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Research Article

Intermittent Behavior and Synchronization of Two Coupled Noisy Driven Oscillators

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The coupled system of two forced Liénard-type oscillators has applications in diode-based electric circuits and phenomenological models for the heartbeat. These systems typically exhibit intermittent transitions between laminar and chaotic states; what affects their performance and, since noise is always present in such systems, dynamical models should include these effects. Accordingly, we investigated numerically the effect of noise in two intermittent phenomena: the intermittent transition to synchronized behavior for identical and unidirectionally coupled oscillators, and the intermittent transition to chaos near a periodic window of bidirectionally coupled oscillators. We found that the transition from a nonsynchronized to a synchronized state exhibits a power-law scaling with exponent 3/2 characterizing on-off intermittency. The inclusion of noise adds an exponential tail to this scaling.

1. Introduction

Intermittency is a ubiquitous phenomenon in nonlinear dynamics. It consists of the intermittent switching between a laminar phase of regular behavior and irregular bursts. In one-dimensional quadratic-type maps it was first associated with a saddle-node bifurcation by Pomeau and Manneville, who also described its scaling characteristics [1]. A comprehensive theory of intermittency for such systems is now available [2]. Another context in which intermittency appears is related to the synchronization of coupled nonlinear oscillators.

Synchronization of nonlinear oscillators is a subject with a venerable history dating back from the early observation by Huygens that two pendula suspended from the same frame—which provides the mechanical coupling—can synchronize their librations so as to become antiphase [3]. Most recently, the probability of synchronizing chaotic oscillations has opened a wide horizon of applications ranging from electronic circuits [4] and lasers [5] to biological rhythms like heartbeat [6].

A paradigmatic example of a nonlinear oscillator is the van der Pol equation [7] $\ddot{x} + \mu(x^2 - 1)\dot{x} + \omega_0^2 x = \rho \sin(\omega t)$, which was originally introduced to model the behavior of a AC-driven circuit with a nonlinear resistance, such as that yielded by a triode vacuum tube [8] and by Zener diodes [9] but which has also been used in other contexts, like in the phenomenological description of the heartbeat [10]. Actually the van der Pol equation belongs to a more general class of Liénard-type oscillators, whose dynamics in the presence of external forcing has been investigated, showing a rich behavior including crises, intermittency, and chaos [11].

The question of how two or more Liénard-type oscillators can synchronize their motions arises in the study of coupled vacuum-tube circuits [12] but is also relevant to the understanding of the mechanisms coupling the heartbeat with the nerve conduction [11]. The synchronization of two coupled Liénard-type oscillators, one of them being forced, has been considered in an earlier work of the authors, where we show the existence of different types of synchronization, according to the forcing and coupling parameters used [13].

In this work we focus on the influence of parametric noise in intermittency phenomena numerically observed in two Liénard-type forced oscillators. The first case is related to the intermittent transition to synchronized behavior in such systems in the presence of noise. Bearing in mind the usefulness of Liénard-type oscillators to model vacuum-tube circuits, we can regard the presence of parametric noise as unavoidable, since virtually each circuit component has a fluctuating magnitude (like resistances, capacitances, or inductances) within a given noise level [14]. The second case to be treated here is the influence of noise in the intermittent behavior related to a periodic window existing for a parameter range where chaotic behavior is dominant.

The rest of the paper is organized as follows. Section 2 introduces the theoretical model, as well as the basic concepts to be used in the discussion of synchronization. Section 3 considers the case of intermittent transition to synchronization in two identical oscillators with unidirectional coupling and the presence of noise. Section 4 is devoted to the case of intermittent transition to chaos in the vicinity of a periodic window, for two nonidentical oscillators with a bidirectional coupling. Our conclusions are left to the last section.

2. Coupled Oscillators and Their Synchronization

The triode circuit is a standard textbook example of the Van der Pol equation [12, 15]. However, a more realistic description of such a circuit does not lead us to the Van der Pol but instead to a more general class of Liénard-type equations:

$$\ddot{x} + \mu\left(x^2 + \sigma x - 1\right)\dot{x} + \omega_0^2 x = \rho \sin(\omega t), \tag{2.1}$$

where ω_0 is the frequency of the unforced oscillations; ρ, ω, are, respectively, the amplitude and frequency of an external AC-voltage, and μ, σ are coefficients whose values are drawn from the triode characteristic curve, supposed a third-order polynomial in the grid voltage, whose normalized form is represented by the variable x [15]. The particular case $\sigma = 0$ yields the usual AC-driven Van der Pol equation.

Moreover, we can rewrite (2.1) in the form

$$\ddot{x} + \mu(x - w_1)(x - w_2)\dot{x} + \omega_0^2 x = \rho \sin(\omega t), \tag{2.2}$$

where

$$w_{1,2} \equiv -\frac{\sigma}{2} \pm \left[\left(\frac{\sigma}{2}\right)^2 + 1\right]^{1/2} \tag{2.3}$$

are the roots of the quadratic resistance, usually of opposite signs.

