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Research Article

Warped Product Semi-Invariant Submanifolds in Almost Paracontact Riemannian Manifolds

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We show that there exist no proper warped product semi-invariant submanifolds in almost paracontact Riemannian manifolds such that totally geodesic submanifold and totally umbilical submanifold of the warped product are invariant and anti-invariant, respectively. Therefore, we consider warped product semi-invariant submanifolds in the form $N = N_{\perp} \times_f N_T$ by reversing two factor manifolds N_T and $N_{\perp}$. We prove several fundamental properties of warped product semi-invariant submanifolds in an almost paracontact Riemannian manifold and establish a general inequality for an arbitrary warped product semi-invariant submanifold. After then, we investigate warped product semi-invariant submanifolds in a general almost paracontact Riemannian manifold which satisfy the equality case of the inequality.

1. Introduction

It is well known that the notion of warped products plays some important role in differential geometry as well as physics. The geometry of warped product was introduced by Bishop and O'Neill [1]. Many geometers studied different objects/structures on manifolds endowed with an warped product metric (see [2–6]).

Recently, Chen has introduced the notion of CR-warped product in Kaehlerian manifolds and showed that there exist no proper warped product CR-submanifolds in the form $N = N_{\perp} \times_f N_T$ in Kaehlerian manifolds. Therefore, he considered warped product CR-submanifolds in the form $N = N_T \times_f N_{\perp}$ which is called CR-warped product, where N_T is an invariant submanifold, and $N_{\perp}$ is an anti-invariant submanifold of Kaehlerian manifold $\overline{M}$ (see [2, 7, 8]). Analogue results have been obtained for Sasakian ambient as the odd dimensional version of Kaehlerian manifold by Hasegawa and Mihai in [3] and Munteanu in [9].

Almost paracontact manifolds and almost paracontact Riemannian manifolds were defined and studied by Şatō [10]. After then, many authors studied invariant and anti-invariant submanifolds of the almost paracontact Riemannian manifold M with the structure (F, g, ξ, η), when ξ is tangent to the submanifold, and ξ is not tangent to the submanifold [11].

We note that CR-warped products in Kaehlerian manifold are corresponding warped product semi-invariant submanifolds in almost paracontact Riemannian manifolds. In this paper, we showed that there exist no warped product semi-invariant submanifolds in the form $N = N_T \times_f N_\perp$ in contrast to Kaehlerian manifolds (see Theorem 3.1). So, from now on we consider warped product semi-invariant submanifolds in the form $N = N_\perp \times_f N_T$, where $N_\perp$ is an anti-invariant submanifold, and N_T is an invariant submanifold of an almost paracontact Riemannian manifold M by reversing the two factor manifolds N_T and $N_\perp$, and it simply will be called warped product semi-invariant submanifold in the rest of this paper (see Example 3.3 and Theorem 3.4).

2. Preliminaries

Although there are many papers concerning the geometry of semi-invariant submanifolds of almost paracontact Riemannian manifolds (see [11–13]), there is no paper concerning the geometry of warped product semi-invariant submanifolds of almost paracontact Riemannian manifolds in literature so far. So the purpose of the present paper is to study warped product semi-invariant submanifolds in almost paracontact Riemannian manifolds. We first review basic formulas and definitions for almost paracontact Riemannian manifolds and their submanifolds, which we shall use for later.

Let M be an $(m+1)$-dimensional differentiable manifold. If there exist on M a $(1,1)$ type tensor field F, a vector field ξ, and 1-form η satisfying

$$F^2 = I - \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

then M is said to be an almost paracontact manifold, where $\otimes$, the symbol, denotes the tensor product. In the almost paracontact manifold, the following relations hold good:

$$F\xi = 0, \quad \eta \circ F = 0, \quad \operatorname{rank}(F) = m. \tag{2.2}$$

An almost paracontact manifold M is said to be an almost paracontact metric manifold if Riemannian metric g on M satisfies

$$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi) \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$ [14], where $\Gamma(TM)$ denotes the differentiable vector field set on M. From (2.2) and (2.3), we can easily derive the relation

$$g(FX, Y) = g(X, FY). \tag{2.4}$$

An almost paracontact metric manifold is said to be an almost paracontact Riemannian manifold with (F, g, ξ, η)-connection if $\overline{\nabla} F = 0$ and $\overline{\nabla}\eta = 0$, where $\overline{\nabla}$ denotes the connection on M. Since $F^2 = I - \eta \otimes \xi$, the vector field ξ is also parallel with respect to $\overline{\nabla}$ [11, 13].

In the rest of this paper, let us suppose that M is an almost paracontact Riemannian manifold with structure (F, g, ξ, η)-connection.

Let M be an almost paracontact Riemannian manifold, and let N be a Riemannian manifold isometrically immersed in M. For each $x \in N$, we denote by D_x the maximal invariant subspace of the tangent space T_xN of N. If the dimension of D_x is the same for all x in N, then D_x gives an invariant distribution D on N.

