Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 653723, 15 pages
doi:10.1155/2009/653723

Research Article

Pursuit-Evasion Differential Game with Many Inertial Players

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Received 18 April 2009; Accepted 17 July 2009

Recommended by Alexander P. Seyranian

We consider pursuit-evasion differential game of countable number inertial players in Hilbert space with integral constraints on the control functions of players. Duration of the game is fixed. The payoff functional is the greatest lower bound of distances between the pursuers and evader when the game is terminated. The pursuers try to minimize the functional, and the evader tries to maximize it. In this paper, we find the value of the game and construct optimal strategies of the players.

1. Introduction and Preliminaries

Many books have been devoted to differential games, such as books by Isaacs [1], Pontryagin [2], Friedman [3], Krasovskii and Subbotin [4].

Constructing the player's optimal strategies and finding the value of the game are of specific interest in studying of differential games.

The pursuit-evasion differential games involving several objects with simple motions take the attention of many authors. Ivanov and Ledyaev [5] studied simple motion differential game of several players with geometric constraints. They obtained sufficient conditions to find optimal pursuit time in $\mathbb{R}^n$, by using the method of the Lyapunov function for an auxiliary problem.

Levchenkov and Pashkov [6] investigated differential game of optimal approach of two identical inertial pursuers to a noninertial evader on a fixed time interval. Control parameters were subject to geometric constraints. They constructed the value function of the game and used necessary and sufficient conditions which a function must satisfy to be the value function [7].

Chodun [8] examined evasion differential game with many pursuers and geometric constraints. He found a sufficient condition for avoidance.

Ibragimov [9] obtained the formula for optimal pursuit time in differential game described by an infinite system of differential equations. In [10] simple motion differential game of many pursuers with geometric constraints was investigated in the Hilbert space l_2.

In the present paper, we consider a pursuit-evasion differential game of infinitely many inertial players with integral constraints on control functions. The duration of the game θ is fixed. The payoff functional of the game is the greatest lower bound of the distances between the evader and the pursuers at θ. The pursuer's goal is to minimize the payoff, and the evader's goal is to maximize it. This paper is close in spirit to [10]. We obtain a sufficient condition to find the value of the game and constructed the optimal strategies of players.

2. Formulation of the Problem

In the space l_2 consisting of elements $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k, \dots)$, with $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, and inner product $(\alpha, \beta) = \sum_{k=1}^{\infty} \alpha_k \beta_k$, the motions of the countably many pursuers P_i and the evader E are defined by the equations

$$\begin{aligned} P_i &: \ddot{x}_i = u_i, \qquad x_i(0) = x_i^0, \qquad \dot{x}_i(0) = x_i^1, \\ E &: \ddot{y} = v, \qquad y(0) = y^0, \qquad \dot{y}(0) = y^1, \end{aligned} \tag{2.1}$$

where $x_i, x_i^0, x_i^1, u_i, y, y^0, y^1, v \in l_2$, $u_i = (u_{i1}, u_{i2}, \dots, u_{ik}, \dots)$ is the control parameter of the pursuer P_i, and $v = (v_1, v_2, \dots, v_k, \dots)$ is that of the evader E; here and throughout the following, $i = 1, 2, \dots, m, \dots$. Let θ be a given positive number, and let $I = \{1, 2, \dots, m, \dots\}$.

As a real life example, one may consider the case of a missile catching an aircraft. If the initial positions and speeds (first derivative) of both missile and aircraft are given and the constraints of both missile and aircraft are their available fuel, which could be mathematically interpreted as the mean average of their acceleration function (second derivative), then the corresponding pursuit-evasion problem is described by (2.1).

A ball (resp., sphere) of radius r and center at the point x_0 is denoted by $H(x_0, r) = \{x \in l_2 : \|x - x_0\| \le r\}$ (resp., by $S(x_0, r) = \{x \in l_2 : \|x - x_0\| = r\}$).

Definition 2.1. A function $u_i(\cdot)$, $u_i : [0, \theta] \to l_2$, such that $u_{ik} : [0, \theta] \to R^1$, $k = 1, 2, \dots$, are Borel measurable functions and

$$\|u_i(\cdot)\|_2 = \left(\int_0^{\theta} \|u_i(s)\|^2 \, ds \right)^{1/2} \le \rho_i, \qquad \|u_i\| = \left(\sum_{k=1}^{\infty} u_{ik}^2 \right)^{1/2}, \tag{2.2}$$

where ρ_i is given positive number, is called an *admissible control of the ith pursuer*.

Definition 2.2. A function $v(\cdot)$, $v : [0, \theta] \to l_2$, such that $v_k : [0, \theta] \to R^1$, $k = 1, 2, \dots$, are Borel measurable functions and

$$\|v(\cdot)\|_2 = \left(\int_0^{\theta} \|v(s)\|^2 \, ds \right)^{1/2} \le \sigma, \tag{2.3}$$

where σ is a given positive number, is called an *admissible control of the evader*.