We consider two such circuits, of which only one is driven by an AC-voltage, and they are supposed to be almost identical, except for their natural frequencies $b_1 = \omega_{01}^2$ and $b_2 = \omega_{02}^2$, which will be considered as variable parameters of the coupled system, whose equations are

$$\dot{x}_1 = x_2, \tag{2.4}$$

$$\dot{x}_2 = -\mu(x_1 - w_1)(x_1 - w_2)x_2 - b_1 x_1 + \rho \sin(\omega t) + c_1(x_3 - x_1), \tag{2.5}$$

$$\dot{x}_3 = x_4, \tag{2.6}$$

$$\dot{x}_4 = -\mu(x_3 - w_1)(x_3 - w_2)x_4 - b_2 x_3 + c_2(x_1 - x_3), \tag{2.7}$$

where the pairs $x_{1,2}$ and $x_{3,4}$ stand for each coupled circuit. We have assumed an asymmetric diffusive coupling whose strengths, c_1 and c_2, take on different values. If $c_1 = 0$, we have a unidimensional coupling, or master-slave configuration, whereas for $c_1 > 0$ (but small) and $c_2 > 0$, the coupling is bidirectional, although it is strongly asymmetric.

The rationale for using such coupling schemes lies in the modeling of the interaction between the heart pacemakers, the sino-atrial (SA), and atrio-ventricular (AV) nodes [13]. The SA node is the primary pacemaker of the heart, and the electrical impulse it generates spreads out through the myocardium, reaching the AV node. Hence, the coupling should be taken either unidirectional or bidirectional but strongly asymmetric. In both cases we use to call the (x_1, x_2) oscillator the driving one, whereas (x_3, x_4) the response oscillator, corresponding to the SA and AV nodes, respectively.

We will use throughout this work the following values for the system parameters: $\mu = -1.45$, $w_1 = -0.2$, $w_2 = 1.9$, $\rho = 0.95$, and $\omega = 1.0$, letting the normal mode frequencies $(b_{1,2})$ and the coupling constants $(c_{1,2})$ to be the parameters to be varied. We have integrated numerically the coupled system of first-order differential equations (2.4)–(2.7) by using a predictor-corrector routine based on the Adams method [16].

For all the cases studied in this paper the system asymptotic behavior will consist of a periodic or chaotic orbit which encircles the points $(x_1, x_2) = (0, 0)$ and $(x_3, x_4) = (0, 0)$ for the driving and response oscillator, respectively. In such cases we can define geometrical phases as

$$\phi_1(t) = \arctan\left[\frac{x_2(t)}{x_1(t)}\right], \qquad \phi_2(t) = \arctan\left[\frac{x_4(t)}{x_3(t)}\right]. \tag{2.8}$$

In cases, however, for which we do not have an orbit encircling a point, as in funnel attractor, for example, the phase should be defined using other methods, like Poincaré sections or Hilbert transforms [17].

The winding number, defined as

$$\Omega_i = \langle \dot{\phi}_i(t) \rangle_T, \quad (i = 1, 2), \tag{2.9}$$

is the average time rate of the phases of both oscillators and stands for their endogenous frequencies. A weak type of synchronization between the driving and response oscillator consists in the equality of their winding numbers, $\Omega_1(t) = \Omega_2(t)$, regardless of the actual value each phase takes on, and called frequency synchronization (FS) [18]. When the phases themselves are equal, we speak of phase synchronization (PS): $\phi_1(t) = \phi_2(t)$ [19].

By way of contrast, the strongest type of synchronization is complete synchronization (CS), for which the positions and velocities themselves (and not only the phases) are equal: $x_1(t) = x_3(t)$, $x_2(t) = x_4(t)$ [4]. However, CS occurs only if the coupled oscillators are identical [17]. If the coupled oscillators have slightly distinct parameters and the coupling is not too strong, it is possible to find a weaker effect called lag synchronization (LS), defined as the approximate equality of the state variables, delayed by a given time lag τ: $x_1(t) \approx x_3(t-\tau)$, $x_2(t) \approx x_4(t-\tau)$ [19].

Finally, if the oscillator parameters are widely different, as in the case we are investigating here, there is no longer LS because the oscillator positions and velocities differ by a large amount [13]. Even in this case, however, it is still possible to find generalized synchronization (GS), which is observed when there exists a functional relationship between the amplitudes of the two coupled oscillators: $x_2(t) = F(x_1(t))$ and can occur for nonidentical systems [20]. An even weaker effect is generalized lag synchronization (GLS), when the functional relationship between the variables holds up to a time delay τ: $x_2(t) = F(x_1(t-\tau))$.

