_{j\to+\infty}|f(B^j\omega)| = 0$. □

Then, we will give a characterization of the generalized lowpass filter.

Theorem 3.3. *Suppose ψ is an MRA PFW and φ is a pseudo-scaling function satisfying* (2.11)*. Then, m defined by* (2.11) *is a generalized lowpass filter.*

Proof. Since ψ is an MRA PFW, from (2.4), (2.6), and (2.11), we can obtain

$$\begin{aligned}1 &= \sum_{j\in Z}\left|\widehat{\psi}\left(B^j\omega\right)\right|^2 \\ &= \sum_{j\in Z}\left|m\left(B^{j-1}\omega+\beta\right)\right|^2\left|\widehat{\varphi}\left(B^{j-1}\omega\right)\right|^2 \\ &= \lim_{n\to+\infty}\sum_{j=-n}^{n}\left|m\left(B^{j-1}\omega+\beta\right)\right|^2\left|\widehat{\varphi}\left(B^{j-1}\omega\right)\right|^2 \\ &= \lim_{n\to+\infty}\sum_{j=-n}^{n}\left[1-\left|m\left(B^{j-1}\omega\right)\right|^2\right]\left|\widehat{\varphi}\left(B^{j-1}\omega\right)\right|^2 \\ &= \lim_{n\to+\infty}\left\{\left|\widehat{\varphi}\left(B^{-n-1}\omega\right)\right|^2 - \left|\widehat{\varphi}(B^n\omega)\right|^2\right\}.\end{aligned} \tag{3.5}$$

Since $\varphi \in L^2(R^n)$, Lemma 3.2 implies $\lim_{n\to+\infty}|\widehat{\varphi}(B^n\omega)|^2 = 0$ for a.e. ω. This shows that for a.e. ω, $\lim_{n\to+\infty}|\widehat{\varphi}(B^{-n}\omega)| = 1$, thus, by Lemma 3.1, m is a generalized lowpass filter.

The following theorem provides a way of constructing MRA PFWs from generalized lowpass filters. □

In order to obtain main result in this part, we firstly introduce a result in [16, Lemma 2.8].

Lemma 3.4. *Let $B \in M_n(Z)$ be an expanding matrix. Let μ be a Z^n-periodic, unimodular function. Then there exists a unimodular function ν that satisfies*

$$\mu(\omega) = \nu(B\omega)\overline{\nu(\omega)}, \quad \text{a.e. } \omega. \tag{3.6}$$

Theorem 3.5. *Suppose that m is a generalized lowpass filter. Then, there exist a pseudo-scaling function φ and an MRA PFW ψ such that they satisfy* (2.11).

Proof. Suppose that m is a generalized lowpass filter. Consider first the signum function μ for m:

$$\mu(\omega) = \begin{cases} \dfrac{m(\omega)}{|m(\omega)|}, & m(\omega) \neq 0, \\ 1, & m(\omega) \neq 0. \end{cases} \tag{3.7}$$

Clearly, μ is a measurable unimodular function such that, for all ω,

$$m(\omega) = \mu(\omega)|m(\omega)|. \tag{3.8}$$

By Lemma 3.4, there exists a unimodular measurable function ν such that

$$\nu(B\omega)\overline{\nu(\omega)} = \mu(\omega), \quad \text{a.e. } \omega. \tag{3.9}$$

Now take the function $\widehat{\varphi_{|m|}}$ constructed from $|m|$ by (2.9), and put

$$\widehat{\varphi}(\omega) := \nu(\omega)\widehat{\varphi_{|m|}}(\omega). \tag{3.10}$$

Obviously, $\varphi \in L^2(R^n)$. Using (2.9), (3.6), (3.10), and the definition of the signum function μ, we find

$$\begin{aligned} \widehat{\varphi}(B\omega) &= \nu(B\omega)\widehat{\varphi_{|m|}}(B\omega) \\ &= \nu(B\omega)|m(\omega)|\widehat{\varphi_{|m|}}(\omega) \\ &= \nu(B\omega)|m(\omega)|\overline{\nu(\omega)}\widehat{\varphi}(\omega) \\ &= \mu(\omega)|m(\omega)|\widehat{\varphi}(\omega) \\ &= m(\omega)\widehat{\varphi}(\omega). \end{aligned} \tag{3.11}$$

Thus, φ is a pseudo-scaling function.

