Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 530287, 15 pages
doi:10.1155/2009/530287

Research Article

A Galerkin-Parameterization Method for the Optimal Control of Smart Microbeams

Marwan Abukhaled and Ibrahim Sadek

Department of Mathematics and Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, UAE

Correspondence should be addressed to Marwan Abukhaled, mabukhaled@aus.edu

Received 18 April 2009; Revised 1 September 2009; Accepted 22 September 2009

Recommended by Mohammad Younis

A proposed computational method is applied to damp out the excess vibrations in smart microbeams, where the control action is implemented using piezoceramic actuators. From a mathematical point of view, we wish to determine the optimal boundary actuators that minimize a given energy-based performance measure. The minimization of the performance measure over the actuators is subjected to the full motion of the structural vibrations of the micro-beams. A direct state-control parametrization approach is proposed where the shifted Legendre polynomials are employed to solve the optimization problem. Legendre operational matrix and the properties of Kronecker product are utilized to find the approximated optimal trajectory and optimal control law of the lumped parameter systems with respect to the quadratic cost function by solving linear algebraic equations. Numerical examples are provided to demonstrate the applicability and efficiency of the proposed approach.

1. Introduction

Devices of Microelectromechanical systems (MEMSs) find wide applications as sensors and actuators. The analysis of methods of actuation and sensing has been a topic of interest over the past several years. Different actuation and sensing properties such as piezosensitive, piezoelectric, electrostatic, thermal, electromagnatic, and optical have been used [1]. Piezo materials can be integrated in various structural components as distributed sensors or actuators.

The use of the piezoelectric actuators has been proved to be effective control devices for the control of structural vibrations in a wide range of engineering applications. One of the most widely used piezo materials in active control is piezoceramic such as PZT. This due to their large bandwidth, mechanical simplicity, and their mechanical ability in producing controlling forces [2].

In this contest, the microelectromechanical systems (MEMSs) seem to be attractive in improving the mechanical efficiency of structural active control. Borovic et al. [3] highlight some MEMS control issues and provide an overview of MEMS control.

Piezoelectrically actuated microbeams have recently received considerable attention. The behavior of electrically actuated microbeams has been studied using different models and approaches [4–8]. With the growth of using microbeams MEMS, it is necessary to study their dynamic behavior. In this paper, the studied microsystem is a microbeam totally covered by a piezoelectric, PZT, film. While it is possible to cover the entire structure with piezoelectric material, it is not possible to do so on large structures. The optimal boundary vibration control for microbeams is investigated. The control action is implemented using piezoceramic actuators to damp out the vibrations of microbeams where the control function appears in the boundary conditions in the form of a moment. In the control problem, we wish to determine the optimal boundary control actuators that minimize a given energy-based performance measure. The minimization of the performance measure over the actuators is subjected to the equation of motion governing the structural vibration, the imposed initial conditions, and the boundary conditions. The performance measure is specified as a quadratic functional of displacement and velocity along with a suitable penalty term involving the boundary control function. For the determination of the optimal boundary controls, it is necessary to convert the problem from one in which there is boundary control into one in which there are distributed controls. The Galerkin-based approach is used then to reduce the modified problem to the optimal control of a linear time-invariant lumped-parameter system. In contrast to standard optimal control or variational methods for lumped parameter systems, a direct state-control parameterization by orthogonal polynomial expansion is employed to solve the modal space optimization problem.

In general, the approach is based on converting differential equations into integral equations through integration, approximating state and control involved in the equation by finite term series, and using the operational matrix of integration to eliminate the integral operations. This method has been used in obtaining the continuous control of various distributed and lumped parameter system modals [9–11]. Typical examples are the applications of Walsh functions [12], block-pulse functions [13], Bessel functions [14], Harr function [15], Chebyshev polynomials [16], Laguerre polynomials [17], Fourier Series [18], Taylor series [19], Jacobi series [20, 21], Hermite polynomials [22], and wavelets [23–29].

A computational method is proposed to sole the modal optimization problem with quadratic performance index. The method is based on parametrizing the state and control variables by finite-term Legendre series whose coefficient values will be determined. The Legendre operational matrix and the properties of Kronecker product are used to relate the unknown coefficients of control variables to coefficients of the state variables. The performance index, as a result, can be expressed in terms of the unknown coefficients of the state variables. The necessary condition for optimality is derived as a system of linear algebraic equations in terms of the unknown coefficients of the state variables. Of the attractive features of our proposed approach is that we can avoid difficult integral equations, which are produced by variational methods [30]. This is achieved by reducing the problem to the solution of algebraic system of equations. Moreover, solving a system of coupled initial boundary-terminal-value problems as a requirement for the maximum principle [31] can now be avoided. Numerical simulations are presented to assess the effectiveness and the capabilities of piezo actuation by means of moments to damp out the vibration of microbeams with a minimum level of voltage applied on the piezo actuators.

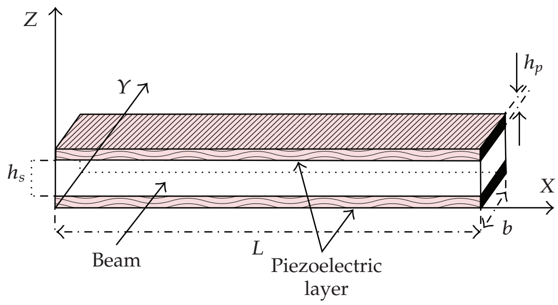

Figure 1: Microbeam with distributed actuator layers.