Once the players' admissible controls $u_i(\cdot)$ and $v(\cdot)$ are chosen, the corresponding motions $x_i(\cdot)$ and $y(\cdot)$ of the players are defined as

$$x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{ik}(t), \ldots), \qquad y(t) = (y_1(t), y_2(t), \ldots, y_k(t), \ldots),$$
$$x_{ik}(t) = x_{ik}^0 + t x_{ik}^1 + \int_0^t \int_0^s u_{ik}(r)\, dr\, ds, \qquad y_k(t) = y_k^0 + t y_k^1 + \int_0^t \int_0^s v_k(r)\, dr\, ds. \tag{2.4}$$

One can readily see that $x_i(\cdot), y(\cdot) \in C(0,\theta; l_2)$, where $C(0,\theta; l_2)$ is the space of functions

$$f(t) = (f_1(t), f_2(t), \ldots, f_k(t), \ldots) \in l_2, \quad t \geq 0, \tag{2.5}$$

such that the following conditions hold:

(1) $f_k(t),\ 0 \leq t \leq \theta,\ k = 1, 2, \ldots,$ are absolutely continuous functions;

(2) $f(t),\ 0 \leq t \leq \theta,$ is a continuous function in the norm of l_2.

Definition 2.3. A function $U_i(t, x_i, y, v),\ U_i : [0, \infty) \times l_2 \times l_2 \times l_2 \to l_2$, such that the system

$$\ddot{x}_i = U_i(t, x_i, y, v), \qquad x_i(0) = x_i^0, \qquad \dot{x}_i(0) = x_i^1,$$
$$\ddot{y} = v, \qquad y(0) = y^0, \qquad \dot{y}(0) = y^1, \tag{2.6}$$

has a unique solution $(x_i(\cdot), y(\cdot))$, with $x_i(\cdot), y(\cdot) \in C(0,\theta; l_2)$, for an arbitrary admissible control $v = v(t),\ 0 \leq t \leq \theta$, of the evader E, is called a *strategy of the pursuer*P_i. A strategy U_i is said to be *admissible* if each control formed by this strategy is admissible.

Definition 2.4. Strategies U_{i0} of the pursuers P_i are said to be *optimal* if

$$\inf_{U_1,\ldots,U_m,\ldots} \Gamma_1(U_1, \ldots, U_m, \ldots) = \Gamma_1(U_{10}, \ldots, U_{m0}, \ldots), \tag{2.7}$$

where $\Gamma_1(U_1, \ldots, U_m, \ldots) = \sup_{v(\cdot)} \inf_{i \in I} \|x_i(\theta) - y(\theta)\|$, U_i are admissible strategies of the pursuers P_i, and $v(\cdot)$ is an admissible control of the evader E.

Definition 2.5. A function $V(t, x_1, \ldots, x_m, \ldots, y), V : [0, \infty) \times l_2 \times \cdots \times l_2 \times \cdots \times l_2 \to l_2$, such that the countable system of equations

$$\ddot{x}_k = u_k, \quad x_k(0) = x_k^0, \quad \dot{x}_k(0) = x_k^1, \quad k = 1, 2, \ldots, m, \ldots,$$
$$\ddot{y} = V(t, x_1, \ldots, x_m, \ldots, y), \qquad y(0) = y^0, \qquad \dot{y}(0) = y^1, \tag{2.8}$$

has a unique solution $(x_1(\cdot), \ldots, x_m(\cdot), \ldots, y(\cdot))$, with $x_i(\cdot), y(\cdot) \in C(0, \theta, l_2)$, for arbitrary admissible controls $u_i = u_i(t),\ 0 \leq t \leq \theta$, of the pursuers P_i, is called a *strategy* of the evader E. If each control formed by a strategy V is admissible, then the strategy V itself is said to be *admissible*.

Definition 2.6. A strategy V_0 of the evader E is said to be *optimal* if $\sup_V \Gamma_2(V) = \Gamma_2(V_0)$, where $\Gamma_2(V) = \inf_{u_1(\cdot),\dots,u_m(\cdot),\dots} \inf_{i\in I} \|x_i(\theta) - y(\theta)\|$, where $u_i(\cdot)$ are admissible controls of the pursuers P_i, and V is an admissible strategy of the evader E.

If $\Gamma_1(U_{10},\dots,U_{m0},\dots) = \Gamma_2(V_0) = \gamma$, then we say that the game has the value γ [7].

It is to find optimal strategies U_{i0} and V_0 of the players P_i and E, respectively, and the value of the game. Instead of differential game described by (2.1) we can consider an equivalent differential game with the same payoff function and described by the following system:

$$\begin{aligned} P_i : \dot{x}_i(t) &= (\theta - t)u_i(t), \qquad x_i(0) = x_{i0} = x_i^1\theta + x_i^0, \\ E : \dot{y}(t) &= (\theta - t)v(t), \qquad y(0) = y_0 = y^1\theta + y^0. \end{aligned} \tag{2.9}$$

Indeed, if the pursuer P_i uses an admissible control $u_i(t) = (u_{i1}(t), u_{i2}(t), \dots)$, $0 \le t \le \theta$, then according to (2.1) we have

$$x_i(\theta) = x_i^0 + x_i^1\theta + \int_0^\theta \int_0^t u_i(s)\,ds\,dt = x_i^0 + x_i^1\theta + \int_0^\theta (\theta - t)u_i(t)\,dt, \tag{2.10}$$

and the same result can be obtained by (2.9)

$$x_i(\theta) = x_{i0} + \int_0^\theta (\theta - t)u_i(t)\,dt = x_i^0 + x_i^1\theta + \int_0^\theta (\theta - t)u_i(t)\,dt. \tag{2.11}$$

Also, for the evader the same argument can be made, therefore in the distance$\|x_i(\theta) - y(\theta)\|$ we can take either the solution of (2.1) or the solution of (2.9).