We will introduce extrinsic noise on the driving oscillator, by adding to (2.5) a term σR_n, where σ is the noise level, and R_n is a pseudorandom variable, with values uniformly distributed in the interval $[-1/2, +1/2]$ and applied at each period of external force. The noisy term is applied at each integration step, so that it plays the role of a stochastic perturbation of a deterministic system rather than a stochastic differential equation, which would need specific integration techniques to be numerically solved [21].

We have considered also this type of noisy term applied on the response oscillator, when both systems are identical, but the results do not differ appreciably from the case we consider here. On the other hand, the inclusion parametric noise (i.e., noise terms applied to system parameters like normal mode frequencies or coupling constants) leads to qualitatively different phenomena, like GLS states, and which we have considered in a recent paper [22].

3. Intermittent Synchronization in Unidirectional Coupling

We initiate our analysis by the case of unidirectional coupling ($c_1 = 0$), for which the response oscillator ($x_3 - x_4$) is slaved under the driving of the master oscillator ($x_1 - x_2$). Moreover, we will consider both oscillators as being identical, in the sense that $b_1 = b_2 = 1.0$, that is, their normal mode frequencies take on exactly the same value. As a consequence, a CS state $x_1 = x_3$, $x_2 = x_4$ is possible and defines a synchronization manifold S in the phase space. If S is transversely asymptotically stable, there follows that small displacements along directions transversal to S will shrink down to zero as time tends to infinity. The conditional

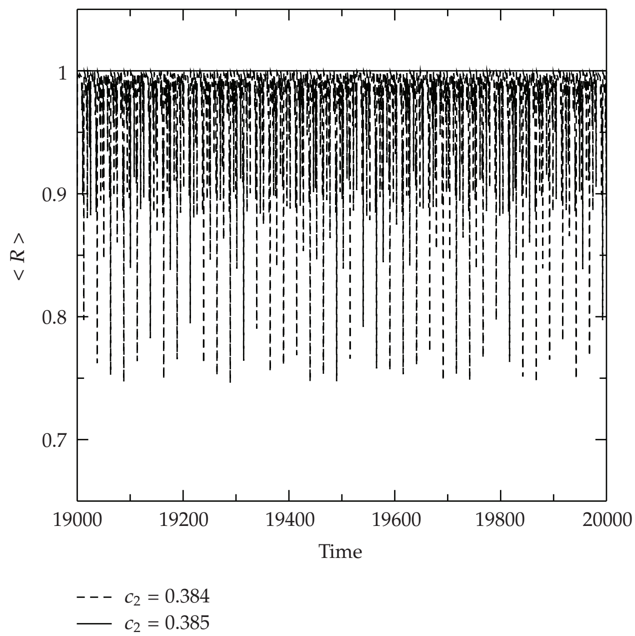


Figure 1: Order parameter for two values of c_2 at the neighborhood of the transition to a synchronized state. The time in the horizontal axis is measured in arbitrary units.

Lyapunov exponents λ_{cond} for the response oscillator are computed under the constraint that the trajectory lies entirely in S for all times. The negativity of all conditional exponents is a necessary condition for the CS state to be transversely asymptotically stable [17].

A useful numerical diagnostic of CS is the order parameter

$$R(t) \equiv \frac{1}{2}\left| e^{2\pi i x_2(t)} + e^{2\pi i x_4(t)} \right|, \tag{3.1}$$

which measures the degree of coherence between oscillators. For completely incoherent evolution R approaches zero, whereas for CS states R goes to the unity. Oscillations of $R(t)$ around values less than unity indicate partial coherence and no synchronization at all. Figure 1 presents the time evolution of the order parameter for two values of c_2, before and after a transition to CS occurring at $c_2 = c_2^{\text{CR}} \approx 0.3845$. For $c_2 \lesssim c_2^{\text{CR}}$ the order parameter oscillates with an average below unity, whereas for $c_2 \gtrsim c_2^{\text{CR}}$ it settles down at unity without noticeable fluctuations.

The approach to the CS state is characterized by the appearance of weaker forms of synchronization, like PS, and their breakdown. This is a particularly interesting point to investigate the role of noise on synchronization properties of our model. The temporal evolution of the phase difference $\phi_1(t) - \phi_2(t)$ between the coupled oscillators is plotted in Figure 2, starting from an arbitrarily chosen time. For $c_2 \gtrsim c_2^{\text{CR}}$ (thick dashed line) the phase difference is zero, which is an obvious consequence of the CS state. On the other hand, for $c_2 \lesssim c_2^{\text{CR}}$ (thin dashed line), the phases increase monotonically with time as well as their