2. Equation of Motion for a Piezoelectric Beam

Consider a microbeam of length l, width b, height h_s and covered by layers of piezoelectric materials of thickness h_p. The dynamical equilibrium of the Euler-Bernoulli beam is defined by [7, 32]

$$\overline{\rho} w_{tt} + E_{c_o} I_{c_o} w_{xxxx} = 0, \quad 0 < x < l, \ 0 < t < t_f, \tag{2.1}$$

where $\overline{\rho}$ is the mass per unit length of the layered beam, $E_{c_o} I_{c_o}$ is the bending stiffness of the beam including the piezoelectric layers, and $w(x,t)$ is the transverse displacement of the beam.

The boundary and initial conditions, respectively, are

$$\begin{aligned} w(0,t) &= w(l,t) = 0, \\ w_{xx}(0,t) &= w_{xx}(l,t) = \frac{Gp}{E_{c_o} I_{c_o}} f(t), \end{aligned} \tag{2.2}$$

$$\begin{aligned} w(x,0) &= w_0(x), \\ w_t(x,0) &= w_l(x), \end{aligned} \tag{2.3}$$

where

$$E_{c_o} I_{c_o} = D_{11} - \frac{B_{11}^2}{A_{11}} \tag{2.4}$$

in which

$$\begin{gathered} A_{11} = (E_s h_s + E_p h_p) b, \qquad B_{11} = \frac{E_p - E_s}{2} h_p h_s b, \\ D_{11} = \frac{b}{12} \left(E_s h_s \left(h_s^2 + 3h_p^2 \right) + E_p h_p \left(h_p^2 + 3h_s^2 \right) \right), \qquad G_p = \left(\frac{h_s}{2} - \frac{B_{11}}{A_{11}} \right) b e_{31} \end{gathered} \tag{2.5}$$

in which $e_{31} = d_{31}E_p$, where E_p and E_s are Young's modulus of the piezoelectric layer and the microbeam, respectively. d_{31} is the actuator piezoelectric constant, and $f(t)$ is the applied voltage.

For convenience, we introduce the nondimensional variables

$$W(X,T) = \frac{w(x,t)}{l}, \qquad X = \frac{x}{l}, \qquad T = \frac{t}{l^2}\sqrt{\frac{E_{c_o}I_{c_o}}{\overline{\rho}}}, \qquad T_f = \frac{t_f}{l^2}\sqrt{\frac{E_{c_o}I_{c_o}}{\rho}}, \tag{2.6}$$

where $W(X,T)$, X, and T are dimensionless transverse displacement, position, and time, respectively. Substituting (2.6) into (2.1)–(2.3) leads to the nondimensional equation of motion

$$W_{TT} + W_{XXXX} = 0, \quad 0 < X < 1, \; 0 < T < T_f \tag{2.7}$$

with boundary conditions

$$W(0,T) = W(1,T) = 0, \qquad W_{XX}(0,T) = W_{XX}(1,T) = F(T) \tag{2.8}$$

and initial conditions

$$W(X,0) = W_0(X), \qquad W_T(X,0) = W_1(X). \tag{2.9}$$

3. Optimal Control Problem

3.1. Problem Statement

Consider the set of all admissible control functions $U_{\mathrm{ad}} = \{F : F \in L^2(0,T_f)\}$. The performance index is given by

$$J(F) = \mu_1\int_0^1 W^2(X,T_f)dX + \mu_2\int_0^1 W_T^2(X,T_f)dX + \mu_3\int_0^{T_f} F^2(T)dT, \tag{3.1}$$

where μ_1, μ_2, and μ_3 are all nonnegative constants and $\mu_1 + \mu_2 > 0$ and $\mu_3 \neq 0$. The last term on the right hand side of (3.1) is a penalty on control energy. The optimal control problem is now formulated:

$$\text{Find } F^* \in U_{\mathrm{ad}} \text{ so that } J(F^*) = \min_{F\in U_{\mathrm{ad}}} J(F), \tag{3.2}$$

and such that $W(X,T)$ satisfies all (2.7)–(2.9). As we will assume the existence of the optimal control (3.2), it can be easily shown that this quadratic control problem has at most one solution.

3.2. Equivalent Optimal Control Problem

Let

$$\overline{W}(X,T) = W(X,T) - \alpha(X)F(T), \tag{3.3}$$

where

$$\alpha(X) = \frac{1}{2}\left(X^2 - X\right). \tag{3.4}$$

Substituting (3.3) into (2.1) gives

$$\overline{W}_{TT}(X,T) + \overline{W}_{XXXX}(X,T) = -\alpha(X)F''(T), \quad 0 < X < L,\ 0 < T < T_f, \tag{3.5}$$

and the new boundary and initial conditions are, respectively,

$$\begin{aligned} \overline{W}(0,T) &= \overline{W}(1,T) = 0, \\ \overline{W}_{XX}(0,T) &= \overline{W}_{XX}(1,T) = 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \overline{W}(X,0) &= W_0(X) - \alpha(X)F(0), \\ \overline{W}_T(X,0) &= W_1(X) - \alpha(X)F'(0). \end{aligned} \tag{3.7}$$

The optimal control problem, that is equivalent to (3.2), is

$$\text{Find } F^* \in U_{\text{ad}} \text{ so that } J(F^*) = \min_{F\in U_{\text{ad}}} J(F), \tag{3.8}$$

and such that $\overline{W}(X,T)$ satisfies (3.5)–(3.7).