The attainability domain of the pursuer P_i at time θ from the initial state x_{i0} at time $t_0 = 0$ is the closed ball $H(x_{i0}, \rho_i(\theta^3/3)^{1/2})$. Indeed, by Cauchy-Schwartz inequality

$$\begin{aligned} \|x_i(\theta) - x_{i0}\| &= \left\| \int_0^\theta (\theta - s)u_i(s)\,ds \right\| \\ &\le \int_0^\theta (\theta - s)\|u_i(s)\|\,ds \\ &\le \left(\int_0^\theta (\theta - s)^2\,ds \right)^{1/2} \cdot \left(\int_0^\theta \|u_i(s)\|^2\,ds \right)^{1/2} \le \rho_i \left(\frac{\theta^3}{3} \right)^{1/2}. \end{aligned} \tag{2.12}$$

On the other hand, if $\overline{x} \in H(x_{i0}, \rho_i(\theta^3/3)^{1/2})$, that is, $\|\overline{x} - x_{i0}\| \le \rho_i(\theta^3/3)^{1/2}$, then for the pursuer's control

$$u_i(t) = \frac{3(\theta - t)}{\theta^3}(\overline{x} - x_{i0}), \quad 0 \le t \le \theta, \tag{2.13}$$

we obtain $x_i(\theta) = \overline{x}$.

The pursuer's control is admissible because

$$\int_0^\theta \|u_i(t)\|^2\,dt = \left(\frac{3}{\theta^3}\right)^2 \frac{\theta^3}{3}\|\overline{x} - x_{i0}\|^2 \le \left(\rho_i\left(\frac{\theta^3}{3}\right)^{1/2}\right)^2 \cdot \frac{3}{\theta^3} = \rho_i^2. \tag{2.14}$$

Likewise, the attainability domain of the evader E at time θ from the initial state y_0 at time $t_0 = 0$ is the closed ball $H(y_0, \sigma(\theta^3/3)^{1/2})$.

3. An Auxiliary Game

In this section we fix the index i and study an auxiliary differential game of two players P_i and E, also for simplicity we drop the index i and use the notion $\rho_i = \rho, x_{i0} = x_0$ and $x_i = x$. Let

$$X = \left\{ z \in l_2 : 2(y_0 - x_0, z) \le \frac{\theta^3}{3}\left(\rho^2 - \sigma^2\right) + \|y_0\|^2 - \|x_0\|^2 \right\}, \quad \rho \ge \sigma, \tag{3.1}$$

if $x_0 \ne y_0$; if $x_0 = y_0$, then

$$X = \left\{ z \in l_2 : (p, z - y_0) \le \rho\left(\frac{\theta^3}{3}\right)^{1/2} \right\}, \tag{3.2}$$

where p is an arbitrary fixed unit vector.

Consider the one-pursuer game described by the equations

$$\begin{aligned} &P : \dot{x} = (\theta - t)u(t), \qquad x(0) = x_0, \\ &E : \dot{y} = (\theta - t)v(t), \qquad y(0) = y_0, \end{aligned} \tag{3.3}$$

with the state of the evader E being subject to $y(\theta) \in X$. The goal of the pursuer P is to realize the equality $x(\tau) = y(\tau)$ at some τ, $0 \le \tau \le \theta$, and that of the evader E is opposite.

We define the pursuer's strategy as follows: if $x_0 = y_0$, then we set

$$u(t) = v(t), \quad 0 \le t \le \theta, \tag{3.4}$$

and if $x_0 \ne y_0$, then we set

$$u(t) = v(t) - (v(t), e)e + e\left(\frac{3}{\theta^3}(\theta - t)^2\left(\rho^2 - \sigma^2\right) + (v(t), e)^2\right)^{1/2}, \quad 0 \le t \le \tau, \tag{3.5}$$

where $e = (y_0 - x_0)/\|y_0 - x_0\|$, and

$$u(t) = v(t), \quad \tau < t \le \theta, \tag{3.6}$$

where τ, $0 \le \tau \le \theta$, is the time instant at which $x(\tau) = y(\tau)$ for the first time.

Lemma 3.1. *If $\sigma \le \rho$ and $y(\theta) \in X$, then the pursuer's strategy* (3.4), (3.5), *and* (3.6) *in the game* (3.3) *ensures that $x(\theta) = y(\theta)$.*

Proof. If $x_0 = y_0$, then from (3.4) we have $x(t) = y(t),\ 0 \le t \le \theta$, because

$$\begin{aligned} x(t) &= x_0 + \int_0^t (\theta - s) u(s)\, ds \\ &= y_0 + \int_0^t (\theta - s) v(s)\, ds = y(t). \end{aligned} \tag{3.7}$$

In particular, $x(\theta) = y(\theta)$.

