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Research Article

Free Vibration Analysis of Rectangular Orthotropic Membranes in Large Deflection

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This paper reviewed the research on the vibration of orthotropic membrane, which commonly applied in the membrane structural engineering. We applied the large deflection theory of membrane to derive the governing vibration equations of orthotropic membrane, solved it, and obtained the power series formula of nonlinear vibration frequency of rectangular membrane with four edges fixed. The paper gave the computational example and compared the two results from the large deflection theory and the small one, respectively. Results obtained from this paper provide some theoretical foundation for the measurement of pretension by frequency method; meanwhile, the results provide some theoretical foundation for the research of nonlinear vibration of membrane structures and the response solving of membrane structures under dynamic loads.

1. Introduction

The membrane structure is a new structure system, which has been rising during the recent several dozens of years. Because of its economy, beauty, and less dead weight, it is widely applied to large span structures, such as large-scale stadium, exhibition center, works of decoration, and so on. The application of tensile membrane structure was the most extensive in each kind of membrane structure systems. The tensile membrane structure's stiffness is formed by the zonal and meridional pretensions. The tallying degree of the actual pretension value and the design pretension value directly influences the normal utilization and safety after the construction finished [1, 2]. Therefore, the nondestructive monitoring of the membrane structure's pretension is very important and necessary after the construction finished.

At present, most researches focused on design and construction of membrane structures; however, there is little research on the pretension measurement of membrane structures [1, 2]. The main methods of the pretension measurement include strain method, frequency method, deflection method, and "cable analogy" method [3]. If we study the application of frequency method, the vibration theory of membrane must be involved. Many scholars studied about the vibration theory of membrane. Their researches involve the problem of free vibration of a confocal composite elliptical membrane [4], the problem of fundamental frequency of rectangular membranes with an internal oblique support [5], the problem of free vibration of composite rectangular membranes with an oblique and a bent interface [6, 7], the problem of free in-plane vibration of an axially moving membrane [8]. However, these researches did not aim at this kind of orthotropic membrane that was used in construction field. Moreover, these researches have not obtained the power series formula of nonlinear vibration frequency of the orthotropic membrane.

In this paper, we studied the vibration of orthotropic membranes according to the large deflection theory of membrane [9, 10] and obtained the power series formula of nonlinear vibration frequency of rectangular membranes with four edges fixed. The paper gave the computational example and compared the two results from the large deflection theory and the small one, respectively.

2. Governing Equations and Boundary Conditions

The studied rectangular membrane is orthotropic. Its two orthogonal directions are the two principal fiber directions, and the material characteristics of the two principal fiber directions are different. Assume that the studied rectangular membrane is fixed on four edges. The two principal fiber directions are x and y, respectively. a and b denote the length of x and y direction, respectively; N_{0x} and N_{0y} denote initial tension in x and y, respectively, as shown in Figure 1.

According to the large deflection theory and D'Alembert's principle of membranes [8–10], the vibration partial differential equation and consistency equation of orthotropic membrane are

$$\rho\frac{\partial^2 w}{\partial t^2}-\left(N_x+N_{0x}\right)\frac{\partial^2 w}{\partial x^2}-\left(N_y+N_{0y}\right)\frac{\partial^2 w}{\partial y^2}=0$$

$$\frac{1}{E_1 h}\frac{\partial^2 N_x}{\partial y^2}-\frac{\mu_2}{E_2 h}\frac{\partial^2 N_y}{\partial y^2}-\frac{\mu_1}{E_1 h}\frac{\partial^2 N_x}{\partial x^2}+\frac{1}{E_2 h}\frac{\partial^2 N_y}{\partial x^2}-\frac{1}{Gh}\frac{\partial^2 N_{xy}}{\partial x\partial y}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}, \tag{2.1}$$

where ρ denotes aerial density of membrane; N_x and N_y denote tension in x and y, respectively; N_{0x} and N_{0y} denote initial tension in x and y, respectively; N_{xy} denotes shear force; w denotes deflection: $w(x,y,t)$; h denotes membrane's thickness; E_1 and E_2 denote Young's modulus in x and y, respectively; G denotes shearing modulus; μ_1 and μ_2 denote Poisson's ratio in x and y, respectively.