A submanifold N in an almost paracontact Riemannian manifold is called semi-invariant submanifold if there exists on N a differentiable invariant distribution D whose orthogonal complementary $D^\perp$ is an anti-invariant distribution, that is, $F(D^\perp) \subset TN^\perp$, where $TN^\perp$ denotes the orthogonal vector bundle of TN in TM. A semi-invariant submanifold is called anti-invariant (resp., invariant) submanifold if $\dim(D_x) = 0$ (resp., $\dim(D_x^\perp) = 0$). It is called proper semi-invariant submanifold if it is neither invariant nor anti-invariant submanifold.

A semi-invariant submanifold N of an almost paracontact Riemannian manifold M is called a Riemannian product if the invariant distribution D and anti-invariant distribution $D^\perp$ are totally geodesic distributions in N. The geometry notion of the semi-invariant submanifolds has been studied by many geometers in the various type manifolds. Authors researched the fundamental properties of such submanifolds (see references).

Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and f a differentiable function on N_1. Consider the product manifold $N_1 \times N_2$ with its projection $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$. The warped product manifold $N = N_1 \times_f N_2$ is the manifold $N_1 \times N_2$ equipped with the Riemannian metric tensor such that

$$g(X, Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + f^2(\pi_1(x)) g_2(\pi_{2*}X, \pi_{2*}Y) \tag{2.5}$$

for any $X, Y \in \Gamma(TN)$, where $*$ is the symbol for the tangent map. Thus we have $g = g_1 + f^2 g_2$, where f is called the warping function of the warped product. The warped product manifold $N = N_1 \times_f N_2$ is characterized by the fact that N_1 and N_2 are totally geodesic and totally umbilical submanifolds of N, respectively. Hence Riemannian products are special classes of the warped products [4].

In this paper, we define and study a new class of warped product semi-invariant submanifolds in an almost paracontact Riemannian manifolds, namely, we investigate the class of warped product semi-invariant submanifolds, and we establish the fundamental theory of such submanifolds.

Now, let N be an isometrically immersed submanifold in an almost paracontact Riemannian manifold M. We denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections on N and M, respectively. Then the Gauss and Weingarten formulas are, respectively, defined by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \overline{\nabla}_X V &= -A_V X + \nabla_X^\perp V \end{aligned} \tag{2.6}$$

for any $X, Y \in \Gamma(TN)$, $V \in \Gamma(TN^\perp)$, where $\nabla^\perp$ is the connection in the normal bundle $TN^\perp$, h is the second fundamental form of N, and A_V is the shape operator. The second fundamental form h and the shape operator A are related by

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2.7}$$

Now, let N be a differentiable manifold, and we suppose that N is an isometrically immersed submanifold in almost paracontact Riemannian manifold M. We denote by g the metric tensor of M as well as that induced on N. For any vector field X tangent to N, we put

$$FX = tX + nX, \tag{2.8}$$

where tX and nX denote the tangential and normal components of FX, respectively. For any vector field V normal to N, we also put

$$FV = BV + CV, \tag{2.9}$$

where BV and CV denote the tangential and normal components of FV, respectively. The submanifold N is said to be invariant if n is identically zero, that is, $F(TN) = TN$. On the other hand, N is said to be anti-invariant submanifold if t is identically zero, that is, $F(TN) \subset (TN^{\perp})$.

We note that for any invariant submanifold N of an almost paracontact Riemannian manifold M, if ξ is normal to N, then the induced structure from the almost paracontact structure on N is an almost product Riemannian structure whenever t is nontrivial. If ξ is tangent to N, then the induced structure on N is an almost paracontact Riemannian structure.

Furthermore, we say that N is a semi-invariant submanifold if there exist two orthogonal distributions D_1 and D_2 such that

(1) TN splits into the orthogonal direct sum $TN = D_1 \oplus D_2$;

(2) the distribution D_1 is invariant, that is, $F(D_1) \subseteq D_1$;

(3) the distribution D_2 is anti-invariant, that is, $F(D_2) \subseteq TN^{\perp}$.

Given any submanifold N of an almost paracontact Riemannian manifold M, from (2.4) and (2.8) we have

$$g(tX, Y) = g(X, tY), \qquad g(nX, V) = g(X, BV) \tag{2.10}$$

for any $X, Y \in \Gamma(TN)$, $V \in \Gamma(TN^{\perp})$.

From now on we suppose that the vector field ξ is tangent to N.

We recall the following general lemma from [1] for later use.