4. Optimal Control of Lumped Parameter System

In this section, the distributed parameter system optimization problem (3.5) is transformed into a modal Lumped parameter problem by means of Galerkin approach [30].

4.1. Control Problem in Modal Space

A finite orthonormal expansion of $\overline{W}(X,T)$ in terms of a complete basis $\{\varphi_n(x) = \sqrt{2}\sin\lambda_n x\} \in L^2([0,1])$ gives the representation

$$\overline{W}(X,T) = \sum_{n=1}^{N} Z_n(T)\varphi_n(X), \tag{4.1}$$

where $\lambda_n = n\pi$. The orthogonality of the Fourier sine series converts (3.5) into

$$\frac{d^2}{dT^2}Z_n(T) + \lambda_n^4 Z_n(T) = -2\eta_n \frac{d^2}{dT^2}F(T), \tag{4.2}$$

where $\eta_n = \int_0^1 \alpha(X)\sin(\lambda_n X)dX,\ n = 1,2,\ldots,N$.

The performance index takes on the form

$$J_N(F(T)) = \frac{1}{2}\mu_1\sum_{n=1}^{N} Z_n^2(T_f) + \frac{1}{2}\mu_2\sum_{n=1}^{N}\frac{d}{dt}Z_n^2(T_f) + \mu_3\int_0^{T_f} F^2(\tau)d\tau. \tag{4.3}$$

Integrating (4.2) twice over the interval $(0,t)$ gives

$$Z_n(T) - TZ_n'(0) - Z_n(0) + \lambda_n^4\int_0^T\int_0^{\eta} Z_n(\tau)d\tau\, d\eta = -2\eta_n\left[F(T) - F(0) - TF'(0)\right] \tag{4.4}$$

or, in vector notation

$$\overrightarrow{Z}(T) - G(T)\,\frac{d}{dT}\overrightarrow{Z}(0) - \overrightarrow{Z}(0) + \Lambda\int_0^T\int_0^{\eta}\overrightarrow{Z}(\tau)d\tau\, d\eta = \overrightarrow{E}_1 + \overrightarrow{E}_2 + \overrightarrow{E}_3, \tag{4.5}$$

where

$$\begin{gathered}\overrightarrow{Z}(T) = [Z_1(T), Z_2(T), \ldots, Z_N(T)]^{\mathrm{tr}}, \qquad G(t) = TI_{N\times N}, \qquad \Lambda = \operatorname{diag}\left[\lambda_i^4\right]_{N\times N}, \\ \overrightarrow{E}_1 = -2D\overrightarrow{F}(t), \qquad \overrightarrow{E}_2 = 2D\overrightarrow{F}(0), \qquad \overrightarrow{E}_1 = 2tD\overrightarrow{F'}(0), \qquad D = \operatorname{diag}\left[\eta_i\right]_{N\times N},\end{gathered} \tag{4.6}$$

where "tr" stands for the matrix transpose. The new optimal control problem is now formulated

$$\text{Find } F^*(T) \in U_{\mathrm{ad}} \text{ so that } J_N(F^*(T)) = \min_{F(T)\in U_{\mathrm{ad}}} J_N(F(T)) \tag{4.7}$$

subject to the integral equation (4.5).

4.2. State-Control Parametrization

In this section, a direct method for solving the modal control problem (4.7) is developed by parametrizing the state variables $Z_i(t)$ and the control variable $F(t)$.

Let

$$Z_i(T) = \sum_{j=0}^{m-1} a_{ij}S_j(T), \qquad F(T) = \sum_{j=0}^{m-1} b_jS_j(T), \tag{4.8}$$

where

$$a_{ij} = \frac{2j+1}{T_f}\int_0^{T_f} Z_i(\tau)S_j(\tau)d\tau, \qquad b_j = \frac{2j+1}{T_f}\int_0^{T_f} F(\tau)S_j(\tau)d\tau \tag{4.9}$$

$0 \le j \le m-1,\ 1 \le i \le N$, and $S_j(T)$ are shifted Legendre polynomials on the interval $[0, T_f]$ (see [33]). Using the expansions (4.8), we write

$$\begin{aligned}
\overrightarrow{Z}(T) &= A_{N\times m}\overrightarrow{S}_{m\times 1} \quad \text{or} \quad \overrightarrow{Z}(T) = S_{N\times Nm}\overrightarrow{A}_{Nm\times 1},\\
\overrightarrow{F}(T) &= B^1_{N\times m}\overrightarrow{S}_{m\times 1} \quad \text{or} \quad \overrightarrow{F}(T) = S_{N\times Nm}\overrightarrow{B^1}_{Nm\times 1},\\
\overrightarrow{F}(0) &= B^2_{N\times m}\overrightarrow{S}_{m\times 1} \quad \text{or} \quad \overrightarrow{F}(0) = S_{N\times Nm}\overrightarrow{B^2}_{Nm\times 1},\\
\overrightarrow{F'}(0) &= B^3_{N\times m}\overrightarrow{S}_{m\times 1} \quad \text{or} \quad \overrightarrow{F'}(0) = S_{N\times Nm}\overrightarrow{B^3}_{Nm\times 1},\\
G(T)\frac{d}{dt}\overrightarrow{Z}(0) &= C_{N\times m}\overrightarrow{S}_{m\times 1} \quad \text{or} \quad G(T)\frac{d}{dt}\overrightarrow{Z}(0) = S_{N\times Nm}C_{Nm\times 1},\\
\overrightarrow{Z}(0) &= W_{N\times m}\overrightarrow{S}_{m\times 1} \quad \text{or} \quad \overrightarrow{Z}(0) = S_{N\times Nm}\overrightarrow{W}_{Nm\times 1},
\end{aligned} \tag{4.10}$$