While the membrane is in vibration, the effect of shearing stress is so small that we may take $N_{xy}=0$. Introducing the stress function and letting $N_x=h(\partial^2\varphi/\partial y^2)$, $N_y=h(\partial^2\varphi/\partial x^2)$,

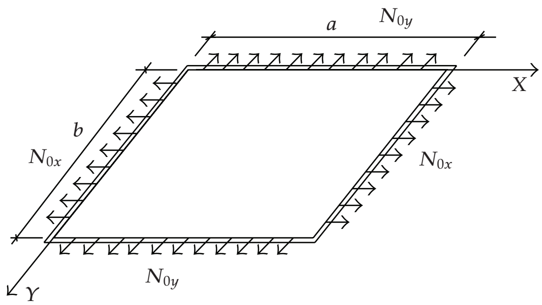

Figure 1: Rectangular membrane with four edges fixed.

$N_{0x} = h \cdot \sigma_{0x}$, and $N_{0y} = h \cdot \sigma_{0y}$, (2.1) can be simplified as follows:

$$\frac{\rho}{h}\frac{\partial^2 w}{\partial t^2} - \left(\sigma_{0x} + \frac{\partial^2 \varphi}{\partial y^2}\right)\frac{\partial^2 w}{\partial x^2} - \left(\sigma_{0y} + \frac{\partial^2 \varphi}{\partial x^2}\right)\frac{\partial^2 w}{\partial y^2} = 0, \tag{2.2}$$

$$\frac{1}{E_1}\frac{\partial^4 \varphi}{\partial y^4} + \frac{1}{E_2}\frac{\partial^2 \varphi}{\partial x^4} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}, \tag{2.3}$$

where φ denotes stress function: $\varphi(x, y, t)$; σ_{0x} and σ_{0y} denote initial tensile stress in x and y, respectively.

The corresponding boundary conditions can be expressed as follows:

$$\begin{aligned} w(0, y, t) &= 0, \qquad w(a, y, t) = 0, \\ w(x, 0, t) &= 0, \qquad w(x, b, t) = 0. \end{aligned} \tag{2.4}$$

3. Solution of Free Vibration Frequency

Functions that satisfy the boundary conditions (2.4) are taken as follows:

$$\begin{aligned} w(x, y, t) &= W(x, y)T(t), \\ \varphi(x, y, t) &= \phi(x, y)T^2(t), \end{aligned} \tag{3.1}$$

where $W(x, y)$ is the given mode shape function, and $\phi(x, y)$ and $T(t)$ are the unknown functions.

Assume that the mode shape function is as follows:

$$W(x, y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{3.2}$$

where m and n are integers and denote the sine half-wave number in x and y, respectively. Equation (3.2) satisfies the boundary conditions automatically.

Substituting (3.1) into (2.3) yields

$$\frac{1}{E_1}\frac{\partial^4\phi}{\partial y^4}+\frac{1}{E_2}\frac{\partial^2\phi}{\partial x^4}=\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}. \tag{3.3}$$

Substituting (3.2) into (3.3) yields

$$\frac{1}{E_1}\frac{\partial^4\phi}{\partial y^4}+\frac{1}{E_2}\frac{\partial^2\phi}{\partial x^4}=\frac{m^2n^2\pi^4}{2a^2b^2}\left(\cos\frac{2m\pi x}{a}+\cos\frac{2n\pi y}{b}\right). \tag{3.4}$$

Assume that the solution of (3.4) is

$$\phi(x,y)=\alpha\cdot\cos\frac{2m\pi x}{a}+\beta\cdot\cos\frac{2n\pi y}{b}. \tag{3.5}$$