Let $x_0 \ne y_0$. By (3.5) and (3.6), we have $y(t) - x(t) = ef(t)$, where

$$\begin{aligned} f(t) = {} & \|y_0 - x_0\| + \int_0^t (\theta - s)(v(s), e)\, ds \\ & - \int_0^t (\theta - s) \left(\frac{3}{\theta^3} (\theta - s)^2 \left(\rho^2 - \sigma^2\right) + (v(s), e)^2 \right)^{1/2} ds. \end{aligned} \tag{3.8}$$

Obviously $f(0) = \|y_0 - x_0\| > 0$. Now we show that $f(\theta) \le 0$. This will imply that $f(\tau) = 0$ for some $\tau \in [0, \theta]$.

To this end we consider the following two-dimensional vector function:

$$g(t) = \left(\left(\frac{3}{\theta^3} \right)^{1/2} (\theta - t)^2 \left(\rho^2 - \sigma^2\right)^{1/2}, (\theta - t)(v(t), e) \right), \quad 0 \le t \le \theta. \tag{3.9}$$

For the last integral of (3.8) we have

$$\begin{aligned} & \int_0^\theta (\theta - s) \left(\frac{3}{\theta^3} (\theta - s)^2 \left(\rho^2 - \sigma^2\right) + (v(s), e)^2 \right)^{1/2} ds \\ & \quad = \int_0^\theta \left(\frac{3}{\theta^3} (\theta - s)^4 \left(\rho^2 - \sigma^2\right) + (\theta - s)^2 (v(s), e)^2 \right)^{1/2} ds \\ & \quad = \int_0^\theta |g(s)|\, ds \\ & \quad \ge \left| \int_0^\theta g(s)\, ds \right| = \left| \left(\int_0^\theta \left(\frac{3}{\theta^3} \right)^{1/2} (\theta - s)^2 \left(\rho^2 - \sigma^2\right)^{1/2} ds, \int_0^\theta (\theta - s)(v(s), e)\, ds \right) \right| \\ & \quad = \left| \left(\left(\frac{\theta^3}{3} \right)^{1/2} \left(\rho^2 - \sigma^2\right)^{1/2}, \int_0^\theta (\theta - s)(v(s), e)\, ds \right) \right| \\ & \quad = \left(\frac{\theta^3}{3} \left(\rho^2 - \sigma^2\right) + \left(\int_0^\theta (\theta - s)(v(s), e)\, ds \right)^2 \right)^{1/2}. \end{aligned} \tag{3.10}$$

Then

$$f(\theta) \leq \|y_0 - x_0\| + \int_0^\theta (\theta - s)(v(s), e)\, ds - \left(\frac{\theta^3}{3}\left(\rho^2 - \sigma^2\right) + \left(\int_0^\theta (\theta - s)(v(s), e) ds \right)^2 \right)^{1/2}. \tag{3.11}$$

By assumption, $y(\theta) \in X$, therefore

$$2(y_0 - x_0, y(\theta)) \leq \frac{\theta^3}{3}\left(\rho^2 - \sigma^2\right) + \|y_0\|^2 - \|x_0\|^2, \tag{3.12}$$

so $(e, y(\theta)) \leq d$, where

$$d = \frac{(\theta^3/3)(\rho^2 - \sigma^2) + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|}. \tag{3.13}$$

As $(e, y(\theta)) = (e, y_0 + \int_0^\theta (\theta - s) v(s) ds) \leq d$, then we obtain

$$\int_0^\theta (\theta - s)(v(s), e)\, ds \leq d - (y_0, e). \tag{3.14}$$

On the other hand, $\varphi(t) = \|y_0 - x_0\| + t - ((\theta^3/3)(\rho^2 - \sigma^2) + t^2)^{1/2}$ is an increasing function on $(-\infty, \infty)$. Then it follows from (3.11) and (3.14) that

$$f(\theta) \leq \|y_0 - x_0\| + d - (y_0, e) - \left(\frac{\theta^3}{3}\left(\rho^2 - \sigma^2\right) + (d - (y_0, e))^2 \right)^{1/2}. \tag{3.15}$$

Now we show that the right-hand side of the last inequality is equal to zero. We show

$$\|y_0 - x_0\| + d - (y_0, e) = \left(\frac{\theta^3}{3}\left(\rho^2 - \sigma^2\right) + (d - (y_0, e))^2 \right)^{1/2}. \tag{3.16}$$

The left part of this equality is positive, since

$$\begin{aligned}
&\|y_0 - x_0\| + d - (y_0, e) \\
&\quad = \frac{(\theta^3/3)(\rho^2 - \sigma^2) + \|y_0\|^2 - \|x_0\|^2 + 2\left(\|x_0\|^2 - 2(x_0, y_0) + \|y_0\|^2\right) - 2\|y_0\|^2 + 2(x_0, y_0)}{2\|y_0 - x_0\|} \\
&\quad = \frac{(\theta^3/3)(\rho^2 - \sigma^2) + \|x_0\|^2 - 2(x_0, y_0) + \|y_0\|^2}{2\|y_0 - x_0\|} \\
&\quad = \frac{(\theta^3/3)(\rho^2 - \sigma^2) + \|x_0 - y_0\|^2}{2\|y_0 - x_0\|} > 0.
\end{aligned} \tag{3.17}$$