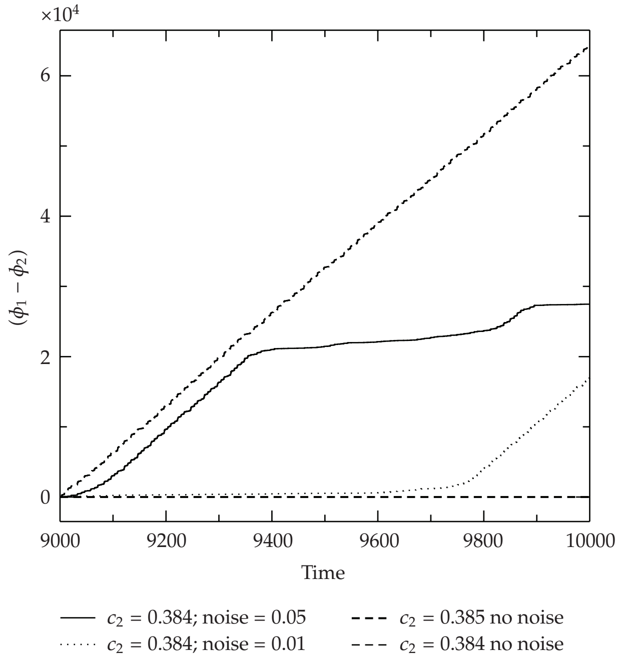


Figure 2: Time evolution of the phase difference between the coupled oscillator with and without noise, for two values of c_2 at the neighborhood of the transition to a synchronized state.

difference, due to absence of synchronization. This phase diffusion can be described by a Gaussian dependence with time

$$\langle \phi_1(t) - \phi_2(t) \rangle \approx 2Dt, \tag{3.2}$$

where D is the diffusion coefficient, and $\langle \cdots \rangle$ stands for an average over many randomly chosen initial condition. From Figure 2 we estimate that $D \sim 30$.

If we add a noise level $\sigma = 0.05$ to the precritical case (solid thin line), the phases initially diffuse but suddenly the phase difference stays for some time at a nearly constant value, indicating a transient PS state which breaks down and yields phase diffusion again, and so on. These phase slips, as they are usually called, are characteristic of the onset of desynchronization [23]. The average duration of laminar PS states increases as the noise level is diminished (see the dashed line for $\sigma = 0.01$ in Figure 2).

In the absence of noise, we observed that the oscillators present FS, characterized by $\Omega_1 - \Omega_2 = 0$, irrespective of exhibiting a CS state (Figure 3(a)). In such case, even though the dynamics for each oscillator is chaotic, their phases are correlated enough to warrant the equality of their time rates. Hence FS can be considered more robust than CS or even PS in our system. This is even more evident when noise is added to the system (Figure 3(b)). The FS state survives with small spikes for the interval of c_2-values studied, except for a narrow interval centered at 0.5 for which there is a (still low) peak of winding number mismatch of $\sim 10^{-3}$.

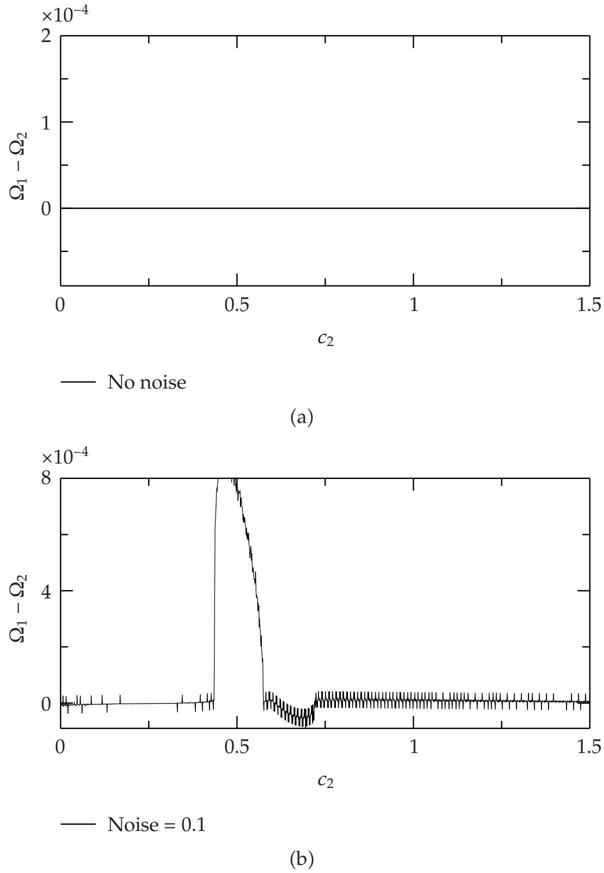


Figure 3: Winding number difference between the coupled oscillator with and without noise, as a function of the coupling strength.

Another dynamical feature observed in the neighborhood of the synchronization transition at $c_2 = c_2^{CR} \approx 0.3845$ is the presence of intermittent bursts of nonsynchronized behavior intercalated with laminar intervals of synchronized behavior. These features are characteristic of the so-called *on-off intermittency*, where there is an invariant manifold on which the system would lie during laminar intervals of duration τ_i, the burst representing excursions off this manifold [24]. On-off intermittency has been observed in coupled systems, like coupled map lattices, whose invariant manifold is the synchronization manifold [25, 26].

