

Research Article

Singular Positone and Semipositone Boundary Value Problems of Nonlinear Fractional Differential Equations

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We present some new existence results for singular positone and semipositone nonlinear fractional boundary value problem $D_{0+}^{\alpha} u(t) = \mu a(t) f(t, u(t))$, $0 < t < 1$, $u(0) = u(1) = u'(0) = u'(1) = 0$, where $\mu > 0$, a , and f are continuous, $\alpha \in (3, 4]$ is a real number, and D_{0+}^{α} is Riemann-Liouville fractional derivative. Throughout our nonlinearity may be singular in its dependent variable. Two examples are also given to illustrate the main results.

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1. Introduction

Fractional calculus has played a significant role in engineering, science, economy, and other fields. Many papers and books on fractional calculus, and fractional differential equations have appeared recently, (see [1–9]). It should be noted that most of papers and books on fractional calculus, are devoted to the solvability of initial fractional differential equations (see [3, 4]). Here, we consider positive solutions of nonlinear fractional differential equation conjugate boundary value problem involving Riemann-Liouville derivative:

$$\begin{aligned} D_{0+}^{\alpha} u(t) &= \mu a(t) f(t, u(t)), \quad 0 < t < 1, \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \tag{1.1}$$

where $\mu > 0$, a , and f are continuous. $\alpha \in (3, 4]$ is a real number, and D_{0+}^{α} is the Riemann-Liouville fractional derivative.

It is well known that in mechanics the boundary value problem (1.1) where $\alpha = 4$ describes the deflection of an elastic beam rigidly fixed at both ends. The integer order boundary value problem (1.2) has been studied extensively. For details, see for instance, the papers [10–13] and the references therein. In [10, 12], Yao considered

$$\begin{aligned} u''''(t) &= \lambda f(t, u(t)), \quad 0 < t < 1, \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \quad (1.2)$$

and using a Krasnosel'skii fixed-point theorem, derived a λ -interval such that, for any λ lying in this interval, the beam equation has existence and multiplicity on positive solution. In this paper, we will consider a more general situation, namely, the boundary value problem (1.1). To the best of our knowledge, there have been few papers which deal with the boundary value problem (1.1) for nonlinear fractional differential equation.

In this paper, in analogy with boundary value problems for differential equations of integer order, we firstly derive the corresponding Green's function named the fractional Green' function. Consequently problem (1.1) is reduced to an equivalent Fredholm integral equation of the second kind. Finally, using Krasnosel'skii's fixed-point theorems, the existence of positive solutions are obtained.

2. Preliminaries

For completeness, in this section, we will demonstrate and study the definitions and some fundamental facts of Riemann-Liouville derivatives of fractional order which can be found in [5].

Definition 2.1 (see [5, Definition 2.1]). The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0, \quad (2.1)$$

where $\alpha > 0$, is called the Riemann-Liouville fractional integral of order α .

Definition 2.2 (see [5, page 36-37]). For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad (2.2)$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α , is called the Riemann-Liouville fractional derivative of order α .

From the definition of Riemann-Liouville derivative, for $\mu > -1$, we have

$$D_{0+}^{\alpha} x^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}, \quad (2.3)$$

giving in particular $D_{0+}^{\alpha} x^{\alpha-m} = 0$, $m = 1, 2, 3, \dots, N$, where N is the smallest integer greater than or equal to α .

Lemma 2.3. Let $\alpha > 0$, then the differential equation

$$\mathbf{D}_{0+}^{\alpha} u(t) = 0 \quad (2.4)$$

has solutions $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, as unique solutions, where n is the smallest integer greater than or equal to α .

As $\mathbf{D}_{0+}^{\alpha} I_{0+}^{\alpha} u = u$, from Lemma 2.3, we deduce the following statement.

Lemma 2.4. Let $\alpha > 0$, then

$$I_{0+}^{\alpha} \mathbf{D}_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad (2.5)$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, n is the smallest integer greater than or equal to α .

The following Krasnosel'skii's fixed-point theorem will play a major role in our next analysis.

Theorem 2.5 (see [6]). Let X be a Banach space, and let $P \subset X$ be a cone in X . Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \rightarrow P$ be a completely continuous operator such that, either

- (1) $\|S\omega\| \leq \|\omega\|$, $\omega \in P \cap \partial\Omega_1$, $\|S\omega\| \geq \|\omega\|$, $\omega \in P \cap \partial\Omega_2$, or
- (2) $\|S\omega\| \geq \|\omega\|$, $\omega \in P \cap \partial\Omega_1$, $\|S\omega\| \leq \|\omega\|$, $\omega \in P \cap \partial\Omega_2$.

Then S has a fixed-point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Green's Function and Its Properties

In this section, we derive the corresponding Green's function for boundary-value problem (1.1), and obtain some properties of Green's function.

Lemma 3.1. Let $h(t) \in C[0, 1]$ be a given function, then the boundary-value problem,

$$\begin{aligned} \mathbf{D}_{0+}^{\alpha} u(t) &= h(t), \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \quad (3.1)$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) h(s) ds, \quad (3.2)$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-2}(1-s)^{\alpha-2}[(s-t) + (\alpha-2)(1-t)s] + (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-2}(1-s)^{\alpha-2}[(s-t) + (\alpha-2)(1-t)s], & 0 \leq t \leq s \leq 1. \end{cases} \quad (3.3)$$

Here $G(t, s)$ is called Green's function of boundary-value problem (3.1).