Therefore taking square we have

$$\|y_0 - x_0\|^2 + (d - (y_0, e))^2 + 2\|y_0 - x_0\|(d - (y_0, e)) = \frac{\theta^3}{3}\left(\rho^2 - \sigma^2\right) + (d - (y_0, e))^2, \tag{3.18}$$

then

$$\|y_0 - x_0\|^2 + 2\|y_0 - x_0\|\left(\frac{(\theta^3/3)(\rho^2 - \sigma^2) + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|} - (y_0, e)\right) = \frac{\theta^3}{3}\left(\rho^2 - \sigma^2\right). \tag{3.19}$$

The above equality is true since

$$\|y_0 - x_0\|^2 + \|y_0\|^2 - \|x_0\|^2 - 2\|y_0 - x_0\|\|y_0\| = 0. \tag{3.20}$$

So $f(\theta) = 0$, consequently $f(\tau) = 0$ for some $\tau,\ 0 \le \tau \le \theta$. Therefore, $x(\tau) = y(\tau)$. Further, by (3.6), $u(t) = v(t)$ at $\tau < t \le \theta$. Then

$$\begin{aligned}
x(\theta) &= x(\tau) + \int_\tau^\theta (\theta - s)u(s)ds \\
&= y(\tau) + \int_\tau^\theta (\theta - s)v(s)ds = y(\theta),
\end{aligned} \tag{3.21}$$

and the proof of the lemma is complete. □

4. Main Result

Now consider the game (2.9). We will solve the optimal pursuit problem under the following assumption.

Assumption 4.1. *There exists a nonzero vector p_0 such that $(y_0 - x_{i0}, p_0) \geq 0$ for all $i \in I$.*

Let

$$\gamma = \inf\left\{ l \geq 0 : H\left(y_0, \sigma\left(\frac{\theta^3}{3}\right)^{1/2} \right) \subset \bigcup_{i=1}^{\infty} H\left(x_{i0}, \rho_i\left(\frac{\theta^3}{3}\right)^{1/2} + l \right) \right\}. \quad (4.1)$$

Theorem 4.2. *If Assumption 4.1 is true and $\sigma \leq \rho_i + \gamma(3/\theta^3)^{1/2}$ for all $i \in I$, then the number γ given by* (4.1) *is the value of the game* (2.9).

Proof of the above theorem relies on the following lemmas.

Consider the sphere $S(y_0, r)$ and finitely or countably many balls $H(x_{i0}, R_i)$ and $H(y_0, r)$, where $x_{i0} \neq y_0$ and r and R_i, $i \in I$ are positive numbers.

Lemma 4.3 (see [10]). *Let*

$$X_i = \left\{ z \in l_2 : 2(y_0 - x_{i0}, z) \leq R_i^2 - r^2 + \|y_0\|^2 - \|x_{i0}\|^2 \right\}, \quad (4.2)$$

if $x_{i0} \neq y_0$, and

$$X_i = \{ z \in l_2 : (z - y_0, p_0) \leq R_i \}, \quad (4.3)$$

if $x_{i0} = y_0$. If Assumption 4.1 is valid and

$$H(y_0, r) \subset \bigcup_{i \in I} H(x_{i0}, R_i), \quad (4.4)$$

then $H(y_0, r) \subset \bigcup_{i \in I} X_i$.

Lemma 4.4 (see [10]). *Let $\inf_{i \in I} R_i = R_0 > 0$. If Assumption 4.1 is true and for any $0 < \varepsilon < R_0$ the set $\bigcup_{i \in I} H(x_{i0}, R_i - \varepsilon)$ does not contain the ball $H(y_0, r)$, then there exists a point $\overline{y} \in S(y_0, r)$ such that $\|\overline{y} - x_{i0}\| \geq R_i$ for all $i \in I$.*

Proof of Theorem 4.2. We prove this theorem in three parts.

(1)*Construction of the Pursuers' Strategies*. We introduce counterfeit pursuers z_i, whose motions are described by the equations

$$\begin{gathered} \dot{z}_i = (\theta - t) w_i^{\varepsilon}, \qquad z_i(0) = x_{i0}, \\ \left(\int_0^{\theta} \|w_i^{\varepsilon}(s)\|^2 \, ds \right)^{1/2} \leq \overline{\rho}_i(\varepsilon) = \rho_i + \gamma\left(\frac{3}{\theta^3}\right)^{1/2} + \frac{\varepsilon}{k_i}\left(\frac{3}{\theta^3}\right)^{1/2}, \end{gathered} \quad (4.5)$$

where $k_i = \max\{1, \rho_i\}$ and ε, $0 < \varepsilon < 1$, is an arbitrary positive number. It is obvious that the attainability domain of the counterfeit pursuer z_i at time θ from an initial state x_{i0} is the ball $H(x_{i0}, \overline{\rho}_i(\varepsilon)(\theta^3/3)^{1/2}) = H(x_{i0}, \rho_i(\theta^3/3)^{1/2} + \gamma + \varepsilon/k_i)$.