On-off intermittency has a numerical signature, which is the scaling obeyed by the statistical distribution $P(\tau)$ of the durations of laminar (or interburst) intervals of synchronization: $P(\tau) \sim \tau^{-3/2}$. The presence of noise in on-off intermittency scenarios introduces an additional tail of exponential dependence [27]. We actually observed this universal scaling for values slightly after and before the transition at $c_2 = c_2^{CR}$, with nonzero noise levels. Our results are shown in Figures 4(a) and 4(b) for $c_2 \lesssim c_2^{CR}$ and $c_2 \gtrsim c_2^{CR}$, respectively, with a 1% noise level, where the diamonds stand for numerical data, with two different scaling regions: one power-law scaling for small interburst intervals, with exponent $-3/2$ within the numerical accuracy, and an exponential tail for large interburst intervals.

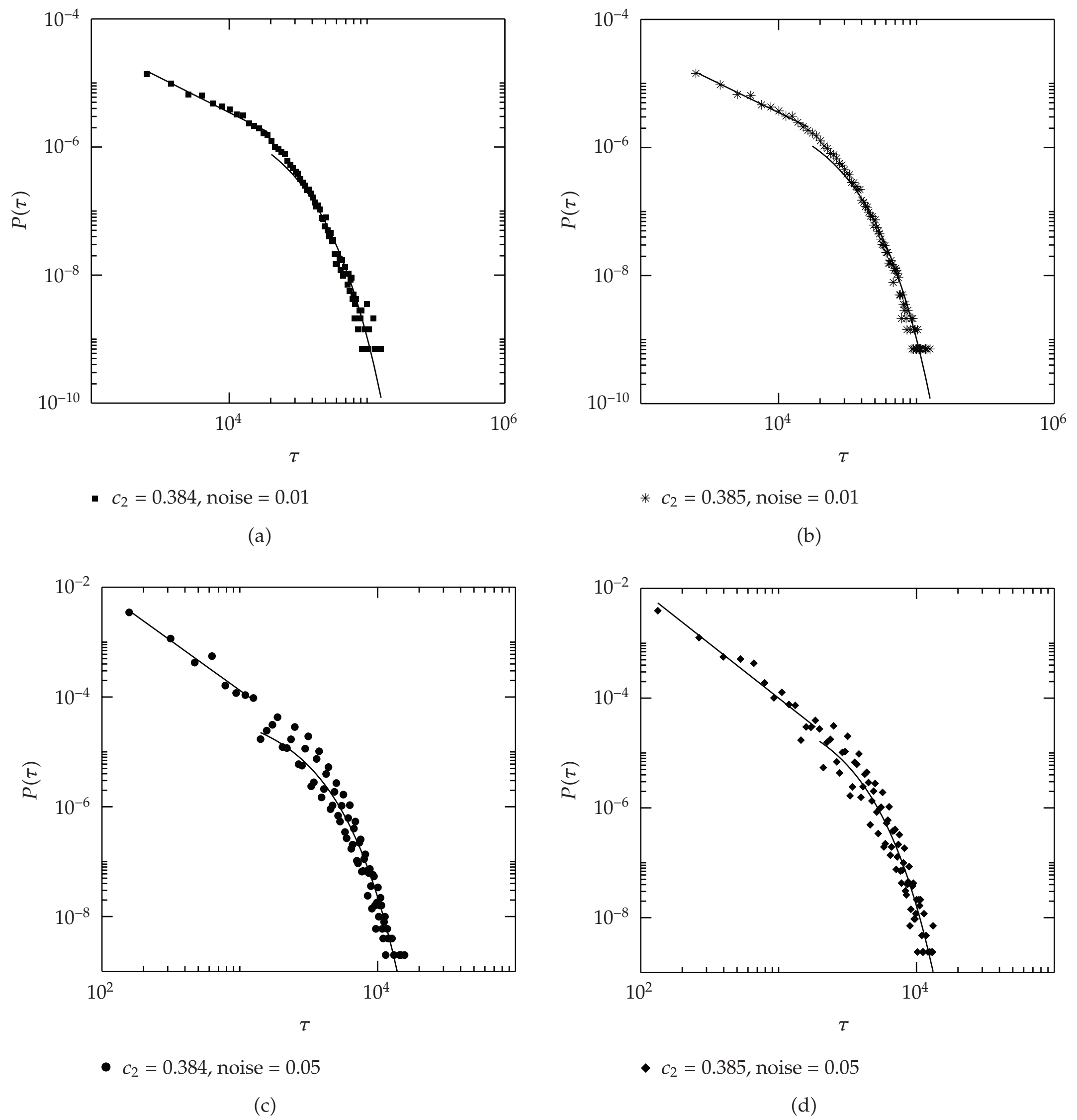


Figure 4: Probability distribution for the duration of laminar synchronized intervals between consecutive bursts of nonsynchronized behavior, for (a) $c_2 = 0.384 \gtrsim c_2^{\mathrm{CR}} c_2 = 0.384 \gtrsim c_2^{\mathrm{CR}}$, and (b) $c_2 = 0.385 \gtrsim c_2^{\mathrm{CR}} c_2 = 0.385 \gtrsim c_2^{\mathrm{CR}}$ with a 1%1% noise level. (c) and (d) correspond to (a) and (b), respectively, with a 5%5% noise level. The lines shown are least-squares fits evidencing two different scalings for numerical data, and the symbols stand for the numerical results.

The two scalings are roughly separated by a shoulder which, according to the general theory of noisy on-off intermittency, defines a crossover time whose value depends on the noise level [27]. We verified this point by considering, in Figures 4(c) and 4(d), the cases before and after the synchronization transition, respectively, but with a higher (5%) noise level. We observe that the crossover time, which is *circa* 10^4 for weak noise, decreases to 10^3 for stronger noise, diminishing the noiseless power-law scaling, as expected.

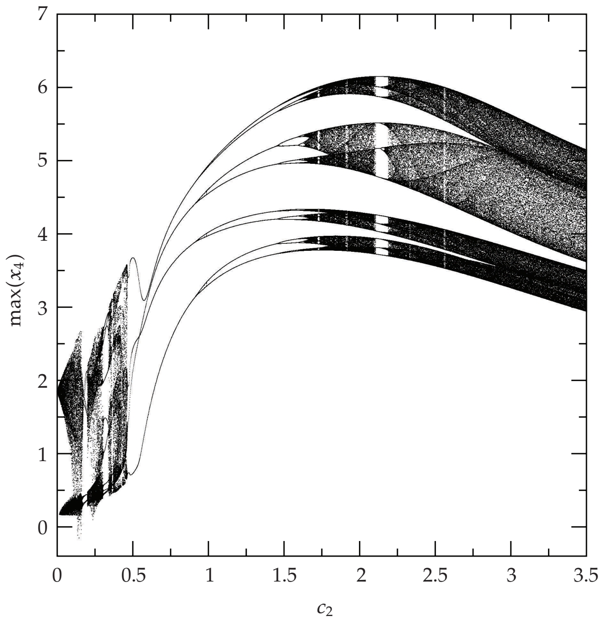


Figure 5: Bifurcation diagram for $x_n = \max(x_4)$, the local maxima of $x_3(t)$ versus c_2.

4. Intermittent Chaotic Bursts in Bidirectional Coupling