Substituting (3.5) into (3.4) yields

$$\alpha=\frac{E_2n^2a^2}{32m^2b^2},\qquad \beta=\frac{E_1m^2b^2}{32n^2a^2}. \tag{3.6}$$

Substituting (3.1) into (2.2), according to the Galerkin method [8, 10], yields

$$\begin{aligned}
&\iint_S\left(\frac{\rho}{h}\frac{\partial^2 w}{\partial t^2}-\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)W(x,y)ds\\
&=\iint_S\left[\frac{\rho}{h}W\frac{\partial^2 T(t)}{\partial t^2}-\left(\sigma_{0x}\frac{\partial^2 W}{\partial x^2}+\sigma_{0y}\frac{\partial^2 W}{\partial y^2}\right)T(t)\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right)T^3(t)\right]W(x,y)ds\\
&=0.
\end{aligned} \tag{3.7}$$

Obviously, (3.7) is a nonlinear differential equation with respect to $T(t)$, and it can be expressed as follows:

$$A\cdot\frac{d^2T(t)}{dt^2}-B\cdot T(t)-C\cdot T^3(t)=0, \tag{3.8}$$

where

$$\begin{aligned}
A&=\iint_S\frac{\rho}{h}W^2ds=\iint_S\frac{\rho}{h}\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}ds=\frac{\rho ab}{4h},\\
B&=\iint_S\left(\sigma_{0x}\cdot\frac{\partial^2 W}{\partial x^2}+\sigma_{0y}\cdot\frac{\partial^2 W}{\partial y^2}\right)W\,ds=-\frac{\pi^2 ab}{4}\left(\frac{m^2}{a^2}\sigma_{0x}+\frac{n^2}{b^2}\sigma_{0y}\right),\\
C&=\iint_S\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right)W\,ds=-\frac{ab\pi^4}{64}\left(\frac{E_1m^4}{a^4}+\frac{E_2n^4}{b^4}\right).
\end{aligned} \tag{3.9}$$

Substituting the values of A, B, and C into (3.8) and dividing by $\rho ab/4h$ on two ends yields

$$\frac{d^2T(t)}{dt^2} + \frac{h\pi^2}{\rho}\left(\frac{m^2}{a^2}\sigma_{0x} + \frac{n^2}{b^2}\sigma_{0y}\right)T(t) + \frac{h\pi^4}{16\rho}\left(\frac{E_1 m^4}{a^4} + \frac{E_2 n^4}{b^4}\right)T^3(t) = 0. \tag{3.10}$$

Letting $M = (h\pi^2/\rho)((m^2/a^2)\sigma_{0x} + (n^2/b^2)\sigma_{0y})$, $N = (\pi^4 h/16\rho)((E_1 m^4/a^4) + (E_2 n^4/b^4))$ yields

$$\frac{d^2T(t)}{dt^2} + M\cdot T(t) + N\cdot T^3(t) = 0. \tag{3.11}$$

Integrating (3.11) yields

$$\left(\frac{dT(t)}{dt}\right)^2 + M\cdot T^2(t) + \frac{N}{2}\cdot T^4(t) = H. \tag{3.12}$$

In (3.12), the value of H is determined by the initial conditions. Assume that the initial displacement is $T|_{t=0} = T_0$. T_0 is the amplitude of the membrane, so the initial velocity is

$$\left.\frac{dT(t)}{dt}\right|_{t=0} = 0. \tag{3.13}$$

Substituting $T|_{t=0} = T_0$ and (3.13) into (3.12) yields $H = MT_0^2 + (N/2)T_0^4$. Then substituting $H = MT_0^2 + (N/2)T_0^4$ into (3.12) yields

$$\frac{dT(t)}{dt} = T_0^2\sqrt{\frac{M}{T_0^2} + \frac{N}{2}}\cdot\sqrt{\left(1 - \frac{T^2}{T_0^2}\right)\cdot\left(1 + \frac{NT_0^2}{2M + NT_0^2}\cdot\frac{T^2}{T_0^2}\right)}, \tag{3.14}$$

letting $\lambda = T_0^2\sqrt{M/T_0^2 + N/2}$, $k^2 = NT_0^2/(2M + NT_0^2)$.