Proof. By means of the Lemma 2.4, we can reduce (3.1) to an equivalent integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + c_4 t^{\alpha-4} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds. \quad (3.4)$$

From $u(0) = u(1) = u'(0) = u'(1) = 0$, we have $c_3 = c_4 = 0$ and

$$c_1 = \int_0^1 \frac{(1-s)^{\alpha-2}(2s-\alpha s-1)}{\Gamma(\alpha)} h(s) ds, \quad (3.5)$$

$$c_2 = \int_0^1 \frac{(\alpha-1)(1-s)^{\alpha-2}s}{\Gamma(\alpha)} h(s) ds.$$

Therefore, the unique solution of (3.1) is

$$\begin{aligned} u(t) &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}(2s-\alpha s-1)}{\Gamma(\alpha)} h(s) ds + \int_0^1 \frac{t^{\alpha-2}(\alpha-1)(1-s)^{\alpha-2}s}{\Gamma(\alpha)} h(s) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \left[t^{\alpha-1}(1-s)^{\alpha-2}(2s-\alpha s-1) + t^{\alpha-2}(\alpha-1)(1-s)^{\alpha-2}s + (t-s)^{\alpha-1} \right] h(s) ds \\ &\quad + \int_t^1 \left[t^{\alpha-1}(1-s)^{\alpha-2}(2s-\alpha s-1) + t^{\alpha-2}(\alpha-1)(1-s)^{\alpha-2}s \right] h(s) ds \\ &= \int_0^1 G(t,s) h(s) ds. \end{aligned} \quad (3.6)$$

The proof is finished. □

Lemma 3.2. *The function $G(t,s)$ defined by (3.3) has the following properties:*

- (1) $G(t,s) = G(1-s,1-t)$, for $t,s \in [0,1]$;
- (2) $t^{\alpha-2}(1-t)^2 q(s) \leq G(t,s) \leq (\alpha-1)q(s)$ and $G(t,s) \leq ((\alpha-1)(\alpha-2)/\Gamma(\alpha))t^{\alpha-2}(1-t)^2$ for $t,s \in [0,1]$,

where $q(s) = ((\alpha-2)/\Gamma(\alpha))s^2(1-s)^{\alpha-2}$.

Proof. Observing the expression of $G(t,s)$, it is clear that $G(t,s) = G(1-s,1-t)$ for $t,s \in [0,1]$. In the following, we consider $\Gamma(\alpha)G(t,s)$.

For $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned}
\Gamma(\alpha)G(t, s) &= (t-s)^{\alpha-1} - (t-ts)^{\alpha-2}(t-s) + (\alpha-2)(1-t)(t-ts)^{\alpha-2}s \\
&= (t-s) \left[(t-s)^{\alpha-2} - (t-ts)^{\alpha-2} \right] + (\alpha-2)(1-t)(t-ts)^{\alpha-2}s \\
&= -(t-s)(\alpha-2) \int_{t-s}^{t-ts} x^{\alpha-3} dx + (\alpha-2)(t-ts)^{\alpha-2}(1-t)s \\
&\geq -(t-s)(\alpha-2)(t-ts)^{\alpha-3} [(t-ts) - (t-s)] + (\alpha-2)(t-ts)^{\alpha-2}(1-t)s \quad (3.7) \\
&= -(t-s)(\alpha-2)(t-ts)^{\alpha-3}(1-t)s + (\alpha-2)(t-ts)^{\alpha-2}(1-t)s \\
&= (\alpha-2)(t-ts)^{\alpha-3}(1-t)s[-(t-s) + (t-ts)] \\
&\geq (\alpha-2)(t-ts)^{\alpha-2}(1-t)^2s^2 \\
&= (\alpha-2)t^{\alpha-2}(1-t)^2s^2(1-s)^{\alpha-2},
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(\alpha)G(t, s) &= -(t-s)(\alpha-2) \int_{t-s}^{t-ts} x^{\alpha-3} dx + (\alpha-2)(t-ts)^{\alpha-2}(1-t)s \\
&\leq -(t-s)(\alpha-2)(t-s)^{\alpha-3} [(t-ts) - (t-s)] + (\alpha-2)(t-ts)^{\alpha-2}(1-t)s \\
&= -(t-s)(\alpha-2)(t-s)^{\alpha-3}(1-t)s + (\alpha-2)(t-ts)^{\alpha-2}(1-t)s \\
&= (\alpha-2)(1-t)s \left[(t-ts)^{\alpha-2} - (t-s)^{\alpha-2} \right] \quad (3.8) \\
&\leq (\alpha-2)^2(1-t)^2s^2(t-ts)^{\alpha-3} \\
&\leq (\alpha-1)(\alpha-2)t^{\alpha-3}(1-t)^2s^2(1-s)^{\alpha-3} \\
&\leq (\alpha-1)(\alpha-2)s^2(1-s)^{\alpha-2}.
\end{aligned}$$