Lemma 2.1. *Let $N = N_1 \times_f N_2$ be a warped product manifold with warping function f, then one has*

(1) $\nabla_X Y \in \Gamma(TN_1)$ *for each* $X, Y \in \Gamma(TN_1)$,

(2) $\nabla_X Z = \nabla_Z X = X(\ln f)Z$, *for each* $X \in \Gamma(TN_1)$, $Z \in \Gamma(TN_2)$,

(3) $\nabla_Z W = \nabla^{N_2}_Z W - g(Z, W)\operatorname{grad} f/f$, *for each* $Z, W \in \Gamma(TN_2)$,

where ∇ and ∇^{N_2} denote the Levi-Civita connections on N and N_2, respectively.

Let N be a semi-invariant submaniold of an almost paracontact Riemannian manifold M. We denote by the invariant distribution D and anti-invariant distribution $D^{\perp}$. We also denote the orthogonal complementary of $F(D^\perp)$ in $TN^\perp$ by ν, then we have direct sum

$$TN^\perp = F\left(D^\perp\right) \oplus \nu. \tag{2.11}$$

We can easily see that ν is an invariant subbundle with respect to F.

3. Warped Product Semi-Invariant Submanifolds in an Almost Paracontact Riemannian Manifold

Useful characterizations of warped product semi-invariant submanifolds in almost paracontact Riemannian manifolds are given the following theorems.

Theorem 3.1. *If $N = N_T \times_f N_\perp$ is a warped product semi-invariant submanifold of an almost paracontact Riemannian manifold M such that N_T is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold of M, then N is a usual Riemannian product.*

Proof. Let ξ be normal to N. Taking into account that h is symmetric and using (2.3), (2.6), (2.7), and considering Lemma 2.1(2), for $X \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_\perp)$, we have

$$\begin{aligned} g(\nabla_X Z, W) = g(\nabla_Z X, W) &= g\left(\overline{\nabla}_Z X, W\right) = g\left(F\overline{\nabla}_Z X, FW\right) + \eta\left(\overline{\nabla}_Z X\right)\eta(W), \\ X\ln(f)g(Z,W) = g\left(\overline{\nabla}_Z FX, FW\right) &= g(h(Z,FX), FW) = g\left(\overline{\nabla}_{FX} Z, FW\right) \\ &= g\left(\overline{\nabla}_{FX} FZ, W\right) = -g(A_{FZ}FX, W) = -g(h(FX,W), FZ) \\ &= -g\left(\overline{\nabla}_W FX, FZ\right) = -g\left(\overline{\nabla}_W X, Z\right) = -X\ln(f)g(W,Z), \end{aligned} \tag{3.1}$$

which implies that $X\ln(f) = 0$.

If ξ is tangent to N, then ξ can be written as follows:

$$\xi = \xi_1 + \xi_2, \tag{3.2}$$

where $\xi_1 \in \Gamma(TN_T)$ and $\xi_2 \in \Gamma(TN_\perp)$. Since $\overline{\nabla}_X \xi = 0$, from the Gauss formulae, we have

$$h(X,\xi) = 0, \qquad \nabla_X \xi = 0 \tag{3.3}$$

for any $X \in \Gamma(TN)$. Considering Lemma 2.1(2), we get

$$\begin{gathered} \nabla_Z \xi_1 = \xi_1(\ln f)Z = 0, \qquad \nabla_X \xi_2 = X(\ln f)\xi_2 = 0, \\ \nabla_{\xi_2}\xi_1 = \nabla_{\xi_1}\xi_2 = \xi_1(\ln f)\xi_2 = 0 \end{gathered} \tag{3.4}$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\perp)$. If ξ_2 is identically zero, then from Lemma 2.1 we have

$$\nabla_Z \xi_1 = \nabla_{\xi_1} Z = \xi_1(\ln f) Z = 0, \qquad \nabla_X \xi_1 \in \Gamma(TN_T). \tag{3.5}$$

It follows that the warping function f is a constant and N is usual Riemannian product. Hence the proof is complete.

If the warping function is constant, then the metric on the "second" factor could be modified by an homothety, and hence, the warped product becomes a direct product. □

Now, we give two examples for almost paracontact Riemannian manifold and their submanifolds in the form $N = N_\perp \times_f N_T$ to illustrate our results. Firstly, we construct an almost paracontact metric structure on $\mathbb{R}^{2n+1}$ (see Example 3.2) and after give an example which is concerning its submanifold (see Example 3.3).