For any unimodular function $s \in L^2(T^n)$, we define

$$\widehat{\psi}(B\omega) = e^{2\pi\omega i}s(B\omega)\overline{m(\omega+\beta)}\widehat{\varphi}(\omega), \quad \text{a.e. } \omega. \tag{3.12}$$

We affirm that $\psi(x)$ is an MRA PFW. Lemma 2.4 shows that it is enough to prove that $\widehat{\psi}$ satisfies (2.4) and (2.5).

From the fact that m is a generalized lowpass filter, we can deduce that (3.1) holds. Using Lemma 3.2, we obtain

$$\begin{aligned}
\sum_{j\in Z}\left|\widehat{\psi}\left(B^j\omega\right)\right|^2 &= \sum_{j\in Z}\left|m\left(B^{j-1}\omega+\beta\right)\right|^2\left|\widehat{\varphi}\left(B^{j-1}\omega\right)\right|^2 \\
&= \lim_{n\to+\infty}\sum_{j=-n}^{n}\left|m\left(B^{j-1}\omega+\beta\right)\right|^2\left|\widehat{\varphi}\left(B^{j-1}\omega\right)\right|^2 \\
&= \lim_{n\to+\infty}\sum_{j=-n}^{n}\left[1-\left|m\left(B^{j-1}\omega\right)\right|^2\right]\left|\widehat{\varphi}\left(B^{j-1}\omega\right)\right|^2 \\
&= \lim_{n\to+\infty}\left\{\left|\widehat{\varphi}\left(B^{-n-1}\omega\right)\right|^2-\left|\widehat{\varphi}(B^n\omega)\right|^2\right\} \\
&= \lim_{n\to+\infty}\left|\widehat{\varphi}\left(B^{-n-1}\omega\right)\right|^2.
\end{aligned} \tag{3.13}$$

This computation shows that the first characterizing condition (2.4) in Lemma 2.4 is satisfied precisely when (3.1) holds.

Let us now prove that the function ψ given by above satisfies (2.5).

Let us fix ω and $q = Bk+\alpha, k\in Z^n,\ \alpha\in Z^n/BZ^n$, and write

$$\sum_{j=0}^{+\infty}\widehat{\psi}\left(B^j\omega\right)\overline{\widehat{\psi}\left(B^j(\omega+q)\right)} = \widehat{\psi}(\omega)\overline{\widehat{\psi}((\omega+q))} + \sum_{j=1}^{+\infty}\widehat{\psi}\left(B^j\omega\right)\overline{\widehat{\psi}\left(B^j(\omega+q)\right)}. \tag{3.14}$$

To compute the first term on the right-hand side of (3.14), by the fact $e^{2\pi iB^{-1}\alpha} = e^{2\pi i\beta} = -1$,and (2.6), we have

$$\begin{aligned}
\widehat{\psi}(\omega)\overline{\widehat{\psi}(\omega+q)} &= e^{2\pi iB^{-1}\omega}s(\omega)\overline{m(B^{-1}\omega+\beta)}\widehat{\varphi}\left(B^{-1}\omega\right) \\
&\quad\times e^{-2\pi iB^{-1}(\omega+q)}\overline{s(\omega+q)}m\left(B^{-1}(\omega+q)+\beta\right)\overline{\widehat{\varphi}(B^{-1}(\omega+q))} \\
&= e^{-2\pi iB^{-1}q}|s(\omega)|^2 m\left(B^{-1}\omega\right)\widehat{\varphi}\left(B^{-1}\omega\right)\overline{m(B^{-1}\omega+k+B^{-1}\alpha)\widehat{\varphi}(B^{-1}(\omega+q))} \\
&= e^{-2\pi iB^{-1}\alpha}\widehat{\varphi}(\omega)\overline{m(B^{-1}(\omega+q))\widehat{\varphi}(B^{-1}(\omega+q))} \\
&= -\widehat{\varphi}(\omega)\overline{\widehat{\varphi}(\omega+q)}.
\end{aligned} \tag{3.15}$$