where

$$A = \begin{bmatrix} a_{10} & a_{11} & \cdots & a_{1,m-1} \\ a_{20} & a_{21} & \cdots & a_{2,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{N,m-1} \end{bmatrix}, \qquad \overrightarrow{A} = \begin{bmatrix} \overrightarrow{A_1} \\ \overrightarrow{A_2} \\ \vdots \\ \overrightarrow{A_N} \end{bmatrix},$$

$$C = \begin{bmatrix} c & c & 0 & \cdots & 0 \\ c & c & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c & c & 0 & \cdots & 0 \end{bmatrix}, \qquad \overrightarrow{C} = \begin{bmatrix} \overrightarrow{C_1} \\ \overrightarrow{C_2} \\ \vdots \\ \overrightarrow{C_N} \end{bmatrix},$$

$$\overrightarrow{S} = \begin{bmatrix} S_0(T) \\ S_1(T) \\ \vdots \\ S_{m-1}(T) \end{bmatrix}, \qquad S = \begin{bmatrix} \overrightarrow{S}^{\mathrm{tr}}(T) & & & \\ & \overrightarrow{S}^{\mathrm{tr}}(T) & & \\ & & \ddots & \\ & & & \overrightarrow{S}^{\mathrm{tr}}(T) \end{bmatrix},$$

$$W = \begin{bmatrix} Z_1(0) & 0 & \cdots & 0 \\ Z_1(0) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ Z_1(0) & 0 & \cdots & 0 \end{bmatrix}, \qquad \overrightarrow{W} = \begin{bmatrix} \overrightarrow{W_1} \\ \overrightarrow{W_2} \\ \vdots \\ \overrightarrow{W_N} \end{bmatrix},$$

$$B^1 = \begin{bmatrix} b_0 & b_1 & \cdots & b_{m-1} \\ b_0 & b_1 & \cdots & b_{m-1} \\ \vdots & \vdots & \vdots & \vdots \\ b_0 & b_1 & \cdots & b_{m-1} \end{bmatrix},$$

$$B^2 = \begin{bmatrix} b_0 & -b_1 & b_2 & -b_3 & \cdots & (-1)^{m-1} b_{m-1} \\ b_0 & -b_1 & b_2 & -b_3 & \cdots & (-1)^{m-1} b_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_0 & -b_1 & b_2 & -b_3 & \cdots & (-1)^{m-1} b_{m-1} \end{bmatrix},$$

$$B^3 = \begin{bmatrix} 0 & 2b_1 & -6b_2 & 12b_3 & -20b_4 & \cdots & (-1)^m (m-1) m b_{m-1} \\ 0 & 2b_1 & -6b_2 & 12b_3 & -20b_4 & \cdots & (-1)^m (m-1) m b_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 2b_1 & -6b_2 & 12b_3 & -20b_4 & \cdots & (-1)^m (m-1) m b_{m-1} \end{bmatrix}, \tag{4.11}$$

in which $c = (T_f/2)(d/dt)Z_1(0)$, $\overrightarrow{A_i} = [a_{i0}, a_{i1}, \ldots, a_{i,m-1}]^{\mathrm{tr}}$, $\overrightarrow{C_i} = [c, c, 0, \ldots, 0]^{\mathrm{tr}}$, and $\overrightarrow{W_i} = [Z_1(0), 0, \ldots, 0]^{\mathrm{tr}}$.

The double integral on the left-hand side of (4.5) is simplified as

$$\int_0^T \int_0^\eta \overrightarrow{Z}(\tau) d\tau\, d\eta = A \int_0^T \int_0^\eta \overrightarrow{S}(\tau) d\tau\, d\eta = AH^2 \overrightarrow{S}(T), \tag{4.12}$$

where H is the shifted Legendre operational matrix and is given by

$$H = \frac{T}{2} \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \dfrac{-1}{3} & 0 & \dfrac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \dfrac{-1}{5} & 0 & \dfrac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \dfrac{-1}{2m-3} & 0 & \dfrac{1}{2m-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \dfrac{-1}{2m-1} & 0 \end{bmatrix}. \tag{4.13}$$

Using (4.10) and (4.12), (4.5) takes on the form

$$A\vec{S} - C\vec{S} - W\vec{S} + \Lambda A H^2 \vec{S} = D\left[-2\underbrace{B^1\vec{S}}_{F(t)} + 2\underbrace{B^2\vec{S}(0)}_{F(0)}\underbrace{\beta\vec{S}}_{1} + 2\underbrace{B^3\vec{S}'(0)}_{F'(0)}\underbrace{\alpha\vec{S}}_{t}\right], \tag{4.14}$$

where

$$\beta_{1\times m} = [\beta_0, \beta_1, \ldots, \beta_{m-1}], \qquad \alpha_{1\times m} = [\alpha_0, \alpha_1, \ldots, \alpha_{m-1}] \tag{4.15}$$

are the coefficients of shifted Legendre polynomials resulting from the expansions of 1 and T, respectively [33].