Example 3.2. Let

$$\mathbb{R}^{2n+1} = \{(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, t) \mid x_i, y_i, t \in \mathbb{R},\ i = 1, 2, \ldots, n\}. \tag{3.6}$$

The almost paracontact Riemannian structure (F, g, ξ, η) is defined on $\mathbb{R}^{2n+1}$ in the following way:

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad F\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial t}\right) = 0, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = dt. \tag{3.7}$$

If $Z = \lambda_i(\partial/\partial x_i) + \mu_i(\partial/\partial y_i) + \nu(\partial/\partial t) \in T(\mathbb{R}^{2n+1})$, then we have

$$g(Z, Z) = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i=1}^{n} \mu_i^2 + \nu^2. \tag{3.8}$$

From this definition, it follows that

$$g(Z, \xi) = \eta(Z) = \nu, \quad g(FZ, FZ) = g(Z, Z) - \eta^2(Z), \quad F\xi = 0, \quad \eta(\xi) = 1 \tag{3.9}$$

for an arbitrary vector field Z. Thus $(\mathbb{R}^{2n+1}, F, g, \xi, \eta)$ becomes an almost paracontact Riemannian manifold, where g and $\{\partial/\partial x_i, \partial/\partial y_i, \partial/\partial t\}$ denote usual inner product and standard basis of $T(\mathbb{R}^{2n+1})$, respectively.

Example 3.3. Let N be a submanifold in $\mathbb{R}^5$ with coordinates (x_1, x_2, y_1, y_2, t) given by

$$x_1 = v \cos\theta, \qquad x_2 = v \sin\theta, \qquad y_1 = v \cos\beta, \qquad y_2 = v \sin\beta, \qquad t = \sqrt{2}u. \tag{3.10}$$

It is easy to check that the tangent bundle of N is spanned by the vectors

$$\begin{aligned}
Z_1 &= \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \cos\beta\frac{\partial}{\partial y_1} + \sin\beta\frac{\partial}{\partial y_2},\\
Z_2 &= -v\sin\theta\frac{\partial}{\partial x_1} + v\cos\theta\frac{\partial}{\partial x_2},\\
Z_3 &= -v\sin\beta\frac{\partial}{\partial y_1} + v\cos\beta\frac{\partial}{\partial y_2},\\
Z_4 &= \sqrt{2}\frac{\partial}{\partial t}.
\end{aligned} \tag{3.11}$$

We define the almost paracontact Riemannian structure of $\mathbb{R}^5$ by

$$F\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}, \qquad F\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial y_i}, \qquad F\left(\frac{\partial}{\partial t}\right) = 0, \qquad \eta = \frac{1}{\sqrt{2}}dt. \tag{3.12}$$

Then with respect to the almost paracontact Riemannian structure F of $\mathbb{R}^5$, the space $F(TN)$ becomes

$$\begin{aligned}
FZ_1 &= -\cos\theta\frac{\partial}{\partial x_1} - \sin\theta\frac{\partial}{\partial x_2} + \cos\beta\frac{\partial}{\partial y_1} + \sin\beta\frac{\partial}{\partial y_2},\\
FZ_2 &= v\sin\theta\frac{\partial}{\partial x_1} - v\cos\theta\frac{\partial}{\partial x_2},\\
FZ_3 &= -v\sin\beta\frac{\partial}{\partial y_1} + v\cos\beta\frac{\partial}{\partial y_2},\\
FZ_4 &= 0.
\end{aligned} \tag{3.13}$$

Since FZ_1 and FZ_4 are orthogonal to N and FZ_2, FZ_3 are tangent to N, D and $D^{\perp}$ can be taken subspace $\mathrm{sp}\{Z_1, Z_4\}$ and subspace $\mathrm{sp}\{Z_2, Z_3\}$, respectively, where ξ can be taken as Z_4 for $FZ_4 = 0$ and $\eta(Z_4) = 1$. Furthermore, the metric tensor of N is given by

$$g = 2\left(du^2 + dv^2\right) + v^2\left(d\theta^2 + d\beta^2\right) = 2g_{N_\perp} + v^2 g_{N_T}. \tag{3.14}$$

Thus N is a warped product semi-invariant submanifold with dimensional 5 of almost paracontact manifold $\mathbb{R}^5$ with warping function $f = v^2$.

Now, let $N = N_\perp \times_f N_T$ be a warped product semi-invariant submanifold of an almost paracontact Riemannian manifold M, where $N_\perp$ is an anti-invariant submanifold, and N_T is an invariant submanifold of M. If we denote the Levi-Civita connections on M and N by $\overline{\nabla}$ and ∇, respectively, by using (2.6) and (2.8), we have

$$\begin{gathered}\overline{\nabla}_X FY = F\overline{\nabla}_X Y,\\ \nabla_X tY + h(X,tY) - A_{nY}X + \nabla^{\perp}_X nY = t(\nabla_X Y) + n(\nabla_X Y) + Bh(X,Y) + Ch(X,Y)\end{gathered} \tag{3.15}$$

for any $X, Y \in \Gamma(TN)$. Taking into account the tangential and normal components of (3.15), respectively, we have

$$(\nabla_X t)Y = A_{nY}X + Bh(X,Y), \tag{3.16}$$

$$(\nabla_X n)Y = Ch(X,Y) - h(X,tY), \tag{3.17}$$

where the derivatives of t and n are, respectively, defined by

$$(\nabla_X t)Y = \nabla_X tY - t(\nabla_X Y), \qquad (\nabla_X n)Y = \nabla^{\perp}_X nY - n(\nabla_X Y). \tag{3.18}$$

Next, we are going to investigate the geometric properties of the leaves of the warped product semi-invariant submanifolds in an almost paracontact Riemannian manifold.