Now we present results considering the case where $c_1 \neq 0$ and $c_2 \neq 0$, what characterizes a bidirectional form of coupling. We choose low values of $c_1 \gtrsim 0.01$, because of the strong asymmetrical character of the bidirectional coupling and kept c_2 in the $[2.0, 2.2]$ interval. In this section, also, we relax the hypothesis we previously made of identical oscillators and allow for a mismatch of the normal mode frequencies: $b_1 = 1.0$, and $b_2 = 0.66$, the other parameter values being the same as those used in the previous section.

The dependence of $x_n = \max(x_4)$, which are the local maxima of $x_4(t)$, is plotted in Figure 5 as a function of c_2. We can distinguish two different parts in this diagram: (i) a quasiperiodic region for $0 < c_2 < c_{II} = 0.45507$; (ii) a period-doubling cascade for $c_2 > c_{II}$. In the latter, we focus on the narrow period-12 window starting at $c_2 = c_I = 2.101503$. We will consider the effects of noise in the vicinity of the critical points c_I and c_{II}, since they are very sensitive with respect to variations in the coupling strength and are related with sudden changes in the dynamical behavior of the system.

In the neighborhood of the period-12 window, that is, at $c_2 = c_I$, the four-band chaotic attractor suddenly disappears, and a saddle-node bifurcation occurs forming a stable and an unstable orbit of period-12. This can be explicitly verified in Figures 6(a) and 6(b), where we plot the 12th return map for the variable $x_n = \max(x_4)$, respectively, just before and just after the saddle-node bifurcation. For $c_2 = 2.101504 \gtrsim c_I$ (Figure 6(a)) the periodic points are the intersections of the return map with the 45° line, corresponding to the regular time series in Figure 6(c). When $c_2 = 2.101470 \lesssim c_I$ (Figure 6(b)) there are no intersections with the 45° line, but since we are too close to the bifurcation point, there exists a narrow bottleneck between the curves. When the trajectory enters such bottlenecks, the resulting behavior is the laminar interval, followed by chaotic bursts when the trajectory eventually exits the nozzle and is randomly reinjected to its vicinity (Figure 6(d)).