To compute the second term on the right-hand side of (3.14), by (2.6), (2.7), and (2.11), we have

$$\begin{aligned}
\sum_{j=1}^{+\infty}\widehat{\psi}\left(B^j\omega\right)\overline{\widehat{\psi}(B^j(\omega+q))} &= \sum_{j=1}^{+\infty}e^{2\pi i B^{j-1}\omega}s\left(B^j\omega\right)\overline{m(B^{j-1}\omega+\beta)}\widehat{\varphi}\left(B^{j-1}\omega\right) \\
&\quad\times e^{-2\pi i B^{j-1}(\omega+q)}\overline{s(B^j(\omega+q))}m\left(B^{j-1}(\omega+q)+\beta\right)\overline{\widehat{\varphi}(B^{j-1}(\omega+q))} \\
&= \sum_{j=1}^{+\infty}e^{-2\pi i B^{j-1}q}\left|m\left(B^{j-1}\omega+\beta\right)\right|^2\widehat{\varphi}\left(B^{j-1}\omega\right)\overline{\widehat{\varphi}(B^{j-1}(\omega+q))} \\
&= \sum_{j=1}^{+\infty}\left(1-\left|m\left(B^{j-1}\omega\right)\right|^2\right)\widehat{\varphi}\left(B^{j-1}\omega\right)\overline{\widehat{\varphi}(B^{j-1}(\omega+q))} \\
&= \sum_{j=1}^{+\infty}\left\{\widehat{\varphi}\left(B^{j-1}\omega\right)\overline{\widehat{\varphi}(B^{j-1}(\omega+q))}-\widehat{\varphi}\left(B^{j}\omega\right)\overline{\widehat{\varphi}(B^{j}(\omega+q))}\right\} \\
&= \widehat{\varphi}(\omega)\overline{\widehat{\varphi}(\omega+q)}-\lim_{p\to+\infty}\widehat{\varphi}(B^p\omega)\overline{\widehat{\varphi}(B^p(\omega+q))} \\
&= \widehat{\varphi}(\omega)\overline{\widehat{\varphi}(\omega+q)}.
\end{aligned} \tag{3.16}$$

This shows that the expression in (3.14) is equal to 0. Hence, $\widehat{\psi}$ satisfies (2.5), and ψ is a PFW. □

4. The Multiplier Classes Associated with PFWs

In this section, we will describe the multiplier classes associated with PFWs in $L^2(R^n)$. We firstly give their definitions.

Definition 4.1.

(1) A PFW multiplier ν is a function such that $\widetilde{\psi}=(\widehat{\psi}\nu)^{\check{}}$ is a PFW whenever ψ is a PFW.

(2) An MRA PFW multiplier is a function ν such that $\widetilde{\psi}=(\widehat{\psi}\nu)^{\check{}}$ is an MRA PFW whenever ψ is an MRA PFW.

(3) A pseudo-scaling function multiplier is a function ν such that $\widetilde{\varphi}=(\widehat{\varphi}\nu)^{\check{}}$ is a pseudo-scaling function associated with an MRA PFW whenever φ has the same property.

(4) A generalized lowpass filter multiplier is a function ν such that $\widetilde{m}=m\nu$ is a generalized lowpass filter whenever m is a generalized lowpass filter.

At first, we will obtain a property of PFW multiplier.

Property 4.2. If a measurable function ν is a PFW multiplier, then, ν is unimodular.

Proof. Let ν be a PFW multiplier; that is, there exist PFWs ψ and $\widetilde{\psi}$ such that $\widehat{\widetilde{\psi}}=\widehat{\psi}\nu$.

Let $\psi \in L^2(R^n)$ be any wavelet satisfying $|\widehat{\psi}(\omega)| > 0$ for a.e. ω. By assumption, for every $l \geq 1$, we easily deduce that $(\widehat{\psi}\nu^l)^{\vee}$ is a PFW and, thus, satisfies (2.4):

$$\sum_{j\in Z}\left|\nu(B^j\omega)\right|^{2l}\left|\widehat{\psi}(B^j\omega)\right|^2 = 1, \quad \text{a.e. } \omega \in R^n. \tag{4.1}$$