By letting $\overline{D} = 2D$ and using the orthogonality of $\vec{S}(T)$, (4.14) becomes

$$A + \Lambda A H^2 = C + W - \overline{D}B^1 + \overline{D}B^2\vec{S}(0)\beta + \overline{D}B^3\vec{S}'(0)\alpha. \tag{4.16}$$

Now using the vec notation and kronecker product [34], (4.16) takes on the form

$$\vec{A} + (\Lambda \otimes \Gamma)\vec{A} = \vec{C} + \vec{W} - \left(\overline{D} \otimes I_m\right)\vec{B^1} + \left(\overline{D} \otimes \Omega^1\right)\vec{B^2} + \left(\overline{D} \otimes \Omega^2\right)\vec{B^3}, \tag{4.17}$$

where $\vec{A}$, $\vec{C}$, $\vec{W}$, $\vec{B^1}$, $\vec{B^2}$, and $\vec{B^3}$ are $Nm \times 1$ vectors, Γ, Ω^1, Ω^2, and I_m are $m \times m$ matrices in which $\Gamma = (H^2)^{\mathrm{tr}}$, $\Omega^1 = (\vec{S}(0)\beta)^{\mathrm{tr}}$, $\Omega^2 = (\vec{S}'(0)\alpha)^{\mathrm{tr}}$, and I_m is the $m \times m$ identity matrix, and $\overline{D}$ and Λ are $N \times N$ matrices.

Using the substitutions

$$X = I_{Nm} + (\Lambda \otimes \Gamma), \qquad U^1 = \overline{D} \otimes I_m, \qquad U^2 = \overline{D} \otimes \Omega^1, \qquad U^3 = \overline{D} \otimes \Omega^2, \qquad \vec{Y} = \vec{C} + \vec{W}, \tag{4.18}$$

equation (4.17) becomes

$$X\vec{A} = \vec{Y} + U^1\vec{B^1} + U^2\vec{B^2} + U^3\vec{B^3} = \vec{Y} + K\vec{B^1}, \tag{4.19}$$

where

$$K = U^1 + U^2 I^1 + U^3 I^2 \tag{4.20}$$

in which

$$I^1 = \operatorname{diag}\left[(-1)^{i+1}\right]_{i=1}^{m}, \qquad I^2 = \operatorname{diag}\left[(-1)^{i}(i-1)i\right]_{i=1}^{m}. \tag{4.21}$$

Solving (4.19) for $\overrightarrow{B^1}$, we obtain

$$\overrightarrow{B^1} = M\overrightarrow{A} + \overrightarrow{N}, \tag{4.22}$$

where $M = K^{-1}X$ and $\overrightarrow{N} = -K^{-1}\overrightarrow{Y}$.

4.3. Approximation of Modal Performance Index

The cost functional (4.3) can be transformed into modal space

$$\begin{aligned} J_N\left(\overrightarrow{F}\right) &= \frac{1}{2}\sum_{n=1}^{N}\overrightarrow{Z_n}^{\mathrm{tr}}(T_f)R_1\overrightarrow{Z_n}(T_f) + \frac{1}{2}\sum_{n=1}^{N}\frac{d}{dT}\overrightarrow{Z_n}^{\mathrm{tr}}(T_f)R_2\frac{d}{dT}\overrightarrow{Z_n}(T_f) \\ &\quad + \frac{1}{N}\int_0^{T_f}\overrightarrow{F}^{\mathrm{tr}}(\tau)R_3\overrightarrow{F}(\tau)d\tau, \end{aligned} \tag{4.23}$$

where R_i =diag $[\mu_i]$, i=1, 2, 3. Inserting (4.10) into (4.23) gives

$$\begin{aligned} J_N\left(\overrightarrow{F}\right) &= \frac{1}{2}\left[\overrightarrow{A}^{\mathrm{tr}}S^{\mathrm{tr}}(T_f)R_1S(T_f)\overrightarrow{A} + \overrightarrow{A}^{\mathrm{tr}}\frac{d}{dT}S^{\mathrm{tr}}(T_f)R_2\frac{d}{dT}S(T_f)\overrightarrow{A}\right] \\ &\quad + \frac{1}{N}\int_0^{T_f}\overrightarrow{B}^{1\mathrm{tr}}S^{\mathrm{tr}}(\tau)R_3S(\tau)\overrightarrow{B^1}d\tau, \end{aligned} \tag{4.24}$$

or by using (4.22),

$$J_N(A) = \frac{1}{2}\left[\overrightarrow{A}^{\mathrm{tr}}P_1(T_f)\overrightarrow{A} + \overrightarrow{A}^{\mathrm{tr}}P_2(T_f)\overrightarrow{A}\right] + \frac{1}{N}\int_0^{T_f}\left(M\overrightarrow{A} + \overrightarrow{N}\right)^{\mathrm{tr}}P_3(\tau)\left(M\overrightarrow{A} + \overrightarrow{N}\right)d\tau, \tag{4.25}$$

where

$$\begin{aligned} P_1(T) &= S^{\mathrm{tr}}(T)R_1S(T), \\ P_2(T) &= \frac{d}{dT}S^{\mathrm{tr}}(T)R_2\frac{d}{dT}S(T), \\ P_3(T) &= S^{\mathrm{tr}}(T)R_3S(T). \end{aligned} \tag{4.26}$$