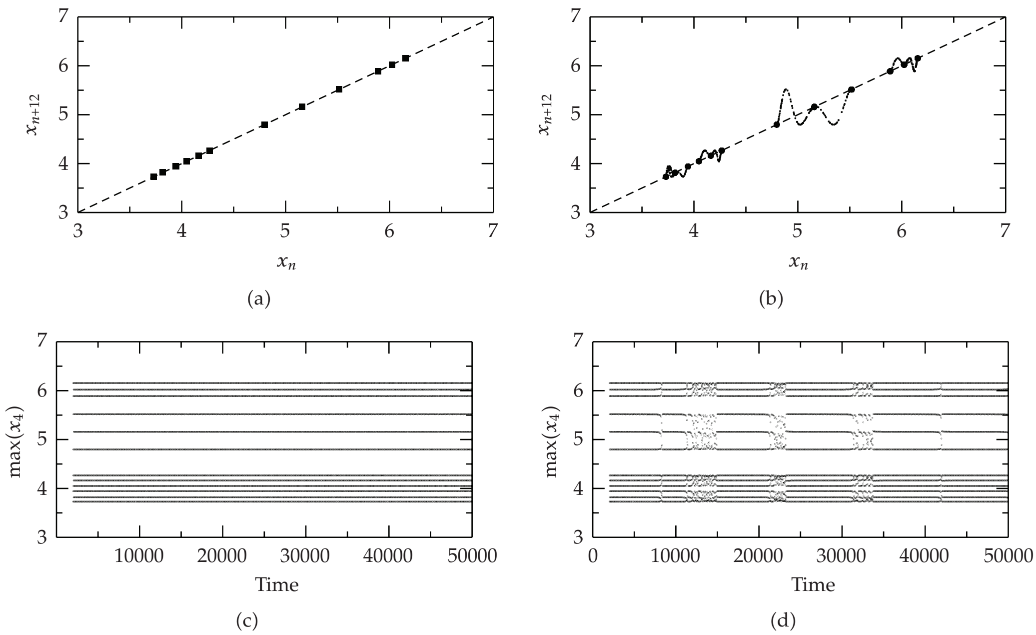
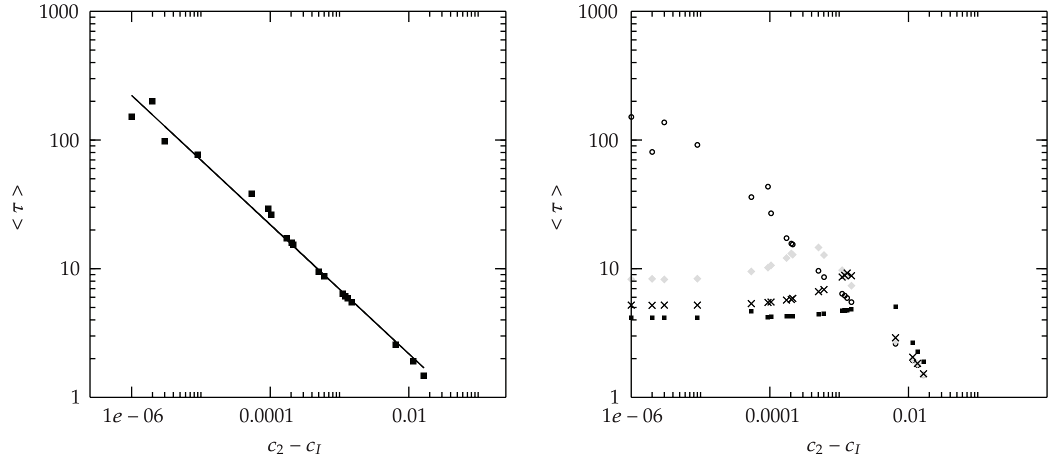


Figure 6: Return maps for $x_{n+12}x_{n+12}$ versus $x_n x_n$ for (a) $c_2 = 2.101504 \gtrsim c_I c_2 = 2.101504 \gtrsim c_I$ and (b) $c_2 = 2.101470 \lesssim c_I c_2 = 2.101470 \lesssim c_I$. (c) and (d) are the corresponding time series for the cases (a) and (b), respectively.

Let $\langle \tau_{\text{int}} \rangle$ be the average duration of the laminar intervals between consecutive bursts. If we approach the critical value c_I from below, this average duration increases according to a power-law: $\langle \tau_{\text{int}} \rangle \sim (c_2 - c_I)^{-\varpi}$ (Figure 7(a)), where the scaling exponent $\varpi = 1/2$ within the numerical precision. This agrees with the Pomeau-Manneville type-I intermittency scenario and shows that the essential dynamics leading to intermittent behavior near c_I is one-dimensional [1].