For $0 \leq t \leq s \leq 1$, since $\alpha > 3$, we have

$$\begin{aligned}
\Gamma(\alpha)G(t, s) &= (t-ts)^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s] \\
&\geq (\alpha-2)(t-ts)^{\alpha-2}(1-t)s \\
&= (\alpha-2)t^{\alpha-2}(1-s)^{\alpha-2}(1-t)s \\
&\geq (\alpha-2)t^{\alpha-2}(1-t)^2s^2(1-s)^{\alpha-2},
\end{aligned}$$

$$\begin{aligned}
\Gamma(\alpha)G(t,s) &= t^{\alpha-2}(1-s)^{\alpha-2}[(s-t) + (\alpha-2)(1-t)s] \\
&\leq t^{\alpha-2}(1-s)^{\alpha-2}[s + (\alpha-2)s] \\
&\leq (\alpha-1)t(1-s)^{\alpha-2}s \\
&\leq (\alpha-1)s^2(1-s)^{\alpha-2} \\
&\leq (\alpha-2)(\alpha-1)s^2(1-s)^{\alpha-2}.
\end{aligned} \tag{3.9}$$

Thus $t^{\alpha-2}(1-t)^2q(s) \leq G(t,s) \leq (\alpha-1)q(s)$, for $t, s \in [0, 1]$. Combining $G(t,s) = G(1-s, 1-t)$, we have

$$G(t,s) \leq (\alpha-1)q(1-t) = \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}t^{\alpha-2}(1-t)^2, \text{ for } t, s \in [0, 1]. \tag{3.10}$$

This completes the proof. \square

We note that $u(t)$ is a solution of (1.1) if and only if

$$u(t) = \mu \int_0^1 G(t,s)a(s)f(u(s))ds, \quad 0 \leq t \leq 1. \tag{3.11}$$

For our constructions, we will consider the Banach space $E = C[0,1]$ equipped with standard norm $\|u\| = \max_{0 \leq t \leq 1} \|u(t)\|, u \in X$. We define a cone K by

$$K = \left\{ u \in X \mid u(t) \geq \frac{t^{\alpha-2}(1-t)^2}{\alpha-1} \|u\|, t \in [0, 1], \alpha \in (3, 4] \right\}. \tag{3.12}$$

Define an integral operator $A : K \rightarrow X$ by

$$Au(t) = \mu \int_0^1 G(t,s)a(s)f(u(s))ds, \quad 0 \leq t \leq 1, u \in K. \tag{3.13}$$

Notice from (3.13) and Lemma 3.2 that, for $u \in K$, $Au(t) \geq 0$ on $[0, 1]$ and

$$Au(t) = \mu \int_0^1 G(t,s)a(s)f(s, u(s))ds \leq \mu \int_0^1 (\alpha-1)q(s)a(s)f(s, u(s))ds, \tag{3.14}$$

then $\|Au\| \leq \int_0^1 (\alpha-1)q(s)a(s)f(s, u(s))ds$.

On the other hand, we have

$$\begin{aligned}
 Au(t) &= \mu \int_0^1 G(t,s)a(s)f(s,u(s))ds \\
 &\geq \mu \int_0^1 t^{\alpha-2}(1-t)^2 q(s)a(s)f(s,u(s))ds \\
 &\geq \frac{t^{\alpha-2}(1-t)^2}{\alpha-1} \mu \int_0^1 (\alpha-1)q(s)a(s)f(s,u(s))ds \\
 &\geq \frac{t^{\alpha-2}(1-t)^2}{\alpha-1} \|Au\|.
 \end{aligned} \tag{3.15}$$

Thus, $A(K) \subset K$. In addition, standard arguments show that A is completely continuous.

4. Singular Positone Problems

In this section we present some new result for the singular problem

$$\begin{aligned}
 \mathbf{D}_{0+}^{\alpha} u(t) &= \mu a(t)f(t,u(t)), \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \\
 u(0) &= u(1) = u'(0) = u'(1) = 0,
 \end{aligned} \tag{4.1}$$

where $\mu > 0$ and nonlinearity f may be singular at $u = 0$.

Using Theorem 2.5 we establish the following main result.

Theorem 4.1. *Suppose that the following conditions are satisfied.*

$$a \in C(0,1) \cap L^1[0,1] \text{ with } a > 0 \text{ on } (0,1) \tag{4.2}$$

$$f : [0,1] \times (0,\infty) \longrightarrow (0,\infty) \text{ is continuous,} \tag{4.3}$$

$$\left\{ \begin{array}{l}
 f(t,u) \leq g(u) + h(u) \text{ on } [0,1] \times (0,\infty) \text{ with } g > 0 \\
 \text{continuous and nonincreasing on } (0,\infty), \quad h \geq 0 \\
 \text{continuous on } [0,\infty) \text{ and } \frac{h}{g} \text{ non decreasing on } (0,\infty),
 \end{array} \right. \tag{4.4}$$

$$\exists K_0 \text{ with } g(xy) \leq K_0 g(x)g(y) \quad \forall x > 0, y > 0, \tag{4.5}$$

$$a_0 = \mu(\alpha-1) \int_0^1 q(s)a(s)g(s^{\alpha-2}(1-s)^2)ds < \infty, \tag{4.6}$$

$$\exists r > 0 \text{ with } \frac{r}{g(r) + h(r)} > K_0^2 a_0 g\left(\frac{1}{\alpha-1}\right), \tag{4.7}$$