In particular, for a.e. ω and every $l \in N$,

$$|\nu(\omega)|^l\left|\widehat{\psi}(\omega)\right| \leq 1. \tag{4.2}$$

This is only possible if $|\nu(\omega)| \leq 1$ a.e. since $|\widehat{\psi}(\omega)|$ almost never vanishes. Using (2.4) for PFWs ψ and $\widetilde{\psi}$ such that $\widehat{\widetilde{\psi}} = \widehat{\psi}\nu$, we easily obtain

$$\sum_{j\in Z}\left|\widehat{\psi}\left(B^j\omega\right)\right|^2\left(1-\left|\nu\left(B^j\omega\right)\right|^2\right) = 1, \quad \text{a.e. } \omega \in R^n. \tag{4.3}$$

which is only possible if all terms vanish. Thus, $\nu(\omega) = 1$ a.e. □

The next theorem gives a characterization of PFW multiplier in $L^2(R^n)$.

Theorem 4.3. *If a measurable function ν is unimodular and $\nu(B\omega)\overline{\nu(\omega)}$ is Z^n-periodic, then, this measurable functionν is a PFW multiplier.*

Proof. PFW is characterized as an element of $L^2(R^n)$ satisfying (2.4) and (2.5) in Lemma 2.4.

Let ψ be a PFW, so that (2.4) and (2.5) hold. Let us assume that ν is unimodular and $\nu(B\omega)\overline{\nu(\omega)}$ is a Z^n-periodic function, necessarily unimodular. Let $\widehat{\widetilde{\psi}} = \widehat{\psi}\nu$ For the function $\widetilde{\psi}$, (2.4) obviously holds from the fact that ν is a unimodular function and $|\widehat{\psi}\nu| = |\widehat{\psi}|$. In the following, for the function $\widetilde{\psi}$, let us consider (2.5). Let $k \in Z^n$, $\alpha \in Z^n/BZ^n$, and let $j > 0$. Then

$$\widehat{\widetilde{\psi}}\left(B^j\omega\right)\overline{\widehat{\widetilde{\psi}}\left(B^j(\omega+Bk+\alpha)\right)} = \widehat{\psi}\left(B^j\omega\right)\overline{\widehat{\psi}\left(B^j(\omega+Bk+\alpha)\right)}\nu\left(B^j\omega\right)\overline{\nu\left(B^j(\omega+Bk+\alpha)\right)}. \tag{4.4}$$

If $j \geq 1$, according to Lemma 3.4, there exists a unimodular function μ that satisfies

$$\mu(\omega) = \nu(B\omega)\overline{\nu(\omega)}, \quad \text{a.e. } \omega. \tag{4.5}$$

By Z^n-periodicity and unimodularity of μ, we have

$$\begin{aligned}
&\nu\left(B^j\omega\right)\overline{\nu(B^j(\omega+Bk+\alpha))}\\
&=\frac{\mu(B^{j-1}\omega)\overline{\mu(B^{j-1}(\omega+Bk+\alpha))}}{\overline{\nu(B^{j-1}\omega)}\nu(B^{j-1}(\omega+Bk+\alpha))}\\
&=\mu\left(B^{j-1}\omega\right)\overline{\mu(B^{j-1}(\omega+Bk+\alpha))}\nu\left(B^{j-1}\omega\right)\overline{\nu(B^{j-1}(\omega+Bk+\alpha))}\\
&=\mu\left(B^{j-1}\omega\right)\overline{\mu(B^{j-1}(\omega))}\nu\left(B^{j-1}\omega\right)\overline{\nu(B^{j-1}(\omega+Bk+\alpha))}\\
&=\nu\left(B^{j-1}\omega\right)\overline{\nu(B^{j-1}(\omega+Bk+\alpha))}.
\end{aligned}\tag{4.6}$$

Then, we can repeat the above argument until we obtain

$$\nu\left(B^{j-1}\omega\right)\overline{\nu(B^{j-1}(\omega+Bk+\alpha))}=\nu(\omega)\overline{\nu(\omega+Bk+\alpha)},\quad \text{for } j\geq 2.\tag{4.7}$$

By the equalities (4.4), (4.7), and summing over $j \geq 0$, we obtain

$$\sum_{j=0}^{+\infty}\widehat{\widetilde{\varphi}}\left(B^j\omega\right)\overline{\widehat{\widetilde{\varphi}}(B^j(\omega+Bk+\alpha))}=\nu(\omega)\overline{\nu(\omega+Bk+\alpha)}\sum_{j=0}^{+\infty}\sum_{j=0}^{+\infty}\widehat{\varphi}\left(B^j\omega\right)\overline{\widehat{\varphi}(B^j(\omega+Bk+\alpha))}.\tag{4.8}$$

Since the above equation of the right-hand side is 0 by (2.5) in Lemma 2.4, we conclude that $\widetilde{\varphi}$ also satisfies (2.5).