Integrating (3.11) by the separate variable method, we can obtain the period of the vibration of the membrane:

$$Z = \frac{4}{\lambda}\int_0^{T_0}\left[\left(1 - \frac{T^2}{T_0^2}\right)\left(1 + k^2\cdot\frac{T^2}{T_0^2}\right)\right]^{-1/2} dT. \tag{3.15}$$

Letting $T/T_0 = \sin\theta$, (3.15) can be transformed into

$$Z = \frac{4T_0}{\lambda}\int_0^{\pi/2}\left(1 + k^2\cdot\sin^2\theta\right)^{-1/2} d\theta \quad (0 \le k\cdot\sin\theta \le 1), \tag{3.16}$$

where $(1+k^2\cdot\sin^2\theta)^{-1/2}$ may be spread as a power series with respect to $k\cdot\sin\theta$:

$$(1+k^2\cdot\sin^2\theta)^{-1/2} = 1-\frac{1}{2}k^2\sin^2\theta+\frac{1\cdot 3}{2\cdot 4}k^4\sin^4\theta+\cdots(-1)^n\frac{(2p-1)!!}{(2p)!!}(k\sin\theta)^{2p}. \quad (3.17)$$

Substituting (3.17) into (3.16) then solving (3.16) through integrating item by item [10, 11] yields

$$\begin{aligned} Z &= \frac{4T_0}{\lambda}\int_0^{\pi/2}\left[1-\frac{1}{2}k^2\sin^2\theta+\frac{1\cdot 3}{2\cdot 4}k^4\sin^4\theta+\cdots(-1)^n\frac{(2p-1)!!}{(2p)!!}(k\sin\theta)^{2p}\right]d\theta \\ &= \frac{2\pi T_0}{\lambda}\left[1-\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^2+\cdots+(-1)^p\left(\frac{(2p-1)!!}{(2p)!!}\right)^2k^{2p}\right] \\ &= \frac{2\pi T_0}{\lambda}\sum_{n=0}^{\infty}(-1)^p\left(\frac{(2p-1)!!}{(2p)!!}\right)^2k^{2p}, \end{aligned} \quad (3.18)$$

where $p = 0, 1, 2, 3, \ldots$.

Therefore, the vibration frequency of the membrane is

$$\omega = \frac{2\pi}{Z} = \frac{\sqrt{M+(N/2)T_0^2}}{\sum_{n=0}^{\infty}(-1)^p((2p-1)!!/(2p)!!)^2\left(NT_0^2/(2M+NT_0^2)\right)^p}, \quad (3.19)$$

where $M = (h\pi^2/\rho)((m^2/a^2)\sigma_{0x}+(n^2/b^2)\sigma_{0y})$, $N = (\pi^4 h/16\rho)(E_1m^4/a^4+E_2n^4/b^4)$, and T_0 is the amplitude of the membrane.

In (3.19), letting $T_0 \to 0$, (3.19) can be transformed into the formula obtained according to the small deflection:

$$\omega = \pi\sqrt{\frac{h}{\rho}}\cdot\sqrt{\frac{\sigma_{0x}\cdot m^2}{a^2}+\frac{\sigma_{0y}\cdot n^2}{b^2}}. \quad (3.20)$$

4. Computational Examples and Discussion

Take the membrane material commonly applied in project as an example. Young's moduli in x and y are $E_1 = 1.4\times10^6$ KN/m^2 and $E_2 = 0.9\times10^6$ KN/m^2, respectively; the aerial density of membranes is $\rho = 1.7$ kg/m^2; the membrane's thickness is $h = 1.0$ mm. Calculate the vibration frequency of the membrane according to (3.19).