Theorem 3.4. *Let N be a warped product semi-invariant submanifold of an almost paracontact Riemannian manifold M. Then the invariant distribution D and the anti invariant distribution $D^{\perp}$ are always integrable.*

Proof. From (3.16) and considering Lemma 2.1(1), we have

$$\begin{gathered}\overline{\nabla}_X FU = F\overline{\nabla}_X U,\\ X\ln(f)tU + h(X,tU) = X\ln(f)tU + Bh(X,U) + Ch(X,U)\end{gathered} \tag{3.19}$$

for any $X \in \Gamma(D^{\perp})$ and $U \in \Gamma(D)$. From the tangential and normal components of (3.19), respectively, we arrive at

$$Bh(X,U) = 0, \tag{3.20}$$

$$h(X,tU) = Ch(X,U). \tag{3.21}$$

By using (3.16) and (3.20) we get

$$A_{nX}U = -X(\ln f)tU. \tag{3.22}$$

Furthermore, by using the Gauss-Weingarten formulas and taking into account that $D^{\perp}$ is totally geodesic in N and it is anti-invariant in M, by direct calculations, it is easily to see that

$$A_{nY}X = -Bh(X,Y), \tag{3.23}$$

which is also equivalent to

$$A_{nY}X = A_{nX}Y \tag{3.24}$$

for any $X, Y \in \Gamma(D^{\perp})$. Moreover, using (2.4) and (2.7) and making use of A being self-adjoint, we obtain

$$\begin{aligned} g(A_{nX}Y, Z) &= g(h(Y, Z), nX) = g\left(\overline{\nabla}_Z Y, FX\right) = g\left(\overline{\nabla}_Z FY, X\right) \\ &= -g(A_{nY}Z, X) = -g(A_{nY}X, Z), \end{aligned} \tag{3.25}$$

which gives us

$$A_{nX}Y = -A_{nY}X \tag{3.26}$$

for any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(TN)$. Thus from (3.24) and (3.26), we arrive at

$$A_{nX}Y = 0, \qquad Bh(X, Y) = 0 \tag{3.27}$$

for any $X, Y \in \Gamma(D^{\perp})$. Furthermore, by using (2.6), (2.8), and (2.9) and considering Lemma 2.1(3), we have

$$\begin{aligned} h(U, tV) + \nabla_U tV &= F(\nabla_U V) + Fh(V, U) \\ &= F\left(\nabla'_U V - g(V, U)\frac{\operatorname{grad} f}{f}\right) + Bh(V, U) + Ch(V, U) \\ &= t\left(\nabla'_U V\right) - g(V, U)n\left(\frac{\operatorname{grad} f}{f}\right) + Bh(V, U) + Ch(V, U) \end{aligned} \tag{3.28}$$

for any $V, U \in \Gamma(D)$, where ∇' denote the Levi-Civita connection on D. Taking into account the normal and tangential components of (3.28), respectively, we have

$$h(U, tV) = -g(U, V)n\left(\frac{\operatorname{grad} f}{f}\right) + Ch(U, V), \tag{3.29}$$

$$\nabla'_U tV - g(tV, U)\frac{\operatorname{grad} f}{f} = t\left(\nabla'_U V\right) + Bh(V, U). \tag{3.30}$$

From (3.29), we can easily see that

$$h(U, tV) = h(V, tU) \tag{3.31}$$

for any $U, V \in \Gamma(D)$. Finally, by using (3.17) and (3.31), we have

$$\begin{aligned} n([V,U]) &= n(\nabla_V U - \nabla_U V) = \nabla_V^{\perp} nU - (\nabla_V nU) - \nabla_U^{\perp} nV + (\nabla_U n)V \\ &= (\nabla_U n)V - (\nabla_V n)U = Ch(U,V) - h(U,tV) - Ch(V,U) + h(V,tU) = 0 \end{aligned} \tag{3.32}$$

for any $V, U \in \Gamma(D)$, that is, $[V,U] \in \Gamma(D)$.

In the same way, making use of (3.16) and (3.27) for any $X, Y \in \Gamma(D^{\perp})$, we conclude that

$$\begin{aligned} t([X,Y]) &= t(\nabla_X Y - \nabla_Y X) \\ &= \nabla_X tY - (\nabla_X t)Y - \nabla_Y tX + (\nabla_Y t)X \\ &= (\nabla_Y t)X - (\nabla_X t)Y = A_{nX}Y - A_{nY}X = 0 \end{aligned} \tag{3.33}$$

that is, $[X,Y] \in \Gamma(D^{\perp})$. So we obtain the desired result. □

Since the distributions D and $D^{\perp}$ are integrable, we denote the integral manifolds of D and $D^{\perp}$ by N_T and $N_{\perp}$, respectively.