Hence, from Lemma 2.4, $\widetilde{\varphi}$ is a PFW, thus, ν is a PFW multiplier. □

Remark 4.4. It is proved in [19, Theorem 3.2] that if a measurable function ν is a PFW multiplier, $\nu(2\omega)\overline{\nu(\omega)}$ is Z-periodic. However, it still is a unsolved question whether this conclusion holds in $L^2(R^n)$. We will discuss it in the future. The next result give a characterization of the MRA PFW multipliers. In order to obtain this result, we firstly introduce a result in [16, Lemma 2.8].

Lemma 4.5. *Let $B \in M_n(Z)$ be an expanding matrix such that $|\det B| = 2$. Let $\alpha \in Z^n/BZ^n$ and $\beta = B^{-1}\alpha$. Suppose that s is a Z^n- periodic, unimodular function on R^n. Then there exists a Z^n-periodic, unimodular function t on R^n such that*

$$s(B\omega)=t(\omega)t(\omega+\beta)\overline{t(B\omega)}.\tag{4.9}$$

Theorem 4.6. *A measurable function ν is an MRA PFW multiplier if and only if ν is unimodular, and $\nu(B\omega)\overline{\nu(\omega)}$ is Z^n-periodic.*

Proof. (if) Let ν be unimodular, and let

$$s(\omega)=\nu(B\omega)\overline{\nu(\omega)}\tag{4.10}$$

be Z^n-periodic, necessarily unimodular. We now use Lemma 4.5 to obtain a unimodular, Z^n-periodic function t such that

$$s(B\omega) = t(\omega)t(\omega+\beta)\overline{t(B\omega)}. \tag{4.11}$$

Let

$$\mu(\omega) = \nu(\omega)t\left(B^{-1}\omega\right)t\left(B^{-1}\omega+\beta\right). \tag{4.12}$$

Then μ is unimodular, and

$$\begin{aligned}\mu(B\omega)\overline{\mu(\omega)} &= \nu(B\omega)\overline{\nu(\omega)}t(\omega)t(\omega+\beta)\overline{t(B^{-1}\omega)t(B^{-1}\omega+\beta)}\\ &= s(\omega)\left(t(\omega)\overline{t(B^{-1}\omega)t(B^{-1}\omega+\beta)}\right)t(\omega+\beta)\\ &= s(\omega)\overline{s(\omega)}t(\omega+\beta)\\ &= t(\omega+\beta)\end{aligned} \tag{4.13}$$

is a Z^n-periodic function. In the above computation, we have used (4.9) and (4.10) as well as the unimodularity and the periodicity of t.

Let ψ be an MRA PFW, and let φ be an associated pseudo-scaling function with $m \in \widetilde{F}_{\varphi}$ such that (2.11) holds. Let

$$\begin{aligned}\widetilde{m}(\omega) &= m(\omega)t(\omega+\beta),\\ \widehat{\widetilde{\varphi}}(\omega) &= \widehat{\varphi}(\omega)\mu(\omega).\end{aligned} \tag{4.14}$$

Then, we have

$$\begin{aligned}\widehat{\widetilde{\varphi}}(\omega) &= \widehat{\varphi}(B\omega)\mu(B\omega)\\ &= m(\omega)\widehat{\varphi}(\omega)t(\omega+\beta)\mu(\omega)\\ &= \widetilde{m}(\omega)\widehat{\widetilde{\varphi}}(\omega),\end{aligned} \tag{4.15}$$

where the second equation is deduced by (4.13) and the definition of pseudo-scaling function.

It is obvious that

$$|\widetilde{m}(\omega)|^2 + |\widetilde{m}(\omega+\beta)|^2 = 1 \quad \text{a.e. } \omega \in R^n. \tag{4.16}$$

Thus, we obtain $\widetilde{m}(\omega) \in \widetilde{F}_{\widetilde{\varphi}}$.