Expanding $(M\overrightarrow{A} + \overrightarrow{N})^{\mathrm{tr}}$ and letting $P_3^* = \int_0^{T_f}P_3(t)dt$, the performance index (4.25) takes on the form

$$\begin{aligned} J_N\left(\overrightarrow{A}\right) &= \frac{1}{N}\left[\overrightarrow{A}^{\mathrm{tr}}M^TP_3^*M\overrightarrow{A} + \overrightarrow{N}^{\mathrm{tr}}P_3^*M\overrightarrow{A} + \overrightarrow{A}^{\mathrm{tr}}M^TP_3^*\overrightarrow{N} + \overrightarrow{N}^{\mathrm{tr}}P_3^*\overrightarrow{N}\right] \\ &\quad + \frac{1}{2}\left[\overrightarrow{A}^{\mathrm{tr}}(P_1(T_f) + P_2(T_f))\overrightarrow{A}\right]. \end{aligned} \tag{4.27}$$

To this end, the optimal control problem (3.8) is converted into the mathematical programming problem:

$$\text{Find } \overrightarrow{A^*} \in \mathbb{R}^N \text{ so that } J_N\left(\overrightarrow{A^*}\right) = \min_{\overrightarrow{A} \in \mathbb{R}^N} J_N\left(\overrightarrow{A}\right). \tag{4.28}$$

4.4. Optimal Control Characterization

Let $P_4 = P_1 + P_2$ and $P_5 = M^{\text{tr}} P_3^* M$. By employing the properties of matrix differentiation [34], we can obtain the necessary condition of the optimal control by differentiating the performance index (4.27) with respect to the unknown vector $\overrightarrow{A}$, we obtain

$$\begin{aligned}\frac{\partial J_N\left(\overrightarrow{A}\right)}{\partial \overrightarrow{A}} &= \frac{1}{N}\left[\left(P_5(T_f) + P_5^{\text{tr}}(T_f)\right)\overrightarrow{A} + \left(\overrightarrow{N}^{\text{tr}} P_3^* M\right)^{\text{tr}} + \left(M^{\text{tr}} P_3^* \overrightarrow{N}\right)\right] \\ &\quad + \frac{1}{2}\left(P_4(T_f) + P_4^{\text{tr}}(T_f)\right)\overrightarrow{A}.\end{aligned} \tag{4.29}$$

Let

$$\begin{gathered}P = \frac{1}{2}\left(P_4(T_f) + P_4^{\text{tr}}(T_f)\right) + \frac{1}{n}\left(P_5(T_f) + P_5^{\text{tr}}(T_f)\right), \\ Q = \frac{-1}{N}\left(M^T P_3^{*\text{tr}} + M^{\text{tr}} P_3^*\right).\end{gathered} \tag{4.30}$$

Now we find $\overrightarrow{A^*}$ so that $\partial J_N(\overrightarrow{A^*})/\partial \overrightarrow{A} = 0$, we obtain

$$P\overrightarrow{A^*} - Q\overrightarrow{N} = 0 \tag{4.31}$$

or

$$\overrightarrow{A^*} = P^{-1} Q \overrightarrow{N} \tag{4.32}$$

and from (4.22), we have

$$\overrightarrow{B^*} = M\left(P^{-1} Q \overrightarrow{N}\right) + \overrightarrow{N} = \left(M P^{-1} Q + I\right)\overrightarrow{N}. \tag{4.33}$$

The optimal state variable $W^*(X, T)$ is obtained from (3.3), so we have

$$W^*(X, T) = \overline{W}^*(X, T) + \alpha(X) F^*(T), \tag{4.34}$$

where

$$\overline{W}^{*}(X,T)=\sum_{n=1}^{N}\left(\sum_{j=0}^{m-1}a_{ij}^{*}S_j(T)\right)\varphi_n(X),\qquad F^{*}(T)=\sum_{j=0}^{m-1}b_j^{*}S_j(T) \tag{4.35}$$

in which a_{ij}^{*} and b_j^{*} are the components of $\overrightarrow{A^{*}}$ and $\overrightarrow{B^{*}}$, respectively.

5. Numerical Experiments

Numerical results are given to show the effectiveness of piezo actuators in controlling the system and damping out the vibrations of the microbeam with a minimal use of voltage applied to the piezo actuators at various terminal times, T_f, subject to the initial impact conditions:

$$W(x,0)=\sin\lambda_1 x+\sin\lambda_2 x,\qquad W_T(x,0)=0, \tag{5.1}$$

where $\lambda_i = i\pi$, $i = 1,2$.

For the numerical simulations, we let $N = 2$ and $m = 10$ (the first 10 shifted Legendre polynomials). The measure of the total force used in the control process is given by

$$J_c\left(F^0\right)=\int_0^{T_f}\left(F^0(\tau)\right)^2 d\tau, \tag{5.2}$$

and the controlled energy of the system is defined as

$$E_c\left(F^0\right)=\int_0^1\left\{\mu_1\left[W^0(x,T_f)\right]^2+\mu_2\left[W_T^0(x,T_f)\right]^2\right\}dx, \tag{5.3}$$

where $W^0(x,T)$ corresponds to the optimal displacement of the microbeam. The controlled and uncontrolled energies are denoted by E_c and E_u, respectively, and the force used is denoted by J_c. In Tables 1–3, we study the effect of each two consecutive weighting factors μ_i and μ_{i+1} for $i = 1,3,$ and 5. While we vary each two consecutive $\mu's$, we set the other four as well as the terminal time at 1. In Table 4, we study the effect of the terminal time on the system energies.