$$\left\{ \begin{array}{l} \text{There exists } 0 < \theta < \frac{1}{2} \text{ (choose and fix it) and a continuous,} \\ \text{nonincreasing function } g_1 : (0, \infty) \rightarrow (0, \infty), \text{ and a continuous} \\ \text{function } h_1 : [0, \infty) \rightarrow (0, \infty) \text{ with } \frac{h_1}{g_1} \text{ nondecreasing on } (0, \infty) \\ \text{and with } f(t, u) \geq g_1(u) + h_1(u) \text{ for } (t, u) \in [\theta, 1 - \theta] \times (0, \infty), \end{array} \right. \quad (4.8)$$

$$\left\{ \begin{array}{l} \exists 0 < R_1 < r < R_2 \text{ with } (i = 1, 2), \\ \frac{R_i g_1(\theta^\alpha / \alpha - 1) R_i}{g_1(R_i) g_1((\theta^\alpha / \alpha - 1) R_i) + g_1(R_i) h_1((\theta^\alpha / \alpha - 1) R_i)} < \mu \int_\theta^{1-\theta} G(\sigma, s) a(s) ds, \end{array} \right. \quad (4.9)$$

here $G(t, s)$ is Green's function and

$$\int_\theta^{1-\theta} G(\sigma, s) a(s) ds = \sup_{t \in [0, 1]} \int_\theta^{1-\theta} G(t, s) a(s) ds. \quad (4.10)$$

Then (4.1) has two nonnegative solutions u_i with $R_1 < \|u_1\| < r < \|u_2\| < R_2$ and $u_i(t) > 0$ for $t \in (0, 1), i = 1, 2$.

Proof. First we will show that there exists a solution u_2 to (4.1) with $u_2(t) > 0$ for $t \in (0, 1)$ and $r < \|u_2\| < R_2$. Let

$$\Omega_1 = \{u \in E : \|u\| < r\}, \quad \Omega_2 = \{u \in E : \|u\| < R_2\}. \quad (4.11)$$

We now show

$$\|Au\| < \|u\| \quad \text{for } K \cap \partial\Omega_1. \quad (4.12)$$

To see this, let $u \in K \cap \partial\Omega_1$. Then $\|u\| = \|u\|_{[0, 1]} = r$ and $u(t) \geq (t^{\alpha-2}(1-t)^2/(\alpha-1))r$ for $t \in [0, 1]$. So for $t \in [0, 1]$, we have

$$\begin{aligned} (Au)(t) &= \mu \int_0^1 G(t, s) a(s) f(s, u(s)) ds \\ &\leq \mu \int_0^1 (\alpha-1) q(s) a(s) [g(u(s)) + h(u(s))] ds \\ &= \mu \int_0^1 (\alpha-1) q(s) a(s) g(u(s)) \left\{ 1 + \frac{h(u(s))}{g(u(s))} \right\} ds \end{aligned}$$

$$\begin{aligned}
 &\leq \mu \int_0^1 (\alpha - 1)q(s)a(s)g\left(\frac{s^{\alpha-2}(1-s)^2}{\alpha - 1}r\right) \left\{1 + \frac{h(r)}{g(r)}\right\} ds \\
 &\leq K_0g\left(\frac{r}{\alpha - 1}\right) \left\{1 + \frac{h(r)}{g(r)}\right\} \mu \int_0^1 (\alpha - 1)q(s)a(s)g\left(s^{\alpha-2}(1-s)^2\right) ds \\
 &= a_0K_0^2g\left(\frac{1}{\alpha - 1}\right) [g(r) + h(r)].
 \end{aligned}
 \tag{4.13}$$

This together with (4.7) yields

$$\|Au\| = \|Au\|_{[0,1]} < r = \|u\|,
 \tag{4.14}$$

so (4.12) is satisfied.
 Next we show

$$\|Au\| > \|u\| \quad \text{for } K \cap \partial\Omega_2.
 \tag{4.15}$$

To see this, let $u \in K \cap \partial\Omega_2$ so $\|u\| = \|u\|_{[0,1]} = R_2$, and let $u(t) \geq (t^{\alpha-2}(1-t)^2/(\alpha - 1))R_2$ for $t \in [0, 1]$.

We have

$$\begin{aligned}
 (Au)(\sigma) &= \mu \int_0^1 G(\sigma, s)a(s)f(s, u(s))ds \\
 &\geq \mu \int_\theta^{1-\theta} G(\sigma, s)a(s)[g_1(u(s)) + h_1(u(s))]ds \\
 &= \mu \int_\theta^{1-\theta} G(\sigma, s)a(s)g_1(u(s)) \left\{1 + \frac{h_1(u(s))}{g_1(u(s))}\right\} ds \\
 &\geq g_1(R_2)\mu \int_\theta^{1-\theta} G(\sigma, s)a(s) \left\{1 + \frac{h_1\left(\left(\frac{s^{\alpha-2}(1-s)^2}{(\alpha - 1)}R_2\right)\right)}{g_1\left(\left(\frac{s^{\alpha-2}(1-s)^2}{(\alpha - 1)}R_2\right)\right)}\right\} ds \\
 &\geq g_1(R_2)\mu \int_\theta^{1-\theta} G(\sigma, s)a(s) \left\{1 + \frac{h_1\left(\left(\frac{\theta^\alpha}{(\alpha - 1)}R_2\right)\right)}{g_1\left(\left(\frac{\theta^\alpha}{(\alpha - 1)}R_2\right)\right)}\right\} ds.
 \end{aligned}
 \tag{4.16}$$

This together with (4.9) yields

$$(Au)(\sigma) > R_2 = \|u\|.
 \tag{4.17}$$

Thus $\|Au\| > \|u\|$, so (4.15) is held.