The strategies of the counterfeit pursuers z_i are defined as follows: if $x_{i0} = y_0$, then we set

$$w_i^{\varepsilon}(t) = v(t), \quad 0 \le t \le \theta, \tag{4.6}$$

and if $x_{i0} \ne y_0$, then we set

$$w_i^{\varepsilon}(t) = v(t) - (v(t), e_i)e_i + e_i\left(\frac{3}{\theta^3}(\theta - t)^2\left(\overline{\rho}_i^2(\varepsilon) - \sigma^2\right) + (v(t), e_i)^2\right)^{1/2}, \quad 0 \le t \le \tau_i, \tag{4.7}$$

where $e_i = (y_0 - x_{i0})/\|y_0 - x_{i0}\|$, and

$$w_i^{\varepsilon}(t) = v(t), \quad \tau_i < t \le \theta, \tag{4.8}$$

where τ_i, $0 \le \tau_i \le \theta$, is the time instant at which $z_i(\tau_i) = y(\tau_i)$ for the first time if it exists. Note that τ_i need not to exist in $[0, \theta]$.

Now let us show that the strategies (4.6), (4.7), and (4.8) are admissible. If $x_{i0} = y_0$ and $0 \le t \le \theta$, then

$$\begin{aligned}\int_0^{\theta} \left\|w_i^{\varepsilon}(s)\right\|^2 ds &= \int_0^{\theta} \|v(s)\|^2 ds \le \sigma^2 \\ &\le \left(\rho_i + \gamma\left(\frac{3}{\theta^3}\right)^{1/2}\right)^2 \\ &\le \left(\rho_i + \gamma\left(\frac{3}{\theta^3}\right)^{1/2} + \frac{\varepsilon}{k_i}\left(\frac{3}{\theta^3}\right)^{1/2}\right)^2 = \overline{\rho}_i^2(\varepsilon).\end{aligned} \tag{4.9}$$

If $x_{i0} \ne y_0$ we have

$$\begin{aligned}\int_0^{\theta} \left\|w_i^{\varepsilon}(s)\right\|^2 ds &= \int_0^{\tau_i} \left\|w_i^{\varepsilon}(s)\right\|^2 ds + \int_{\tau_i}^{\theta} \left\|w_i^{\varepsilon}(s)\right\|^2 ds \\ &= \int_0^{\tau_i} \|v(s)\|^2 ds + \frac{3}{\theta^3}\left(\overline{\rho}_i^2(\varepsilon) - \sigma^2\right)\int_0^{\tau_i} (\theta - s)^2 ds + \int_{\tau_i}^{\theta} \|v(s)\|^2 ds \\ &\le \int_0^{\theta} \|v(s)\|^2 ds + \frac{3}{\theta^3}\left(\overline{\rho}_i^2(\varepsilon) - \sigma^2\right)\int_0^{\theta} (\theta - s)^2 ds \le \sigma^2 + \overline{\rho}_i^2(\varepsilon) - \sigma^2 = \overline{\rho}_i^2(\varepsilon).\end{aligned} \tag{4.10}$$

The strategies of the pursuers x_i are defined as follows:

$$u_i(t) = \frac{\rho_i}{\overline{\rho}_i} w_i(t), \quad 0 \le t \le \theta, \tag{4.11}$$

where $\overline{\rho}_i = \overline{\rho}_i(0) = \rho_i + \gamma(3/\theta^3)^{1/2}$ and $w_i(t) = w_i^0(t)$; that is, $w_i(t)$ is given by (4.6), (4.7), and (4.8) with $\varepsilon = 0$ and the same τ_i.

(2)*The value γ is guaranteed for the pursuers*. Let us show that the above-constructed strategies of the pursuers satisfy the inequalities

$$\sup_{v(\cdot)} \inf_{i\in I} \|y(\theta) - x_i(\theta)\| \leq \gamma. \tag{4.12}$$

By the definition of γ, we have

$$H\left(y_0, \sigma\left(\frac{\theta^3}{3}\right)^{1/2}\right) \subset \bigcup_{i=1}^{\infty} H\left(x_{i0}, \rho_i\left(\frac{\theta^3}{3}\right)^{1/2} + \gamma + \frac{\varepsilon}{k_i}\right). \tag{4.13}$$

By Assumption 4.1 the inequality $(y_0 - x_{i0}, p_0) \geq 0$ holds for all $i \in I$. Then it follows from Lemma 4.3 that

$$H\left(y_0, \sigma\left(\frac{\theta^3}{3}\right)^{1/2}\right) \subset \bigcup_{i=1}^{\infty} X_i^{\varepsilon}, \tag{4.14}$$

where

$$X_i^{\varepsilon} = \left\{ z : 2(y_0 - x_{i0}, z) \leq \left(\rho_i\left(\frac{\theta^3}{3}\right)^{1/2} + \gamma + \frac{\varepsilon}{k_i}\right)^2 - \sigma^2\frac{\theta^3}{3} + \|y_0\|^2 - \|x_{i0}\|^2 \right\}, \tag{4.15}$$

if $x_{i0} \neq y_0$, and

$$X_i^{\varepsilon} = \left\{ z : (z - y_0, p_0) \leq \rho_i\left(\frac{\theta^3}{3}\right)^{1/2} + \gamma + \frac{\varepsilon}{k_i} \right\}, \tag{4.16}$$

if $x_{i0} = y_0$. Consequently, the point $y(\theta) \in H(y_0, \sigma(\theta^3/3)^{1/2})$ belongs to some half-space X_s^{ε}, $s = s(\varepsilon) \in I$.