The following observations are made.

(1) The system achieves substantial energy reduction when the weighting factors μ_1 and μ_2 are smaller. This implies that the smaller μ_1 and μ_2, the more emphasis on kinetic energies; see Table 1.

(2) The system energies remain unchanged as the weighting factors μ_3 and μ_4 assume different values; see Table 2.

(3) The weighting factors μ_5 and μ_6 are effective in regulating the system energies, see Table 3.

(4) At any given terminal time, the system is damping the energy out to a desired value; see Table 4.

Table 1: Effect of μ_1 and μ_2.

$\mu_1 = \mu_2$	$E_u\ (F = 0)$	$E_c\ (F^0)$	$J_c\ (F^0)$
0.0625	0.014787	0.000006	0.000001
0.1250	0.029574	0.000010	0.064868
0.2500	0.059149	0.000019	0.691537
0.5000	0.118297	0.000033	0.006049
1.0000	0.236594	0.000043	0.064752
2.0000	0.473188	0.000056	0.013062

Table 2: Effect of μ_3 and μ_4.

$\mu_3 = \mu_4$	$E_u\ (F = 0)$	$E_c\ (F^0)$	$J_c\ (F^0)$
0.0625	0.236594	0.000042	0.010527
0.1250	0.236594	0.000042	0.000251
0.2500	0.236594	0.000042	0.040656
0.5000	0.236594	0.000046	0.263530
1.0000	0.236594	0.000043	0.064752
2.0000	0.236594	0.000044	0.128104

Table 3: Effect of μ_5 and μ_6.

$\mu_5 = \mu_6$	$E_u\ (F = 0)$	$E_c\ (F^0)$	$J_c\ (F^0)$
0.0625	0.236594	0.000003	0.003365
0.1250	0.236594	0.000006	0.005434
0.2500	0.236594	0.000015	0.000076
0.5000	0.236594	0.000032	0.159332
1.0000	0.236594	0.000043	0.064752
2.0000	0.236594	0.000056	0.023356

Table 4: Effect of terminal time (T_f).

T_f	$E_u\ (F = 0)$	$E_c\ (F^0)$	$J_c\ (F^0)$
1	0.236594	0.0000043	0.064752
5	0.029106	0.002017	0.005514
10	0.385400	0.000003	0.006740
15	0.25548	0.000001	0.004564

6. Conclusions

In this paper, a computational approach was presented to optimal boundary control of smart mirco-beams with quadratic performance index. The Galerkin method is first used to reduce the problem to optimal control of lumped parameter system. An algorithm based on parametrizing the state and control variables by shifted Legendre polynomials was employed to solve the lumped parameter optimization problem. The control parameters are obtained from the integrated system state equations as a function of the approximated state parameters, and the performance index was evaluated by an algorithm, which was also proposed in the current study. The optimal control problem in lumped parameter system was converted into a parameter optimization problem, which was quadratic in the unknown parameters. The optimal value of these parameters is obtained by using quadratic programming results. The numerical examples presented support the theoretical study and reveal the usefulness of the proposed approach.

References

[1] R. C. Batra, M. Porfiri, and D. Spinello, "Vibrations of narrow microbeams predeformed by an electric field," *Journal of Sound and Vibration*, vol. 309, no. 3–5, pp. 600–612, 2008.

[2] S. Hurlebans and L. Gaul, "Smart structure dynamics," *Mechanical Systems and Signal Processing*, vol. 20, no. 2, pp. 255–281, 2006.

[3] B. Borovic, F. L. Lewis, W. McCulley, L. Al Qun, E. S. Klesar, and D. O. Popa, "Control issues for microelectromechanical systems," *IEEE Control Systems Magazine*, vol. 26, no. 2, pp. 18–21, 2006.

[4] M. I. Younis, E. M. Abdel-Rahman, and A. Nayfeh, "A reduced-order model for electrically actuated microbeam-based MEMS," *Journal of Microelectromechanical Systems*, vol. 12, no. 5, pp. 672–680, 2003.

[5] S. N. Mahmoodi and N. Jalili, "Piezoelectrically actuated microcantilivers: an experimental nonlinear vibration analysis," *Sensors and Actuators A*, vol. 150, no. 1, pp. 131–136, 2009.

[6] M. Collet, V. Walter, and P. Delobelle, "Active damping with piezoelectric MEMS devices," in *Smart Structures and Materials: Damping and Isolation*, vol. 5386 of *Proceedings of SPIE*, pp. 301–308, San Diego, Calif, USA, March 2004.

[7] M. Collet, V. Walter, and P. Delobelle, "Active damping of a micro-cantilever piezo-composite beam," *Journal of Sound and Vibration*, vol. 260, no. 3, pp. 453–476, 2003.

[8] W. Zhang, G. Meng, and H. Li, "Adaptive vibration control of micro-cantiliver beam with piezoelectric actuator in MEMS," *The International Journal of Advanced Manufacturing Technology*, vol. 28, no. 3-4, pp. 321–327, 2006.

[9] I. S. Sadek and M. A. Bokhari, "Optimal control of a parabolic distributed parameter system via orthogonal polynomials," *Optimal Control Applications and Methods*, vol. 19, no. 3, pp. 205–213, 1998.

[10] R. Y. Chang and S. Y. Yang, "Solution of two-point boundary value problems by generalized orthogonal polynomials and application to optimal control of lumped and distributed parameter systems," *International Journal of Control*, vol. 43, no. 6, pp. 1785–1802, 1986.

