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Research Article

Solving Nonlinear Boundary Value Problems Using He's Polynomials and Padé Approximants

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We apply He's polynomials coupled with the diagonal Padé approximants for solving various singular and nonsingular boundary value problems which arise in engineering and applied sciences. The diagonal Padé approximants prove to be very useful for the understanding of physical behavior of the solution. Numerical results reveal the complete reliability of the proposed combination.

1. Introduction

With the rapid development of nonlinear sciences, many analytical and numerical techniques have been developed by various scientists for solving singular and nonsingular initial and boundary value problems which arise in the mathematical modeling of diversified physical problems related to engineering and applied sciences. The application of these problems involves physics, astrophysics, experimental and mathematical physics, nuclear charge in heavy atoms, thermal behavior of a spherical cloud of gas, thermodynamics, population models, chemical kinetics, and fluid mechanics see [1–68] and the references therein. Several techniques [1–68] including decomposition, variational iteration, finite difference, polynomial spline, differential transform, exp-function and homotopy perturbation have been developed for solving such problems. Most of these methods have their inbuilt deficiencies coupled with the major drawback of huge computational work. He [19–24] developed the homotopy perturbation method (HPM) for solving linear, nonlinear, initial and boundary value problems. The homotopy perturbation method was formulated by merging the standard homotopy with perturbation. Recently, Ghorbani and Saberi-Nadjafi [15, 16] introduced He's polynomials by splitting the nonlinear term and also proved that He's polynomials are fully compatible with Adomian's polynomials but are easier to calculate and are more user friendly. The basic motivation of this paper is to apply He's polynomials coupled with the diagonal Padé approximants for solving singular and nonsingular boundary value problems. The Padé approximants are applied in order to make the work more concise and for the better understanding of the solution behavior. The use of Padé approximants shows real promise in solving boundary value problems in an infinite domain; see [42, 50, 56–59]. It is well known in the literature that polynomials are used to approximate the truncated power series. It was observed [42, 50, 56–59] that polynomials tend to exhibit oscillations that may give an approximation error bounds. Moreover, polynomials can never blow up in a finite plane and this makes the singularities not apparent. To overcome these difficulties, the obtained series is best manipulated by Padé approximants for numerical approximations. Using the power series, isolated from other concepts, is not always useful because the radius of convergence of the series may not contain the two boundaries. It is now well known that Padé approximants [42, 50, 56–59] have the advantage of manipulating the polynomial approximation into rational functions of polynomials. By this manipulation, we gain more information about the mathematical behavior of the solution. In addition, the power series are not useful for large values of x. It is an established fact that power series in isolation are not useful to handle boundary value problems. This can be attributed to the possibility that the radius of convergence may not be sufficiently large to contain the boundaries of the domain. It is therefore essential to combine the series solution with the Padé approximants to provide an effective tool to handle boundary value problems on an infinite or semi-infinite domain. We apply this powerful combination of series solution and Padé approximants for solving a variety of boundary value problems. Precisely the proposed combination is applied on boundary layer problem, unsteady flow of gas through a porous medium, Thomas-Fermi equation, Flierl-Petviashivili (FP) equation, and Blasius problem. It is worth mentioning that Flierl-Petviashivili equation has singularity behavior at $x = 0$ which is a difficult element in this type of equations. We transform the FP equation to a first-order initial value problem and He's polynomials are applied to the reformulated first-order initial value problem which leads the solution in terms of transformed variable. The desired series solution is obtained by implementing the inverse transformation. The fact that the proposed algorithm solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method.

2. Homotopy Perturbation Method and He's Polynomials

To explain the He's homotopy perturbation method, we consider a general equation of the type

$$L(u) = 0, \tag{2.1}$$

where L is any integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{2.2}$$

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u, p) = 0, \tag{2.3}$$

we have

$$H(u,0) = F(u), \qquad H(u,1) = L(u). \tag{2.4}$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$, continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter [15, 16, 19–24, 41–50, 60, 63–68]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [19–24] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{2.5}$$

if $p \to 1$, then (2.5) corresponds to (2.2) and becomes the approximate solution of the form

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{2.6}$$

It is well known that series (2.6) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$; see [19–24]. We assume that (3.2) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to [15, 16], He's HPM considers the nonlinear term $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \cdots, \tag{2.7}$$

where H_n's are the so-called He's polynomials [15, 16], which can be calculated by using the formula

$$H_n(u_0,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i u_i\right)\right)_{p=0}, \quad n = 0,1,2,\ldots \tag{2.8}$$

of various orders.