Now Theorem 2.5 implies that A has a fixed-point $u_2 \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, that is, $r \leq \|u_2\| = \|u_2\|_{[0,1]} \leq R$ and $u_2(t) \geq q(t)r$ for $t \in [0, 1]$. It follows from (4.12) and (4.15) that $\|u_2\| \neq r$, $\|u_2\| \neq R_2$, so we have $r < \|u_2\| < R_2$.

Similarly, if we put

$$\Omega_1 = \{u \in E : \|u\| < R_1\}, \quad \Omega_2 = \{u \in E : \|u\| < r\}, \quad (4.18)$$

we can show that there exists a solution u_1 to (4.1) with $u_1(t) > 0$ for $t \in (0, 1)$ and $R_1 < \|u_1\| < r$.

This completes the proof of Theorem 4.1. \square

Similar to the proof of Theorem 4.1, we have the following result.

Theorem 4.2. *Suppose that (4.2)–(4.8) hold. In addition suppose*

$$\left\{ \begin{array}{l} \exists 0 < R_1 < r \text{ with} \\ \frac{R_1 g_1(\theta^\alpha / (\alpha - 1)) R_1}{g_1(R_1) g_1(\theta^\alpha / (\alpha - 1)) R_1 + g_1(R_1) h_1(\theta^\alpha / (\alpha - 1)) R_1} < \mu \int_{\theta}^{1-\theta} G(\sigma, s) q(s) ds. \end{array} \right. \quad (4.19)$$

Then (4.1) has a nonnegative solution u_1 with $R_1 < \|u_1\| < r$ and $u_1(t) > 0$ for $t \in (0, 1)$.

Remark 4.3. If in (4.19) we have $R_1 > r$, then (4.1) a nonnegative solution u_2 with $r < \|u_2\| < R_2$ and $u_2(t) > 0$ for $t \in (0, 1)$.

It is easy to use Theorem 4.2 and Remark 4.3 to write theorems which guarantee the existence of more than two solutions to (4.1). We state one such result.

Theorem 4.4. *Suppose that (4.2)–(4.6) and (4.8) hold. Assume that $\exists m \in \{1, 2, \dots\}$ and constants $R_i, r_i (i = 1, \dots, m)$, with $r_1 > b_0$, and*

$$0 < R_1 < r_1 < R_2 < r_2 < \dots < R_m < r_m. \quad (4.20)$$

In addition suppose for each $i = 1, \dots, m$ that

$$\frac{r_i}{g(r_i) + h(r_i)} > K_0^2 a_0 g\left(\frac{1}{\alpha - 1}\right), \quad (4.21)$$

and

$$\frac{R_i g_1(\theta^\alpha / (\alpha - 1)) R_i}{g_1(R_i) g_1(\theta^\alpha / (\alpha - 1)) R_i + g_1(R_i) h_1(\theta^\alpha / (\alpha - 1)) R_i} < \mu \int_{\theta}^{1-\theta} G(\sigma, s) a(s) ds \quad (4.22)$$

hold. Then (4.1) has nonnegative solutions y_1, \dots, y_m with $y_i(t) > 0$ for $t \in (0, 1)$.

Example 4.5. Consider the boundary value problem

$$\begin{aligned} D_{0+}^\alpha u(t) &= \sigma \left(u^{-a}(t) + u^b(t) \right), \quad t \in (0, 1), \quad 3 < \alpha \leq 4, \\ u(0) = u(1) = u'(0) = u'(1) &= 0, \quad 0 < a < 1 < b, \end{aligned} \quad (4.23)$$

where $\sigma \in (0, \sigma_0)$ is such that

$$\sigma_0 \leq \frac{1}{2a_1}, \quad (4.24)$$

here

$$a_1 = \frac{(\alpha - 1)(\alpha - 2)}{\Gamma(\alpha)} \int_0^1 s^{\alpha-2}(1-s)^2 \left(s^{\alpha-2}(1-s)^2 \right)^{-\alpha} ds = \int_0^1 s^{(\alpha-2)(1-a)}(1-s)^{2(1-a)} ds < \infty. \quad (4.25)$$

Then (4.23) has two solutions u_1, u_2 with $u_1(t) > 0, u_2(t) > 0$ for $t \in (0, 1), i = 1, 2$.