Let

$$\widehat{\widetilde{\psi}}(\omega) = \widehat{\psi}(\omega)\nu(\omega). \tag{4.17}$$

Since φ is a PFW, we can apply Lemma 2.4 to deduce that $\widetilde{\varphi}$ is a PFW. It follows, that $\widetilde{\varphi}$ is an MRA PFW, since

$$\begin{aligned} e^{2\pi i\omega}s(B\omega)\overline{\widetilde{m}(\omega+\beta)}\widehat{\widetilde{\varphi}}(\omega) &= e^{2\pi i\omega}s(B\omega)\overline{m(\omega+\beta)t(\omega+2\beta)}\widehat{\varphi}(\omega)\mu(\omega) \\ &= \widehat{\varphi}(B\omega)\overline{t(\omega+2\beta)}\mu(\omega) \\ &= \widehat{\varphi}(B\omega)\overline{t(\omega+2\beta)}\mu(B\omega)\overline{t(\omega+\beta)} \\ &= \nu(B\omega)\widehat{\varphi}(B\omega) \\ &= \widehat{\widetilde{\varphi}}(B\omega), \end{aligned} \tag{4.18}$$

where we have used (4.13) and the definition of μ.

(only if) The Haar wavelet is, in particular, an MRA TFW; we can then proceed as in the proof of Property 4.2 to show the unimodularity of ν. The proof that $\nu(B\omega)\overline{\nu(\omega)}$ is Z^n-periodic is similar to Theorem 2 in [18], and we omit it.

Thus, we complete the proof. □

The next theorem characterizes the class of the generalized lowpass filter multipliers. Note that the generalized lowpass filters are the functions $m \in \widetilde{\mathrm{F}}_{\varphi}$ for some pseudo-scaling function φ satisfying Definition 2.5, that is, $m \in \widetilde{\mathrm{F}}_{\varphi}$ and $|N_0(m)| = 0$. We will use this fact in the following result.

Theorem 4.7. *A measurable function ν is a generalized lowpass filter multiplier if and only if ν is unimodular and Z^n-periodic.*

Proof. (if) Suppose that ν is unimodular and Z^n-periodic, and $m(\omega)$ is a generalized lowpass filter.

Let

$$\widetilde{m}(\omega) = m(\omega)\nu(\omega). \tag{4.19}$$

Then, from the definition of the generalized lowpass filter, we have $m \in \widetilde{F}_{\varphi}$ and $|N_0(m)| = 0$. Therefore, according to the function ν being unimodular and Z^n-periodic, it is obvious that these properties are also true for $\widetilde{m}(\omega)$.

(only if) We proceed similarly in the proof of the two previous theorems.

Let $\varphi \in L^2(R^n)$ be any wavelet satisfying $|\widehat{\varphi}(\omega)| > 0$ for a.e. ω, and m the corresponding lowpass filter. Then, it follows that $|m(\omega)| > 0$ a.e. $\omega \in R^n$.

Let

$$\widetilde{m}(\omega) = m(\omega)\nu(\omega). \tag{4.20}$$

By the assumption that the measurable function ν is a generalized lowpass filter multiplier, we have $\widetilde{m} \in \widetilde{F}$. In particular, $\widetilde{m}$ is Z^n-periodic, and so ν is also Z^n-periodic.

Applying ν repetitively, we obtain

$$|\nu(\omega)|^l |m(\omega)| \leq 1, \quad \text{for a.e. } \omega, \ l \geq 1. \tag{4.21}$$

This implies that $|\nu(\omega)| \leq 1$ a.e. Unimodularity follows, since both m and $\widetilde{m}$ are in $\widetilde{F}$ and, thus, satisfy

$$|\widetilde{m}(\omega)|^2 + \left|\widetilde{m}(\omega + \beta)\right|^2 = 1 = |m(\omega)|^2 + \left|m(\omega + \beta)\right|^2. \tag{4.22}$$

In a conclusion, ν is unimodular and Z^n-periodic.

The above results provide description of PFW, MRA PFW, and generalized lowpass filter multipliers. These classes are identical with the respective multiplier classes of wavelets. This fact is basically a consequence of the fact that all of these multiplier operations necessarily preserve the $L^2(R^n)$ norm of the PFW $\psi(\omega)$.