By the assumption of the theorem, $\overline{\rho}_i(\varepsilon) > \sigma$; then it follows from Lemma 3.1 that if z_i uses the strategy (4.6), (4.7), and (4.8), then $z_s(\theta) = y(\theta)$. By taking account of (4.11) we obtain

$$\begin{aligned}
\|y(\theta) - x_s(\theta)\| &= \|z_s(\theta) - x_s(\theta)\| \\
&= \left\| \int_0^{\theta} (\theta - t)\left(w_s^{\varepsilon}(t) - \frac{\rho_s}{\overline{\rho}_s} w_s(t)\right) dt \right\| \\
&\leq \int_0^{\theta} \|(\theta - t)(w_s^{\varepsilon}(t) - w_s(t))\| \, dt + \int_0^{\theta} \left\|(\theta - t)\left(w_s(t) - \frac{\rho_s}{\overline{\rho}_s} w_s(t)\right)\right\| dt.
\end{aligned} \tag{4.17}$$

Now we put aside the right-hand side of the last inequality. Let us show that

$$\lim_{\varepsilon\to 0}\sup_{i\in I}\int_0^{\theta}\left\|(\theta-t)\left(w_i^{\varepsilon}(t)-w_i(t)\right)\right\|dt=0. \tag{4.18}$$

Indeed, if $x_{i0}=y_0$, then by (4.6), $w_i^{\varepsilon}(t)=w_i(t)=v(t)$, and the validity of (4.18) is obvious. Now let $x_{i0}\neq y_0$. If there exists $\tau_i\in[0,\theta]$, mentioned in (4.7) and (4.8), then

$$\begin{aligned}
\int_0^{\tau_i}\left\|w_i^{\varepsilon}(t)-w_i(t)\right\|^2dt &= \int_0^{\tau_i}\Bigg(\left(\frac{3}{\theta^3}(\theta-t)^2\left(\overline{\rho}_i^2(\varepsilon)-\sigma^2\right)+(v(t),e_i)^2\right)^{1/2} \\
&\qquad -\left(\frac{3}{\theta^3}(\theta-t)^2\left(\overline{\rho}_i^2-\sigma^2\right)+(v(t),e_i)^2\right)^{1/2}\Bigg)^2dt \\
&\le \int_0^{\tau_i}\left(\left(\frac{3}{\theta^3}(\theta-t)^2\left(\overline{\rho}_i^2(\varepsilon)-\sigma^2\right)\right)^{1/2}-\left(\frac{3}{\theta^3}(\theta-t)^2\left(\overline{\rho}_i^2-\sigma^2\right)\right)^{1/2}\right)^2dt \\
&\le \int_0^{\theta}\left(\left(\frac{3}{\theta^3}(\theta-t)^2\right)^{1/2}\left(\left(\overline{\rho}_i^2(\varepsilon)-\sigma^2\right)^{1/2}-\left(\overline{\rho}_i^2-\sigma^2\right)^{1/2}\right)\right)^2dt \\
&= \left(\left(\overline{\rho}_i^2(\varepsilon)-\sigma^2\right)^{1/2}-\left(\overline{\rho}_i^2-\sigma^2\right)^{1/2}\right)^2 \\
&= \left(\left(2\frac{\overline{\rho}_i\varepsilon}{k_i}\left(\frac{3}{\theta^3}\right)^{1/2}+\frac{3}{\theta^3}\left(\frac{\varepsilon}{k_i}\right)^2+\overline{\rho}_i^2-\sigma^2\right)^{1/2}-\left(\overline{\rho}_i^2-\sigma^2\right)^{1/2}\right)^2 \\
&\le 2\frac{\overline{\rho}_i\varepsilon}{k_i}\left(\frac{3}{\theta^3}\right)^{1/2}+\frac{3}{\theta^3}\left(\frac{\varepsilon}{k_i}\right)^2\le\left(2\left(\frac{3}{\theta^3}\right)^{1/2}+\frac{6\gamma}{\theta^3}+\frac{3}{\theta^3}\right)\varepsilon.
\end{aligned} \tag{4.19}$$

In the last inequality, we have used the facts that $0<\varepsilon<1$, $k_i\ge 1$ and the inequality

$$\frac{\overline{\rho}_i}{k_i}=\frac{\rho_i}{k_i}+\frac{\gamma\left(3/\theta^3\right)^{1/2}}{k_i}\le 1+\gamma\left(\frac{3}{\theta^3}\right)^{1/2}. \tag{4.20}$$

So

$$\begin{aligned}
\int_0^{\theta}\left\|(\theta-t)\left(w_i^{\varepsilon}(t)-w_i(t)\right)\right\|dt &= \int_0^{\tau_i}\left\|(\theta-t)\left(w_i^{\varepsilon}(t)-w_i(t)\right)\right\|dt \\
&\le \left(\int_0^{\tau_i}(\theta-t)^2\,dt\right)^{1/2}\left(\int_0^{\tau_i}\left\|w_i^{\varepsilon}(t)-w_i(t)\right\|^2dt\right)^{1/2}\le K\varepsilon,
\end{aligned} \tag{4.21}$$

where K is some positive number.