To see this we will apply Theorem 4.1 with (here $0 < R_1 < 1 < R_2$ will be chosen below)

$$g(u) = g_1(u) = u^{-a}, \quad h(u) = h_1(u) = u^b, \quad a(t) = \sigma, \quad K_0 = 1, \theta = \frac{1}{4}. \quad (4.26)$$

Clearly (4.2)–(4.6) and (4.8) hold, and $a_0 = (\sigma a_1 / (\alpha - 1)^a)$. Now (4.7) holds with $r = 1$ since

$$\frac{r}{g(r) + h(r)} = \frac{1}{2} \geq a_1 \sigma_0 > K_0^2 a_0 (\alpha - 1)^a = K_0^2 a_0 g\left(\frac{1}{\alpha - 1}\right). \quad (4.27)$$

Finally notice (4.9) is satisfied for R_1 small and R_2 large since

$$\frac{R_i}{g_1(R_i) \{1 + (h_1)((\theta^\alpha / \alpha - 1)R_i) / (g_1)((\theta^\alpha / \alpha - 1)R_i)\}} = \frac{R_i^{1+a}}{1 + (\alpha - 1)^{-(a+b)} \theta^{\alpha(a+b)} R_i^{a+b}} \rightarrow 0, \quad (4.28)$$

as $R_1 \rightarrow 0, R_2 \rightarrow \infty$, since $b > 1$. Thus all the conditions of Theorem 4.1 are satisfied so existence is guaranteed.

5. Singular Semipositone Problems

In this section we present a new result for the singular semipositone problem:

$$\begin{aligned} \mathbf{D}_{0+}^\alpha u(t) &= \mu a(t) f(t, u(t)), \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \quad (5.1)$$

where $\mu > 0$ and nonlinearity f may be singular at $u = 0$.

Before we prove our main result, we first state a result.

Lemma 5.1. *Suppose $a \in L^1[0, 1]$ with $a > 0$ on $(0, 1)$. Then the boundary value problem,*

$$\begin{aligned} \mathbf{D}_{0+}^\alpha u(t) &= a(t)e(t), \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \quad (5.2)$$

has a solution w with

$$w(t) \leq \frac{t^{\alpha-2}(1-t)^2}{\alpha-1} C_0 \quad \text{for } t \in [0, 1], \quad (5.3)$$

here

$$C_0 = \frac{(\alpha-1)^2(\alpha-2)}{\Gamma(\alpha)} \int_0^1 a(s)e(s)ds. \quad (5.4)$$

In fact, from Lemma 3.1, (5.2) has solution

$$w(t) = \int_0^1 G(t,s)a(s)e(s)ds. \quad (5.5)$$

According to Lemma 3.2, we have

$$w(t) \leq \int_0^1 \frac{(\alpha-1)(\alpha-2)t^{\alpha-2}(1-t)^2}{\Gamma(\alpha)} a(s)e(s)ds = \frac{t^{\alpha-2}(1-t)^2}{\alpha-1} C_0. \quad (5.6)$$

The above Lemma together with Theorem 2.5 establish our main result.

Theorem 5.2. Suppose that the following conditions are satisfied.

$$q \in C(0,1) \cap L^1[0,1] \quad \text{with } q > \text{ on } (0,1). \quad (5.7)$$

$$\left\{ \begin{array}{l} f : [0,1] \times (0,\infty) \rightarrow \mathbf{R} \text{ is continuous and there exists} \\ a \text{ function } e \in C((0,1), (0,+\infty)) \text{ with } f(t,u) + e(t) \geq \quad \text{for } (t,u) \in (0,1) \times (0,\infty), \end{array} \right. \quad (5.8)$$

$$\left\{ \begin{array}{l} f^*(t,u) = f(t,u) + e(t) \leq g(u) + h(u) \text{ on } [0,1] \times (0,\infty) \text{ with } g > 0 \\ \text{continuous and nonincreasing on } (0,\infty), \quad h \geq 0 \\ \text{continuous on } [0,\infty) \text{ and } \frac{h}{g} \text{ nondecreasing on } (0,\infty), \end{array} \right. \quad (5.9)$$

$$\exists K_0 \text{ with } g(xy) \leq K_0 g(x)g(y) \quad \forall x > 0, y > 0, \quad (5.10)$$

$$a_0 = \int_0^1 G(\sigma,s)q(s)g(s^{\alpha-2}(1-s)^2)ds < \infty, \quad (5.11)$$

$$\exists r > \mu C_0 \text{ with } \frac{r}{g((r - \mu C_0)/(\alpha-1))\{1 + (h(r)/g(r))\}} \geq \mu K_0 a_0, \quad (5.12)$$

$$\left\{ \begin{array}{l} \text{There exists } 0 < \theta < \frac{1}{2} \text{ (choose and fix it) and a continuous,} \\ \text{nonincreasing function } g_1 : (0, \infty) \rightarrow (0, \infty), \text{ and a continuous} \\ \text{function } h_1 : [0, \infty) \rightarrow (0, \infty) \text{ with } \frac{h_1}{g_1} \text{ nondecreasing on } (0, \infty) \\ \text{and with } f(t, u) + e(t) \geq g_1(u) + h_1(u) \text{ for } (t, u) \in [\theta, 1 - \theta] \times (0, \infty), \end{array} \right. \quad (5.13)$$

and $\exists R > r$ with

$$\frac{Rg_1((\epsilon\theta^\alpha/(\alpha-1))R)}{g_1(R)g_1((\epsilon\theta^\alpha/(\alpha-1))R) + g_1(R)h_1((\epsilon\theta^\alpha/(\alpha-1))R)} \leq \mu \int_\theta^{1-\theta} G(\sigma, s)q(s)ds, \quad (5.14)$$

here $\epsilon > 0$ is any constant (choose and fix it) so that $1 - (\mu C_0/R) \geq \epsilon$ (note ϵ exists since $R > r > \mu C_0$ in fact we can have $\epsilon = 1 - (\mu C_0/r)$) and $G(t, s)$ is Green's function and

$$\int_\theta^{1-\theta} G(\sigma, s)a(s)ds = \sup_{t \in [0,1]} \int_\theta^{1-\theta} G(t, s)a(s)ds. \quad (5.15)$$

Then (5.1) has a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $y(t) > 0$ for $t \in (0, 1)$.