Let us introduce a notation for the next theorem. For a measurable function ν, let $E = \{\omega : \nu(\omega) \neq 0\}$ and $\mu(\omega) = (\nu(B\omega))/(\nu(\omega))$ on E. □

The following result shows that the situation for pseudo-scaling function multipliers is completely different from others.

Theorem 4.8. *ν is a pseudo-scaling function multiplier if and only if*

(1) $|\nu(B\omega)| \leq |\nu(\omega)|$ *a.e. and* $\lim_{j\to\infty}|\nu(B^j\omega)| = 1$ *a.e.;*

(2) $\mu(\omega)$ *extends to a* Z^n*-periodic function;*

(3) *If* $\xi, \eta \in E$*, and* $\xi - \eta$ *is an odd multiple of* β*, then* $|\mu(\omega)| = |\nu(\omega)| = 1$.

Proof. (if) Let φ be a pseudo-scaling function satisfying (2.6), and suppose that ν satisfies (1)–(3).

Let

$$\widehat{\widetilde{\varphi}}(\omega) = \widehat{\varphi}(\omega)\nu(\omega). \tag{4.23}$$

Using condition (1), we see that $\widetilde{\varphi}$ satisfies

$$\lim_{j\to\infty}\left|\widehat{\widetilde{\varphi}}\left(B^j\omega\right)\right| = \lim_{j\to\infty}\left|\nu\left(B^j\omega\right)\right|\lim_{j\to\infty}\left|\widehat{\varphi}\left(B^j\omega\right)\right| = 1. \tag{4.24}$$

Let us now examine the scale equation $\widehat{\widetilde{\varphi}}(B\omega) = \widetilde{m}(\omega)\widehat{\widetilde{\varphi}}(\omega)$ from (4.15).

We claim that there exists $\widetilde{m} \in \widetilde{F}$ such that

$$\nu(B\omega)m(\omega) = \nu(\omega)\widetilde{m}(\omega). \tag{4.25}$$

If $\omega \overline{\in} E$, then, by (1), $B\omega \overline{\in} E$ and, thus, (4.25) is satisfied automatically.

If $\omega \in E$, then (4.25) is equivalent to

$$\widetilde{m}(\omega) = \frac{\nu(B\omega)}{\nu(\omega)}m(\omega). \tag{4.26}$$

The condition (2) implies that $\tilde{m}(\omega)$ defined on E by (4.26) is Z^n-periodic on E. We will now define a Z^n-periodic extension of $\tilde{m}(\omega)$ to R^n satisfying

$$|\tilde{m}(\omega)|^2 + |\tilde{m}(\omega+\beta)|^2 = 1. \tag{4.27}$$

Let us consider $[-1,1]^n$. Define $\tilde{m}(\omega)$ on (part of) $[-1,1]^n$ by the following: if $\omega + k \in E$ for some $k \in Z^n$, then

$$\tilde{m}(\omega) = \tilde{m}(\omega + k). \tag{4.28}$$

The definition is consistent with the Z^n periodicity of m on E. Let $\xi, \eta \in E$, and $\xi - \eta = \beta$. Then one of the following conditions must hold:

(a) $\exists k, l \in Z^n$, such that $\xi + k \in E$ and $\eta + l \in E$;

(b) $\exists k \in Z^n$ such that $\xi + k \in E$ and for any $l \in Z^n$, $\eta + l \overline{\in} E$;

(c) For any $k, l \in Z^n$, such that $\xi + k \overline{\in} E$ and $\eta + l \overline{\in} E$.

In (a) case, $(\xi + k) - (\eta + l)$ is an odd multiple of β , so by (3), $1 = |\mu(\xi + k)| = |\mu(\eta + l)|$. Thus, we have

$$|\tilde{m}(\xi)| = |\tilde{m}(\xi + k)| = |m(\xi + k)| = |m(\xi)|. \tag{4.29}$$

Similarly, we can deduce $|\tilde{m}(\eta)| = |m(\eta)|$. Since $m \in F$, this implies

$$|\tilde{m}(\xi)|^2 + \left|\tilde{m}(\eta)\right|^2 = 1. \tag{4.30}$$

We have either $\xi + \beta = \eta$ or $\eta + \beta = \xi$, and so (4.27) holds.