We can draw the conclusion from the result of Table 1. If we exchange the two lengths of the two orthogonal directions and consider the orthotropic characteristic of the membrane, the result is dissimilar with the one before the exchange; however, if we only exchange the two lengths of the two orthogonal directions and do not consider the orthotropic characteristic, the result is similar with the one before the exchange. Therefore, we need to consider the orthotropic characteristic of membranes in practical engineering.

We can draw the conclusion from the result of Table 2. The initial displacement (the amplitude) has influenced the vibration frequency of the rectangle membrane when

Table 1: $T_0 = 0.1\,\text{m}$, $\sigma_{0x} = \sigma_{0y} = 5.0 \times 10^3\,\text{KN/m}^2$, and $m = n = 1$.

M	$a = 1$, $b = 1$	$a = 1$, $b = 2$	$a = 2$, $b = 1$	$a = 1$, $b = 3$	$a = 3$, $b = 1$	$a = 1$, $b = 4$	$a = 4$, $b = 1$	$a = 1$, $b = 5$	$a = 5$, $b = 1$
Rad/s	344.231	273.032	249.679	245.268	237.578	260.131	233.909	258.752	232.316
Hz	54.786	43.455	39.738	39.036	37.812	41.401	37.227	41.182	36.974

Table 2: $a = 1\,\text{m}$, $b = 1\,\text{m}$, $\sigma_{0x} = \sigma_{0y} = 5.0 \times 10^3\,\text{KN/m}^2$, and $m = n = 1$.

M	$T_0 = 0.2$	$T_0 = 0.15$	$T_0 = 0.10$	$T_0 = 0.05$	$T_0 = 0.01$	$T_0 = 0.005$	$T_0 = 0.001$	$T_0 = 0.0001$	$T_0 \to 0$
Rad/s	544.651	439.21	344.231	270.86	242.227	286.325	240.962	240.949	240.948
Hz	86.684	69.902	54.786	43.109	38.552	45.570	38.350	38.348	38.348

Table 3: $a = 1\,\text{m}$, $b = 1\,\text{m}$, $\sigma_{0x} = \sigma_{0y} = 5.0 \times 10^3\,\text{KN/m}^2$, and $T_0 = 0.10\,\text{m}$.

Order	$m = 1, n = 1$	$m = 1, n = 2$	$m = 2, n = 1$	$m = 2, n = 2$	$m = 1, n = 3$	$m = 3, n = 1$	$m = 3, n = 3$
Rad/s	344.231	745.731	865.641	1089.30	1486.92	1799.04	2308.23
Hz	54.786	118.687	137.771	173.368	236.650	286.326	367.367

Table 4: $a = 1\,\text{m}$, $b = 1\,\text{m}$, $\sigma_{0x} = \sigma_{0y} = 5.0 \times 10^3\,\text{KN/m}^2$, and $T_0 = 0.05\,\text{m}$.

Order	$m = 1, n = 1$	$m = 1, n = 2$	$m = 2, n = 1$	$m = 2, n = 2$	$m = 1, n = 3$	$m = 3, n = 1$	$m = 3, n = 3$
Rad/s	270.86	499.358	546.065	688.462	880.846	1016.96	1317.63
Hz	43.109	79.475	86.909	109.572	140.191	161.854	209.707

Table 5: $a = 1\,\text{m}$, $b = 1\,\text{m}$, $\sigma_{0x} = \sigma_{0y} = 5.0 \times 10^3\,\text{KN/m}^2$, and $T_0 = 0.01\,\text{m}$.

Order	$m = 1, n = 1$	$m = 1, n = 2$	$m = 2, n = 1$	$m = 2, n = 2$	$m = 1, n = 3$	$m = 3, n = 1$	$m = 3, n = 3$
Rad/s	242.227	386.497	389.087	492.030	556.943	566.445	756.548
Hz	38.552	61.513	61.925	78.309	88.640	90.153	120.408

Table 6: $a = 1\,\text{m}$, $b = 1\,\text{m}$, and $\sigma_{0x} = \sigma_{0y} = 5.0 \times 10^3\,\text{KN/m}^2$.