Proof. To show that (5.1) has a nonnegative solution we will look at the boundary value problem

$$\begin{aligned} D_{0+}^\alpha y(t) &= \mu q(t) f^*(t, y(t) - \phi(t)), \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \\ y(0) &= y(1) = y'(0) = y'(1) = 0, \end{aligned} \quad (5.16)$$

where

$$\phi(t) = \mu M w(t), \quad 0 \leq t \leq 1, \quad (5.17)$$

(w is as in Lemma 5.1).

We will show, using Theorem 2.5, that there exists a solution y_1 to (5.16) with $y_1(t) > \phi(t)$ for $t \in (0, 1)$. If this is true then $u(t) = y_1(t) - \phi(t), 0 \leq t \leq 1$ is a nonnegative solution (positive on $(0, 1)$) of (5.1), since

$$\begin{aligned} D_{0+}^\alpha u(t) &= D_{0+}^\alpha y_1(t) - D_{0+}^\alpha \phi(t) = -\mu q(t) f^*(t, y_1(t) - \phi(t)) + \mu q(t) e(t) \\ &= -\mu q(t) [f(t, y_1(t) - \phi(t)) + e(t)] + \mu q(t) e(t) \\ &= -\mu q(t) f(t, y_1(t) - \phi(t)) \\ &= -\mu q(t) f(t, u(t)), \quad 0 < t < 1. \end{aligned} \quad (5.18)$$

As a result, we will concentrate our study on (5.16). Let E, K as in Section 2, and let

$$\Omega_1 = \{u \in E : \|u\| < r\}, \quad \Omega_2 = \{u \in E : \|u\| < R\}. \quad (5.19)$$

Next let $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow E$ be defined by

$$(Ay)(t) = \mu \int_0^1 G(t,s)q(s)f^*(s,y(s) - \phi(s))ds, \quad 0 \leq t \leq 1. \quad (5.20)$$

In addition, standard argument shows that $A(P) \subset P$ and A is completely continuous.

We now show

$$\|Ay\| \leq \|y\| \quad \text{for } K \cap \partial\Omega_1. \quad (5.21)$$

To see this, let $y \in K \cap \partial\Omega_1$. Then $\|y\| = \|y\|_{[0,1]} = r$ and $y(t) \geq (t^{\alpha-2}(1-t)^2/(\alpha-1))r$ for $t \in [0,1]$. Now for $t \in (0,1)$, the Lemma 5.1 implies

$$y(t) - \phi(t) \geq \frac{t^{\alpha-2}(1-t)^2}{\alpha-1}r - \mu \frac{t^{\alpha-2}(1-t)^2}{\alpha-1}C_0 \geq \frac{t^{\alpha-2}(1-t)^2}{\alpha-1}(r - \mu C_0) > 0, \quad (5.22)$$

so for $t \in [0,1]$ we have

$$\begin{aligned} (Ay)(t) &= \mu \int_0^1 G(t,s)a(s)f^*(s,y(s) - \phi(s))ds \\ &\leq \mu \int_0^1 q(s)a(s)[g(y(s) - \phi(s)) + h(y(s) - \phi(s))]ds \\ &= \mu \int_0^1 q(s)a(s)g(y(s) - \phi(s)) \left\{ 1 + \frac{h(y(s) - \phi(s))}{g(y(s) - \phi(s))} \right\} ds \\ &\leq \mu \int_0^1 q(s)a(s)g\left(\frac{s^{\alpha-2}(1-s)^2}{\alpha-1}(r - \mu C_0)\right) \left\{ 1 + \frac{h(r)}{g(r)} \right\} ds \\ &\leq \mu K_0 g\left(\frac{r - \mu C_0}{\alpha-1}\right) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_0^1 q(s)a(s)g(s^{\alpha-2}(1-s)^2)ds \\ &= \mu K_0 a_0 g\left(\frac{r - \mu C_0}{\alpha-1}\right) \left\{ 1 + \frac{h\{r\}}{g\{r\}} \right\}. \end{aligned} \quad (5.23)$$

This together with (5.12) yields

$$\|Ay\| = \|Ay\|_{[0,1]} \leq r = \|y\|, \quad (5.24)$$

so (5.21) is satisfied.