3. Padé Approximants

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(x)$. The $[L/M]$ Padé approximants to a function $y(x)$ are given by [42, 50, 56–59]

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}, \tag{3.1}$$

where $P_L(x)$ is polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$y(x) = \sum_{i=1}^{\infty} a_i x^i, \tag{3.2}$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = O\left(x^{L+M+1}\right) \tag{3.3}$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation. Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we imposed the normalization condition

$$Q_M(0) = 1.0. \tag{3.4}$$

Finally, we require that $P_L(x)$ and $Q_M(x)$ have noncommon factors. If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\begin{aligned} P_L(x) &= p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L, \\ Q_M(x) &= q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M, \end{aligned} \tag{3.5}$$

then by (3.6) and (3.7), we may multiply (3.3) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (3.5) in more details as

$$\begin{gathered} a_{L+1} + a_L q_1 + \cdots + a_{L-M} q_M = 0, \\ q_{L+2} + q_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0, \\ \vdots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0, \end{gathered} \tag{3.6}$$

$$\begin{gathered} a_0 = p_0, \\ a_0 + a_0 q_1 + \cdots + = p_1, \\ \vdots \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L. \end{gathered} \tag{3.7}$$

To solve these equations, we start with (3.6), which is a set of linear equations for all the unknown q's. Once the q's are known, then (3.7) gives and explicit formula for the unknown p's, which complete the solution. If (3.6) and (3.7) are nonsingular, then we can solve them

Table 1: Numerical values for $\alpha = f''(0)$ for $0 < n < 1$ by using diagonal Padé approximants [51, 59].

n	[2/2]	[3/3]	[4/4]	[5/5]	[6/6]
0.2	−0.3872983347	−0.3821533832	−0.3819153845	−0.3819148088	−0.3819121854
1/3	−0.5773502692	−0.5615999244	−0.5614066588	−0.5614481405	−0.561441934
0.4	−0.6451506398	−0.6397000575	−0.6389732578	−0.6389892681	−0.6389734794
0.6	−0.8407967591	−0.8393603021	−0.8396060478	−0.8395875381	−0.8396056769
0.8	−1.007983207	−1.007796981	−1.007646828	−1.007646828	−1.007792100

directly and obtain (3.8) [42, 50, 56–59], where (3.8) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right] = \frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M}x^j & \sum_{j=M-1}^{L} a_{j-M+1}x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}}. \tag{3.8}$$

To obtain diagonal Padé approximants of different order such as [2/2], [4/4], or [6/6], we can use the symbolic calculus software Maple.

4. Numerical Applications

In this section, we apply He's polynomials for solving boundary layer problem, unsteady flow of gas through a porous medium, Thomas-Fermi equation, Flierl-Petviashivili equation, and Blasius problem. The powerful Padé approximants are applied for making the work more concise and to get the better understanding of solution behavior.

Example 4.1 (see [51, 59]). Consider the following nonlinear third-order boundary layer problem which appears mostly in the mathematical modeling of physical phenomena in fluid mechanics [51, 59]

$$f'''(x) + (n-1)f(x)f''(x) - 2n\left(f'(x)\right)^2 = 0, \quad n > 0, \tag{4.1}$$

with boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \quad n > 0 \tag{4.2}$$

By applying the convex homotopy, we have

$$f_0 + pf_1 + \cdots = f_0(x) - p\iint_0^x \Big((n-1)(f_0 \ + pf_1 + \cdots)(f_0'' \ pf_1'' + \cdots) \\ -2n\ (f_0' + pf_1' + \cdots)^2\Big)dxdx, \quad n > 0, \tag{4.3}$$

comparing the co-efficient of like powers of p, following approximants are made

$$\begin{aligned}
&p^{(0)} : f_0(x) = x,\\
&p^{(1)} : f_1(x) = \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3,\\
&p^{(2)} : f_2(x) = \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5,\\
&p^{(3)} : f_3(x) = \frac{1}{120}\alpha^2(3n+1)x^5 + \frac{1}{720}\alpha\Big(19n^2+18n+3\Big)x^6 + \frac{1}{315}n\Big(2n^2+2n+1\Big)x^7,\\
&p^{(4)} : f_4(x) = \frac{1}{5040}\alpha^2\Big(27n^2+42n+11\Big)x^7 + \frac{1}{40320}\alpha\Big(167n^3+297n^2+161n+15\Big)x^8\\
&\qquad\qquad + \frac{1}{22680}n\Big(13n^3+38n^2+23n+6\Big)x^9,\\
&\qquad\quad \vdots
\end{aligned} \tag{4.4}$$

where $f''(0) = \alpha < 0$ and $p^i s$ are He's polynomials. The series solution is given as