We can now investigate the role of a noise level on this average duration of laminar intervals. Figure 7(b) plots $\langle \tau \rangle$ versus $c_2 - c_I$ for various noise levels. If we are far enough from the critical value c_I, the scaling is essentially the same as in the noiseless case. If the noise level is too small—for example, 2×10^{-10}, represented by circles in Figure 7(b)—the scaling is practically unchanged even very close to the critical point. However, for higher noise levels, the scaling holds only to a certain minimum distance, between 10^{-3} and 10^{-4}, below which the value of $\langle \tau_{\text{int}} \rangle$ is barely affected by the distance $c_2 - c_I$. This maximum distance increases with the noise level [2].

In the context of the heartbeat model described in [13], intermittency can be a highly undesirable feature, since the irregular alternations between laminar and chaotic states make it difficult to either control or suppress pathological rhythms, like those characterizing cardiac arrhythmias [6]. If one tries to control such rhythms, for example, using OGY chaos control, it is required that the system attractor be chaotic enough so as to present recurrence of trajectories, a necessary condition to control the system by applying only small perturbation [28]. As an example of the applicability of our results, let us imagine that an experimentally obtained data series shows an intermittent alternation between regular and arrhythmic heartbeats (this can be done by usual electrocardiographic recording).



(a) (b)

Figure 7: Average duration of interburst laminar intervals versus the difference $c_2 - c_I c_2 - c_I$, where $c_I = 2.101503 c_I = 2.101503$ marks the onset of a period-1212 window. (a) Noiseless case. The solid line is a least squares fit with slope 0.50.5. (b) The effect of various noise levels: $2.0 \times 10^{-10} 2.0 \times 10^{-10}$ (empty circles); $5.0 \times 10^{-10} 5.0 \times 10^{-10}$ (vertical crosses); $8.0 \times 10^{-10} 8.0 \times 10^{-10}$ (inclined crosses); $2.0 \times 10^{-9} 2.0 \times 10^{-9}$ (filled squares).

It is possible, at least in principle, to record the durations of the laminar (interburst) intervals and make a histogram of the laminar times $\overline{P}(\tau_{\text{int}})$, which can be then used to compute an average duration $\langle \overline{\tau}_{\text{int}} \rangle$. Let us also suppose that one has a control parameter p that can be varied over an interval. In an experimental setting this control parameter could be, for example, an external factor like the amount of some chemical influencing arrhythmic behavior. For example, cardiac arrhythmias can be induced by the drug ouabain in rabbit ventricle [29]. Finally, if we were able to record the critical dose p_C for which laminar regions disappear at all and intermittent behavior becomes stable chaos, the characterization of $\langle \overline{\tau}_{\text{int}} \rangle$ as a function of the parameter difference $|p - p_c|$ could provide an empirical verification of the results we obtained from an analytical model. Moreover, if the scaling law for the numerically determined probability distribution $\overline{P}(\tau_{\text{int}})$ was found to be a power-law with exponent 3/2, this would be a strong evidence in favor of a universal on-off intermittency mechanism that, since an invariant manifold is supposed for this to occur, what could shed some light on the dynamical mechanisms underlying intermittent behavior in this system [24].

5. Conclusions

We have studied the effect of parametric noise in the coupled system of two Liénard-type oscillators with external periodic forcing, focusing on two different intermittent phenomena exhibited by the system under distinct types of coupling. In the unidirectional coupling or master-slave configuration, we have analyzed the occurrence of complete synchronization of identical oscillators and have determined the necessary coupling strength for a transition from a nonsynchronized to a synchronized state. Near this transition there is an intermittent switching between laminar phases of synchronized (albeit chaotic) behavior and bursts of nonsynchronized dynamics. We verified the universal 3/2 power-law scaling, obeyed by on-off intermittency, for the statistical distribution of the duration of the synchronized laminar phases. The inclusion of noise modifies this scaling according to the general description by adding an exponential tail (for large times) to the power-law scaling (for short times).

We also verified the presence of other types of synchronization, like phase and frequency synchronization, and observed that the latter is robust in the sense that it is not likely to cease with addition of white noise. In the bidirectional coupling of nonidentical oscillators (because of a mismatch of their natural frequencies), we no longer have synchronization, and the intermittent phenomenon of interest is the transition to chaos in the beginning of a periodic window for a parameter range where chaos is the dominant feature. We verified that this transition obeys the Pomeau-Manneville type-I intermittency scenario, by considering the statistical properties of the average laminar durations as well as evidencing the saddle-node bifurcation which is the mechanism underlying the phenomenon. The addition of noise affects these properties in the way predicted for one-dimensional maps. Finally, the results of this paper can be applied to a number of physical systems described by Liénard-type oscillators. Two representative examples are electronic circuits using tunnel diodes, like Zener diode, and models of the heartbeat. The statistical nature of our numerical results makes them amenable to further comparisons with experimental investigations of intermittent behavior.

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