Next we show

$$\|Ay\| \geq \|y\| \quad \text{for } K \cap \partial\Omega_2. \quad (5.25)$$

To see this let $y \in K \cap \partial\Omega_2$ so $\|y\| = \|y\|_{[0,1]} = R$ and $y(t) \geq (t^{\alpha-2}(1-t)^2/(\alpha-1))R$ for $t \in [0, 1]$. Also for $t \in [0, 1]$ we have

$$\begin{aligned} y(t) - \phi(t) = y(t) - \mu\tau w(t) &\geq \frac{t^{\alpha-2}(1-t)^2}{\alpha-1}R - \mu C_0 \frac{t^{\alpha-2}(1-t)^2}{\alpha-1} \\ &\geq \frac{t^{\alpha-2}(1-t)^2}{\alpha-1}R \left(1 - \frac{\mu C_0}{R}\right) \geq \frac{\epsilon t^{\alpha-2}(1-t)^2}{\alpha-1}R. \end{aligned} \tag{5.26}$$

As a result

$$y(t) - \phi(t) \geq \frac{\epsilon\theta^\alpha}{\alpha-1}R \quad \text{for } t \in [\theta, 1 - \theta]. \tag{5.27}$$

We have

$$\begin{aligned} (Ay)(\sigma) &= \mu \int_0^1 G(\sigma, s)q(s)f^*(s, y(s) - \phi(s))ds \\ &\geq \mu \int_0^{1-\theta} G(\sigma, s)q(s)[g_1(y(s) - \phi(s)) + h_1(y(s) - \phi(s))]ds \\ &= \mu \int_0^{1-\theta} G(\sigma, s)q(s)g_1(y(s) - \phi(s)) \left\{1 + \frac{h_1(y(s) - \phi(s))}{g_1(y(s) - \phi(s))}\right\} ds \\ &\geq \mu g_1(R) \int_0^{1-\theta} G(\sigma, s)q(s) \left\{1 + \frac{h_1((\epsilon\theta^\alpha/(\alpha-1))R)}{g_1((\epsilon\theta^\alpha/(\alpha-1))R)}\right\} ds. \end{aligned} \tag{5.28}$$

This together with (5.14) yields

$$(Ay)(\sigma) \geq R = \|y\|. \tag{5.29}$$

Thus $\|Ay\| \geq \|y\|$, so (5.25) is held.

Now Theorem 2.5 implies that A has a fixed-point $y \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, that is, $r \leq \|y\| = \|y\|_{[0,1]} \leq R$ and $y(t) \geq t^{\alpha-2}(1-t)^2r$ for $t \in [0, 1]$. Thus $y(t)$ is a solution of (5.16) with $y(t) > \phi(t)$ for $t \in (0, 1)$. Thus (5.1) has a positive solution $u(t) = y(t) - \phi(t) > 0$ for $t \in (0, 1)$. \square

Example 5.3. Consider the boundary value problem

$$\begin{aligned} \mathbf{D}_{0+}^\alpha y(t) &= \mu(y^{-a}(t) + y^b(t) - 1), \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \\ y(0) = y(1) = y'(0) = y'(1) &= 0, \quad 0 < a < 1 < b, \end{aligned} \tag{5.30}$$

where $\mu \in (0, \mu_0)$ is such that

$$\frac{(\alpha-1)^2(\alpha-2)\mu_0}{2\Gamma(\alpha)} + (\alpha-1)(2\mu_0 a_0)^{1/\alpha} \leq 1, \tag{5.31}$$

here

$$a_0 = \int_0^1 q(s)a(s)g\left(s^{\alpha-2}(1-s)^2\right)ds = \int_0^1 s^{1-a(\alpha-1)}(1-s)^{(\alpha-1)}ds < \infty. \quad (5.32)$$

Then (5.30) has a solution y with $y(t) > 0$ for $t \in (0, 1)$.

To see this we will apply Theorem 5.2 with (here $R > 1$ will be chosen later, in fact here we choose $R > 1$ so that $\varepsilon = 1/2$ works, i.e., we choose R so that $1 - (\mu/(2\Gamma(\alpha)R)) \geq 1/2$),

$$\begin{aligned} g(y) = g_1(y) = y^{-a}, \quad h(y) = h_1(y) = y^b, \quad a(t) = 1, e(t) = t^{-1/2}, \\ \mu K_0 = 1, \varepsilon = \frac{1}{2}, \quad \theta = \frac{1}{4}, \quad C_0 = \frac{(\alpha-1)^2(\alpha-2)}{\Gamma(\alpha)} \int_0^1 s^{-1/2}ds = \frac{(\alpha-2)^2(\alpha-2)}{2\Gamma(\alpha)}. \end{aligned} \quad (5.33)$$

Clearly (5.7)–(5.11) and (5.13) hold. Now (5.12) holds with $r = 1$ since

$$\begin{aligned} \mu K_0 a_0 = \mu a_0 < \mu_0 a_0 \leq \frac{1}{2} \left(\frac{1-\mu_0 C_0}{\alpha-1} \right)^a \leq \frac{1}{2} \left(\frac{1-\mu c_0}{\alpha-1} \right)^a \\ = \frac{r}{\{1+(h(r)/g(r))\}g((r-\mu M C_0)/(\alpha-1))}, \end{aligned} \quad (5.34)$$

from (5.31). Finally notice (5.14) is satisfied for R large since

$$\frac{R g_1(\varepsilon \theta^\alpha / (\alpha - 1)) R}{g_1(R) g_1((\varepsilon \theta^\alpha / (\alpha - 1)) R) + g_1(R) h_1((\varepsilon \theta^\alpha / (\alpha - 1)) R)} = \frac{R^{1+a}}{1 + (\alpha - 1)^{-(a+b)} [\varepsilon \theta^\alpha]^{a+b} R^{a+b}} \rightarrow 0, \quad (5.35)$$

as $R \rightarrow \infty$, since $b > 1$. Thus all the conditions of Theorem 5.2 are satisfied so existence is guaranteed.

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