In (b) case we extend the definition of $\tilde{m}(\omega)$ to the set of all such $\eta \in [-1,1]^n$ by

$$\tilde{m}(\eta) = \sqrt{1 - |\tilde{m}(\xi)|^2}, \tag{4.31}$$

and so we get (4.27).

In (c) case, we let $\tilde{m}(\zeta) = 1$ if $\zeta \in B^{-1}[-1,1]^n$, and $\tilde{m}(\zeta) = 0$ if $\zeta \in [-1,1]^n \setminus B^{-1}[-1,1]^n$, whenever $\zeta = \xi$ or η.

We have thus extended $\tilde{m}(\omega)$ to the entire interval $[-1,1]^n$ so that (4.27) holds if ξ and η are in $[-1,1]^n$. We now extend $\tilde{m}(\omega)$ to R^n by Z^n-periodicity.

Clearly, $\tilde{m} \in \tilde{F}$, and (4.15) is satisfied.

(only if) Let φ be the Shannon scaling function in $L^2(R^n)$; that is, $\hat{\varphi}(\omega) = \chi_{[-1,1)^n}(\omega)$. Then

$$\hat{\tilde{\varphi}}(\omega) = \hat{\varphi}(\omega)\nu(\omega) = \nu(\omega), \quad \omega \in [-1,1)^n. \tag{4.32}$$

By (2.6) for $\hat{\tilde{\varphi}}(\omega)$, we have

$$\lim_{j\to\infty}\left|\nu\left(B^j\omega\right)\right| = 1. \tag{4.33}$$

This establishes (2), since the right-hand side is Z^n-periodic, The remain proof is similar to ones in [19], and so we omit them. Then, we will provide an example to prove our results. □

Example 4.9. Let A be the quincunx matrix $Q = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right) \in E_n^{(2)}$, then we get $B = Q^t = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right) \in E_n^{(2)}$. Furthermore, we have $B^{-1}C \subseteq C$ and $B^jC \subseteq B^{j+1}C$, for all $j \in N$, where C is the standard unit square in R^2.

Let us define

$$\widehat{\varphi}(\omega) = \begin{cases} \dfrac{1}{2}, & \omega \in B^{-1}C \setminus B^{-2}C, \\ 1, & \omega \in B^{-2}C, \\ 0, & \omega \overline{\in} B^{-1}C, \end{cases}$$
$$m(\omega) = \begin{cases} 0, & \omega \in B^{-1}C \setminus B^{-2}C, \\ \dfrac{1}{2}, & \omega \in B^{-2}C \setminus B^{-3}C, \\ 1, & \omega \in B^{-3}C. \end{cases} \tag{4.34}$$

Now we extend $m(\omega)$ to C such that the equality $|m(\omega)|^2 + |m(\omega+\beta)|^2 = 1$ is satisfied for all $\omega \in C$, where we take $\beta = (1/2, 1/2)$ and extend $m(\omega)$ to R^2 by Z^2-periodicity.

From the definitions of the functions φ and m, we easily deduce that φ is a pseudo-scaling function, and m is a generalized lowpass filter.

Finally, we define

$$\widehat{\psi}(B\omega) = e^{2\pi i\omega}\overline{m(\omega+\beta)}\widehat{\varphi}(\omega). \tag{4.35}$$

Therefore, by Theorem 3.5, we know that we get an MRA Parseval frame wavelet.

However, from [17], we know that this MRA Parseval frame wavelet is not a Parseval frame wavelet associated to MRA, which does not permit fast algorithm.

5. Conclusion

In this paper, we characterize all generalized lowpass filter and MRA Parseval frame wavelets in $L^2(R^n)$ with matrix dilations of the form $(Df)(x) = \sqrt{2}f(Ax)$, where A is an arbitrary expanding $n \times n$ matrix with integer coefficients, such that $|\det A| = 2$. We study some properties of the multipliers of generalized wavelets, scaling functions, and filters in $L^2(R^n)$. Our result is a generalization of the construction of PFWs from generalized low-pass filters that is introduced in [19]. However, our ways are different from original ones. Then, we describe the multiplier classes associated with Parseval frame wavelets in $L^2(R^n)$ and give an example to prove our theory.

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