Order	$m = 1, n = 1$	$m = 1, n = 2$	$m = 2, n = 1$	$m = 2, n = 2$	$m = 1, n = 3$	$m = 3, n = 1$	$m = 3, n = 3$
Rad/s	240.949	380.974	380.974	481.898	538.779	538.779	722.847
Hz	38.348	60.634	60.634	76.697	85.749	85.749	115.045

calculated according to the large deflection theory. The frequency enlarged with the increase of the initial displacement, and the larger the initial displacement is, the larger the effect on the frequency is and vice versa. When the initial displacement approaches zero, the result is consistent with that obtained according to the small deflection theory.

The results from Table 3 to Table 6 are plotted as shown in Figure 2.

We can draw conclusions from the analysis of Figure 2. While order keeps unchanged, all frequency results based on the large deflection theory are larger than the corresponding ones based on the small deflection theory. The larger the initial displacement is, the larger the frequency is and vice versa. Specially, the smaller the initial displacement is, the closer the two results based on large and small deflection theories are.

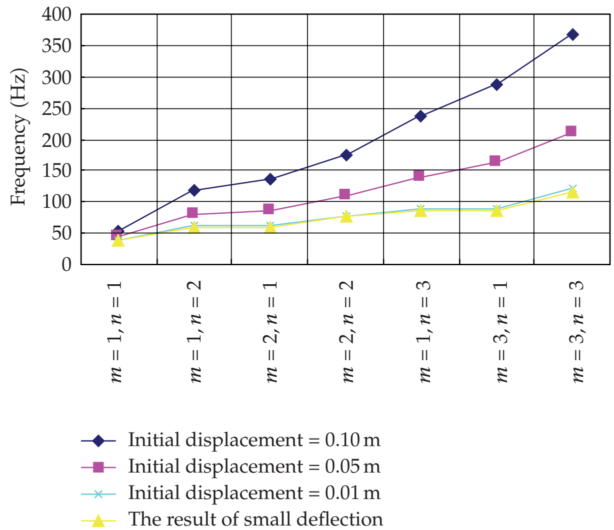

Figure 2: Comparative analysis of the results based on the large deflection theory and the small one.

The above conclusions are analyzed as follows.

(i) We considered the rigidity change caused by geometrical large deflection of membrane when calculating the frequency according to the large deflection theory. When the lateral displacement of membrane increased, the inner force increased, and the lateral rigidity also increased; then the vibration was quickened. Therefore, the membrane's vibration frequency will enlarge with the increase of initial displacement.

(ii) When initial displacement of membrane is very little, the change of lateral rigidity is also very little in the process of vibration, so it is negligible. In this case, the computational result based on the large deflection theory is very close to that based on the small deflection theory, which considers no changes of the lateral rigidity in the process of vibration.

5. Conclusions

(i) We applied the large deflection theory of membranes and D'Alembert's principle derived the governing vibration equations of rectangular membrane with four edges fixed, solved it, and obtained the power series formula of nonlinear vibration frequency of rectangular membranes with four edges fixed.

(ii) The frequency formula obtained in this paper is dependent on the initial conditions, as we considered the change of the lateral rigidity in the process of vibration. Therefore, the formula has reflected the geometric nonlinear characteristic of membrane structure's vibration. This is more tally with the actual situation and more reasonable than the result that is calculated according to the small deflection theory.

(iii) The nonlinear governing equation and the power series formula obtained in this paper provide some theoretical foundation for the measurement of pretension by frequency method; meanwhile, the results provide some theoretical foundation for the research of nonlinear vibration of membrane structures and the response solving of membrane structures under dynamic loads.

According to (3.20) (the frequency formula of the small deflection theory), by calculating the vibration frequency of the membrane, the results are listed in Table 6.

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