For the second integral in (4.17) we have

$$\begin{aligned}\int_0^\theta \left\|(\theta-t)\left(1-\frac{\rho_s}{\overline{\rho}_s}\right)w_s(t)\right\| dt &= \left(1-\frac{\rho_s}{\overline{\rho}_s}\right)\int_0^\theta \|(\theta-t)w_s(t)\|\,dt \\ &\le \left(1-\frac{\rho_s}{\overline{\rho}_s}\right)\left(\int_0^\theta (\theta-t)^2\,dt\right)^{1/2}\left(\int_0^\theta \|w_s(t)\|^2\,dt\right)^{1/2} \\ &= \left(1-\frac{\rho_s}{\overline{\rho}_s}\right)\left(\frac{\theta^3}{3}\right)^{1/2}\overline{\rho}_s = \gamma.\end{aligned} \tag{4.22}$$

Then it follows from (4.17) that $\|y(\theta)-x_s(\theta)\| \le \gamma + K\varepsilon$.

Thus if the pursuers use the strategies (4.11), the inequality (4.12) is true.

(3)*The value γ is guaranteed for the evader*. Let us construct the evader's strategy ensuring that

$$\inf_{u_1(\cdot),\ldots,u_m(\cdot),\ldots}\ \inf_{i\in I}\|y(\theta)-x_i(\theta)\| \ge \gamma, \tag{4.23}$$

where $u_1(\cdot),\ldots,u_m(\cdot),\ldots$ are arbitrary admissible controls of the pursuers. If $\gamma = 0$, then inequality (4.23) is obviously valid for any admissible control of the evader. Let $\gamma > 0$. By the definition of γ, for any $\varepsilon > 0$, the set

$$\bigcup_{i=1}^{\infty} H\left(x_{i0}, \rho_i\left(\frac{\theta^3}{3}\right)^{1/2} + \gamma - \varepsilon\right), \tag{4.24}$$

does not contain the ball $H(y_0,\sigma(\theta^3/3)^{1/2})$. Then, by Lemma 4.4 there exists a point $\overline{y} \in S(y_0,\sigma(\theta^3/3)^{1/2})$, that is, $\|\overline{y}-y_0\| = \sigma(\theta^3/3)^{1/2}$ such that $\|\overline{y}-x_{i0}\| \ge \rho_i(\theta^3/3)^{1/2}+\gamma$. On the other hand

$$\|x_i(\theta)-x_{i0}\| \le \left(\frac{\theta^3}{3}\right)^{1/2}\left(\int_0^\theta \|u_i(t)\|^2\,dt\right)^{1/2} = \rho_i\left(\frac{\theta^3}{3}\right)^{1/2}. \tag{4.25}$$

Consequently

$$\|\overline{y}-x_i(\theta)\| \ge \|\overline{y}-x_{i0}\| - \|x_i(\theta)-x_{i0}\| \ge \rho_i\left(\frac{\theta^3}{3}\right)^{1/2} + \gamma - \rho_i\left(\frac{\theta^3}{3}\right)^{1/2} = \gamma. \tag{4.26}$$

Now by using the control

$$v(t) = \sigma\left(\frac{3}{\theta^3}\right)^{1/2}(\theta-t)e, \quad 0\le t\le\theta,\ e = \frac{\overline{y}-y_0}{\|\overline{y}-y_0\|}, \tag{4.27}$$

we obtain

$$\begin{aligned} y(\theta) &= y_0 + \int_0^\theta (\theta - s) v(s)\, ds \\ &= y_0 + \int_0^\theta (\theta - s)^2 \sigma \left(\frac{3}{\theta^3}\right)^{1/2} e\, ds \\ &= y_0 + \sigma \left(\frac{\theta^3}{3}\right)^{1/2} e = \overline{y}. \end{aligned} \tag{4.28}$$

Then the value of the game is not less than γ, and inequality (4.23) holds. The proof of the theorem is complete. □

5. Conclusion

A pursuit-evasion differential game of fixed duration with countably many pursuers has been studied. Control functions satisfy integral constraints. Under certain conditions, the value of the game has been found, and the optimal strategies of players have been constructed.

The proof of the main result relies on the solution of an auxiliary differential game problem in the half-space. Such method was used by many authors (see, e.g., [5, 6]), but the method used here for this auxiliary problem is different from those of others and requires only basic knowledge of calculus.

It should be noted that the condition given by Assumption 4.1 is relevant. If this condition does not hold, then, in general, we do not have a solution of the pursuit-evasion problem even in a finite dimensional space with a finite number of pursuers.

The present work can be extended by considering higher-order differential equations instead of (2.1). Then differential game can be reduced to an equivalent game, described by (2.1), with $\theta - t$ replaced by another function.

Acknowledgments

The authors would like to thank the referee for giving useful comments and suggestions for the improvement of this paper. The present research was supported by the National Fundamental Research Grant Scheme (FRGS) of Malaysia, no. 05-10-07-376FR.

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