Now, the following theorem characterizes (warped product or Riemannian product) semi-invariant submanifolds in almost paracontact manifolds.

Theorem 3.5. *Let N be a submanifold of an almost paracontact Riemannian manifold M. Then N is a semi-invariant submanifold if and only $nt = 0$.*

Proof. Let us assume that N is a semi-invariant submanifold of an almost paracontact Riemannian manifold M and by Q and P; we denote the projection operators on subspaces $\Gamma(D^{\perp})$ and $\Gamma(D)$, respectively, then we have

$$P + Q = I, \qquad P^2 = P, \qquad Q^2 = Q, \qquad PQ = QP = 0. \tag{3.34}$$

Moreover, by using (2.1), (2.8), and (2.9), if ξ is tangent to N, then we get

$$X - \eta(X)\xi = t^2 X + BnX, \qquad ntX + CnX = 0, \tag{3.35}$$

$$tBV + BCV = 0, \qquad nBV + C^2 V = V \tag{3.36}$$

for any $X \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$. On the other hand, if ξ is normal to N, then (3.35) and (3.36) become, respectively,

$$\begin{aligned} X = t^2 X + BnX, &\qquad ntX + CnX = 0, \\ V - \eta(V)\xi = nBV + C^2 V, &\qquad tBV + BCV = 0. \end{aligned} \tag{3.37}$$

From (2.8), we have

$$\begin{aligned} FX &= FPX + FQX, \\ tX + nX &= tPX + tQX + nPX + nQX \end{aligned} \tag{3.38}$$

for any $X \in \Gamma(TN)$. From the tangential and normal components, we have

$$tX = tPX + tQX, \qquad nX = nPX + nQX. \tag{3.39}$$

Since D is invariant and $D^{\perp}$ is anti-invariant, we get

$$nP = 0, \qquad Qt = 0. \tag{3.40}$$

We have

$$tP = t \tag{3.41}$$

by virtue of $Q = I - P$. Now by using the right-hand side to the second equation of (3.35) and using (3.40) and (3.41), we conclude that

$$nt = 0, \tag{3.42}$$

which is also equivalent to

$$Cn = 0. \tag{3.43}$$

Conversely, for a submanifold N of an almost paracontact Riemannian manifold M, we suppose that $nt = 0$. For any vector fields tangent X to N and V normal to N, by using (2.4) and (3.43), we have

$$\begin{aligned} g(X, FV) &= g(FX, V), \\ g(X, BV) &= g(nX, V), \\ g(X, FBV) &= g(FnX, V), \\ g(X, tBV) &= g(CnX, V) = 0 \end{aligned} \tag{3.44}$$

for all $X \in \Gamma(TN)$. So we have $g(tBV, X) = 0$. Since $X, tBV \in \Gamma(TN)$, it implies $tB = 0$ which is also equivalent to $BC = 0$ from (3.36). Since $F\xi = 0$, we get $t\xi = n\xi = 0$. So, from (3.35) and (3.36), we conclude

$$t^3 = t, \qquad C^3 = C. \tag{3.45}$$

Now, if we put

$$P = t^2, \qquad Q = I - P, \tag{3.46}$$

then we can derive that $P + Q = I$, $P^2 = P$, $Q^2 = Q$, and $PQ = QP = 0$ which show that Q and P are orthogonal complementary projection operators and define complementary distributions $D^{\perp}$ and D, respectively, where D and $D^{\perp}$ denote the distributions which are belong to subspaces TN_T and $TN_{\perp}$, respectively. From (3.42), (3.45), and (3.46) we can derive

$$tP = t, \qquad tQ = 0, \qquad QtP = 0, \qquad nP = 0. \tag{3.47}$$

These equations show that the distribution D is an invariant and the distribution $D^{\perp}$ is an anti-invariant. The proof is complete. □

Theorem 3.6. *Let N be a semi-invariant submanifold of an almost paracontact Riemannian manifold M. Then N is a warped product semi-invariant submanifold if and only if the shape operator of N satisfies*

$$A_{FX}Z = -X(\mu)FZ, \quad X \in \Gamma\left(D^{\perp}\right), \ Z \in \Gamma(D) \tag{3.48}$$

for some function μ on N satisfying $W(\mu) = 0$, $W \in \Gamma(D)$.

Proof. We suppose that N is a warped product semi-invariant submanifold in an almost paracontact Riemannian manifold M. Then from (3.22), we have

$$A_{FX}Z = -X(\ln f)FZ \tag{3.49}$$

for any $X \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D)$. Since f is the only function on $N_{\perp}$, we can easily see that $W(\ln f) = 0$ for all $W \in \Gamma(D)$.