[11] M. Razzaghi and A. Arabshahi, "Optimal control of linear distributed-parameter systems via polynomial series," *International Journal of Systems Science*, vol. 20, no. 7, pp. 1141–1148, 1989.

[12] C. F. Chen and C. H. Hsiao, "A state-space approach to Walsh series solution of linear systems," *International Journal of Systems Science*, vol. 6, no. 9, pp. 833–858, 1975.

[13] N. S. Hsu and B. Cheng, "Analysis and optimal control of time-varying linear systems via block-pulse functions," *International Journal of Control*, vol. 33, no. 6, pp. 1107–1122, 1981.

[14] P. N. Paraskevopoulos, P. G. Sklavounos, and G. Ch. Georgiou, "The operational matrix of integration for Bessel functions," *Journal of the Franklin Institute*, vol. 327, no. 2, pp. 329–341, 1990.

[15] H. R. Karimi, "Optimal vibration control of vehicle engine-body system using Haar functions," *International Journal of Control, Automation, and Systems*, vol. 4, no. 6, pp. 714–724, 2006.

[16] M. L. Nagurka and S.-K. Wang, "A Chebyshev-based state representation for linear quadratic optimal control," *Journal of Dynamic Systems, Measurement and Control*, vol. 115, no. 1, pp. 1–6, 1993.

[17] D. Shih, F. Kung, and C. Chao, "Laguerre series approach to the analysis of a linear control system incorporating observes," *International Journal of Control*, vol. 43, no. 1, pp. 123–128, 1986.

[18] C. Y. Yang and C. K. Chen, "Analysis and optimal control of time-varying systems via Fourier series," *International Journal of Systems Science*, vol. 25, no. 11, pp. 1663–1678, 1994.

[19] M. Razzaghi and M. Razzaghi, "Taylor series direct method for variational problems," *Journal of the Franklin Institute*, vol. 325, no. 1, pp. 125–131, 1988.

[20] C. C. Liu and Y. P. Shih, "System analysis, parameter estimation and optimal regulator design of linear systems via Jacobi series," *International Journal of Control*, vol. 42, no. 1, pp. 211–224, 1985.

[21] M. Razzaghi and M. Razzaghi, "Shifted-Jacobi series direct method for variational problems," *International Journal of Systems Science*, vol. 20, no. 7, pp. 1119–1129, 1989.

[22] K. K. Shyu, *Analysis of dynamic systems using hermite polynomials*, M.S. thesis, Department of Electrical Engineering, National Cheng Kung University, Tainan, Taiwan, 1984.

[23] H. R. Karimi, "A computational method for optimal control problem of time-varying state-delayed systems by Haar wavelets," *International Journal of Computer Mathematics*, vol. 83, no. 2, pp. 235–246, 2006.

[24] H. R. Karimi, B. Moshiri, B. Lohmann, and P. Jabehdar Maralani, “Haar wavelet-based approach for optimal control of second-order linear systems in time domain,” *Journal of Dynamical and Control Systems*, vol. 11, no. 2, pp. 237–252, 2005.

[25] H. R. Karimi, M. Zapateiro, and N. Luo, “Wavelet-based parameter identification of a nonlinear magnetorheological damper,” *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 7, no. 2, pp. 183–198, 2009.

[26] H. R. Karimi, B. Lohmann, P. Jabedar Maralani, and B. Moshiri, “A computational method for solving optimal control and parameter estimation of linear systems using Haar wavelets,” *International Journal of Computer Mathematics*, vol. 81, no. 9, pp. 1121–1132, 2004.

[27] M. Razzaghi and S. Yousefi, “The Legendre wavelets operational matrix of integration,” *International Journal of Systems Science*, vol. 32, no. 4, pp. 495–502, 2001.

[28] M. Razzaghi and S. Yousefi, “Sine-cosine wavelets operational matrix of integration and its applications in the calculus of variations,” *International Journal of Systems Science*, vol. 33, no. 10, pp. 805–810, 2002.

[29] I. Sadek, T. Abualrub, and M. Abukhaled, “Application of wavelets to the optimal control of a parallel system of beams,” *WSEAS Transactions on Mathematics*, vol. 5, no. 6, pp. 728–733, 2006.

[30] I. S. Sadek, “Variational method for the distributed control of a vibrating beam,” *Optimal Control Applications and Methods*, vol. 9, no. 1, pp. 79–85, 1988.

[31] I. Sadek, M. Abukhaled, and T. Abualrub, “Optimal pointwise control for a parallel system of Euler-Bernoulli beams,” *Journal of Computational and Applied Mathematics*, vol. 137, no. 1, pp. 83–95, 2001.

[32] I. Sadek, I. Kucuk, E. Zeini, and S. Adali, “Optimal boundary control of dynamics responses of piezo actuating micro-beams,” *Applied Mathematical Modelling*, vol. 33, no. 8, pp. 3343–3353, 2009.

[33] L.-P. Zhang and F.-W. Yang, “Study on the application of iterative learning control to terminal control of linear time-varying systems,” *Acta Automatica Sinica*, vol. 31, no. 2, pp. 309–313, 2005.

[34] J. W. Brewer, “Kronecker products and matrix calculus in system theory,” *IEEE Transactions on Circuits and Systems*, vol. 25, no. 9, pp. 772–781, 1978.