Conversely, let us assume that N is a semi-invariant submanifold in an almost paracontact Riemannian manifold M satisfying

$$A_{FX}Z = X(\mu)FZ, \quad X \in \Gamma\left(D^{\perp}\right), \ Z \in \Gamma(D) \tag{3.50}$$

for some function μ on N satisiying $W(\mu) = 0$ for all $W \in \Gamma(D)$. Since the ambient space M is an almost paracontact Riemannian manifold and making use of (2.4) and (3.27), we arrive at

$$g(\nabla_X Y, FZ) = g\left(\overline{\nabla}_X Y, FZ\right) = g\left(\overline{\nabla}_X FY, Z\right) = -g(A_{FY}X, Z) = 0 \tag{3.51}$$

for any $X, Y \in \Gamma(D^{\perp})$ and $Z \in (D)$. Thus the anti-invariant distribution $D^{\perp}$ is totally geodesic in N. In the same way, making use of $\overline{\nabla}$ being Levi-Civita connection and (3.22), we have

$$\begin{aligned} g(\nabla_Z W, X) &= g\left(\overline{\nabla}_Z W, X\right) = -g\left(\overline{\nabla}_Z X, W\right) = -g\left(\overline{\nabla}_Z FX, FW\right) \\ &= g(A_{FX}Z, FW) = X(\mu)g(Z, W) \end{aligned} \tag{3.52}$$

for any $Z, W \in \Gamma(D)$ and $X \in \Gamma(D^{\perp})$, where $\mu = \ln(1/f)$. Since the invariant distribution D of semi-invariant submanifold N is always integrable (Theorem 3.4) and $W(\mu) = 0$, for each $W \in \Gamma(TN_T)$, which implies that the integral manifold of D is an extrinsic sphere in N, that is, it is a totally umbilical submanifold and its mean curvature vector field is non-zero and parallel, thus we know that N is a Riemannian warped product $N_{\perp} \times_f N_T$, where $N_{\perp}$ and N_T denote the integral manifolds of the distributions of $D^{\perp}$ and D, respectively, and f is the warping function. So we obtain the desired result. □

In the rest of this section, we are going to obtain an inequality for the squared norm of the second fundamental form by means of the warping function for warped product semi-invariant submanifolds of an almost paracontact Riemannian manifold. Now, we recall that semi-invariant N is said to be mixed geodesic (resp., D-geodesic and $D^{\perp}$-geodesic) submanifold if the second fundamental form h of N satisfies $h(X, Z) = 0$, $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$ (resp., $h(X, Y) = 0$, $X, Y \in \Gamma(D)$ and $h(Z, W) = 0$, $Z, W \in \Gamma(D^{\perp})$).

Now, we are going to give the following lemma for later use.

Lemma 3.7. *Let $N = N_{\perp} \times_f N_T$ be a warped product semi-invariant submanifold of an almost paracontact Riemannian manifold M. Then one has*

(1) $g(h(D^{\perp}, D^{\perp}), FD^{\perp}) = 0$,

(2) $g(h(Z, W), FX) = -X(\ln f) g(tZ, W)$, $Z, W \in \Gamma(D)$, $X \in \Gamma(D^{\perp})$,

(3) $g(h(X, Z), FY) = 0$, *for any* $X, Y \in \Gamma(D^{\perp})$ *and* $Z \in \Gamma(D)$,

(4) $g(h(D, FD), FD^{\perp}) = 0$ *if and only if* $N = N_{\perp} \times_f N_T$ *is a usual Riemannian product, where* D *and* $D^{\perp}$ *denote the leaves of* N_T *and* $N_{\perp}$*, respectively.*

Proof. (1) For any $X, Y, Z \in \Gamma(D^{\perp})$, by using (2.4) and (3.27) and considering that the ambient space is an almost paracontact Riemannian manifold, we have

$$g(h(X, Y), FZ) = g\left(\overline{\nabla}_X Y, FZ\right) = g\left(\overline{\nabla}_X FY, Z\right) = -g(A_{nY} X, Z) = 0. \tag{3.53}$$

(2) Making use of $\overline{\nabla}$ being Levi-Civita connection and Lemma 2.1(2.2), we get

$$g(h(Z, W), FX) = g\left(\overline{\nabla}_W FZ, X\right) = -g\left(\overline{\nabla}_W X, tZ\right) = -X \ln g \cdot g(W, tZ) \tag{3.54}$$

for any $Z, W \in \Gamma(D)$, $X \in \Gamma(D^{\perp})$.

(3) In the same way, we have

$$g(h(X, Z), FY) = g\left(\overline{\nabla}_X Z, FY\right) = g(\nabla_X tZ, Y) = X \ln f g(tZ, Y) = 0 \tag{3.55}$$

for any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D)$.

(4) Considering Lemma 2.1(3) we derive

$$g(h(W, FZ), FX) = g\left(\overline{\nabla}_{FZ} W, FX\right) = g\left(\overline{\nabla}_{FZ} FW, X\right) = -g(tZ, tW) X(\ln f) \tag{3.56}$$

for any$Z, W \in \Gamma(D)$ and $X \in \Gamma(D^{\perp})$. □

Theorem 3.8. *Let $N = N_\perp \times_f N_T$ be a warped product semi-invariant submanifold of an almost paracontact Riemannian manifold M. Then one has the following.*

(1) *The squared norm of the second fundamental form of N in M satisfies*

$$\|h\|^2 \geq \frac{1}{f^2}\|\operatorname{grad} f\|^2 (Tr(t))^2, \tag{3.57}$$

where $Tr(t)$ denote the trace of mapping t.

(2) *If the equality sign of* (3.57) *holds identically, then $N_\perp$ is a totally geodesic, N_T is a totally umbilical submanifolds of M, and N is a mixed geodesic submanifold in M. Furthermore, N is a minimal submanifold M if and only if $Tr(t) = 0$ or $N = N_\perp \times_f N_T$ is a usual Riemannian product.*

Proof. Let $\{e_1, e_2, \ldots, e_p, e^1, e^2, \ldots, e^q, N_1, N_2, \ldots, N_s, \xi\}$ be an orthonormal basis of an almost paracontact Riemannian manifold M such that $\{e_1, e_2, \ldots, e_p\}$ is tangent to $\Gamma(TN_\perp)$, $\{e^1, e^2, \ldots, e^q\}$ is tangent to $\Gamma(TN_T)$, and $\{N_1, N_2, \ldots, N_s\}$ is tangent to $\Gamma(\nu)$. Taking into account Lemma 3.7 and the basic linear algebra rules, by direct calculations, we have

$$\begin{aligned} h(X,Y) &= g(h(X,Y), N_j)N_j, \quad 1 \leq j \leq s, \\ h(Z,W) &= -e_i \ln f(tZ,W)Fe_i + g(h(Z,W), N_j)N_j, \quad 1 \leq i \leq p, \\ h(X,Z) &= g(h(X,Z), N_j)N_j \end{aligned} \tag{3.58}$$

for all $X, Y \in \Gamma(TN_\perp)$ and $Z, W \in \Gamma(TN_T)$. Since

$$\begin{aligned} \|h\|^2 = &\sum_{i,j=1}^{p}\sum_{\ell=1}^{s} g(h(e_i,e_j),N_\ell)^2 + 2\sum_{i=1}^{p}\sum_{k=1}^{q}\sum_{\ell=1}^{s} g\left(h\left(e_i,e^k\right),N_\ell\right)^2 \\ &+ \sum_{r,k=1}^{q}\sum_{i=1}^{p}(e_i \ln f)^2 g\left(te^k,e^r\right)^2 + \sum_{r,k=1}^{q}\sum_{\ell=1}^{s} g\left(h\left(e^k,e^r\right),N_\ell\right)^2, \end{aligned} \tag{3.59}$$

here by direct calculations, we get

$$(e_i(\ln f))^2 = \frac{1}{f^2}\|\operatorname{grad} f\|^2, \qquad \mathrm{Tr}(t) = \sum_{k=1}^{q} g\left(te^k, e^k\right). \tag{3.60}$$

So we conclude that

$$\|h\|^2 \geq \frac{1}{f^2}\|\operatorname{grad} f\|^2 (\mathrm{Tr}(t))^2, \tag{3.61}$$

which proves our assertion.

Now we assume that the equality case of (3.57) holds identically, then from (3.58), respectively, we obtain

$$h(D^{\perp}, D^{\perp}) = 0, \quad h(D, D) \in \Gamma(F(D^{\perp})), \tag{3.62}$$

$$h(D^{\perp}, D) = 0. \tag{3.63}$$

Since $N_{\perp}$ is totally geodesic submanifold in N, the first condition in (3.62) implies that $N_{\perp}$ is totally geodesic submanifold in M. Moreover, Lemma 2.1(3) shows that N_T is totally umbilical submanifold in N. Therefore, the second condition in (3.62) implies that N_T is also totally umbilical submanifold in M. On the other hand, (3.20) and (3.63) imply that N is mixed geodesic submanifold in M. □

Conclusion 3.9. The geometry of the warped products in Riemannian manifolds is totally different from the geometry of the warped products in complex manifolds. Namely, in the complex manifolds, there exists no proper warped product CR-submanifold in the form $N = N_{\perp} \times_f N_T$ (see [2, 8]) while there exists no proper warped product semi-invariant submanifold in the form $N = N_T \times_f N_{\perp}$ in Riemannian manifolds (see Theorem 3.1). The first condition in (3.62) implies that warped product CR-submanifold is minimal in complex manifolds while it does not imply that warped product semi-invariant submanifold is minimal in Riemannian product manifolds.

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