$$\begin{aligned}
f(x) = x &+ \frac{\alpha x^2}{2} + \frac{nx^3}{3} + \left(\frac{1}{8}n\alpha + \frac{1}{24}\alpha\right)x^4 + \left(\frac{1}{30}n^2 + \frac{1}{40}n\alpha^2 + \frac{1}{120}\alpha^2 + \frac{1}{30}n\right)x^5\\
&+ \left(\frac{19}{720}n^2\alpha + \frac{1}{240}\alpha + \frac{1}{40}n\alpha\right)x^6\\
&+ \left(\frac{1}{120}n\alpha^2 + \frac{1}{315}n + \frac{2}{315}n^3 + \frac{11}{5040}\alpha^2 + \frac{3}{560}n^2\alpha^2 + \frac{2}{315}n^2\right)x^7\\
&+ \left(\frac{11}{40320}\alpha^3 + \frac{33}{4480}n^2\alpha + \frac{3}{4480}\alpha^3 n^2 + \frac{23}{5760}n\alpha + \frac{1}{2688}\alpha + \frac{167}{40320}n^3\alpha + \frac{1}{960}\alpha^3 n\right)x^8\\
&+ \left(\frac{1}{3780}n + \frac{527}{362880}n^3\alpha^2 + \frac{19}{11340}n^3 + \frac{709}{362880}n\alpha^2 + \frac{23}{8064}n^2\alpha^2 + \frac{23}{22680}n^2\right.\\
&\qquad \left. + \frac{13}{22680}n^4 + \frac{43}{120960}\alpha^2\right)x^9 + \cdots.
\end{aligned} \tag{4.5}$$

Table 2: Numerical values for $\alpha = f''(0)$ for $n < 1$ by using diagonal Padé approximants [51, 59].

n	α
4	−2.483954032
10	−4.026385103
100	−12.84334315
1000	−40.65538218
5000	−104.8420672

Example 4.2 (see [51, 57]). Consider the following nonlinear differential equation which governsthe unsteady flow of gas through a porous medium

$$y''(x) + \frac{2x}{\sqrt{1-\alpha y}} y'(x) = 0, \quad 0 < \alpha < 1 \tag{4.6}$$

with the following boundary conditions:

$$y(0) = 1, \qquad \lim_{x \to \infty} y(x) = 0. \tag{4.7}$$

By applying the convex homotopy method we have

$$y_0 + py_1 + \cdots = y_0(x) - p \iint_0^x \left(2x\ (1-\alpha)\left(y_0 + py_1 + p^2 y_2 + \cdots\right)^{-1/2}\right) dx dx. \tag{4.8}$$

By comparing the coefficient of like powers of p, the following approximants are obtained:

$$\begin{aligned}
p^{(0)} &: y_0(x) = 1, \\
p^{(1)} &: y_1(x) = Ax, \\
p^{(2)} &: y_2(x) = \frac{A}{3\sqrt{1-\alpha}} x^3, \\
p^{(3)} &: y_3(x) = -\frac{\alpha A^2}{12(1-\alpha)^{3/2}} x^4 + \frac{A}{10(1-\alpha)} x^5, \\
p^{(4)} &: y_4(x) = -\frac{3\alpha^2 A^3}{80(1-\alpha)^{5/2}} x^5 + \frac{\alpha A^2}{15(1-\alpha)^2} x^6 + \cdots, \\
&\vdots
\end{aligned} \tag{4.9}$$

Table 3: [51, 57].

α	$B_{[2/2]} = y'(0)$	$B_{[3/3]} = y'(0)$
0.1	−3.556558821	−1.957208953
0.2	−2.441894334	−1.786475516
0.3	−1.928338405	−1.478270843
0.4	−1.606856838	−1.231801809
0.5	−1.373178096	−1.025529704
0.6	−1.185519607	−0.8400346085
0.7	−1.021411309	−0.6612047893
0.8	−0.8633400217	−0.4776697286
0.9	−0.6844600642	−0.2772628386

where $A = y'(0)$ and $p^i s$ are He's polynomials. The series solution is given as

$$y(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \left(\frac{A}{10(1-\alpha)} - \frac{3\alpha^2 A^3}{80(1-\alpha)^{5/2}}\right)x^5 + \left(\frac{\alpha A^2}{15(1-\alpha)^2} - \frac{\alpha^3 A^4}{48(1-\alpha)^{7/2}}\right)x^6 + \cdots. \tag{4.10}$$

The diagonal Padé approximants [51, 57] can be applied to analyze the physical behavior. Based on this, the [2/2] Padé approximants produced the slope A to be

$$A = -\frac{2(1-\alpha)^{1/4}}{\sqrt{3\alpha}}, \tag{4.11}$$

and by using [3/3] Padé approximants we find

$$A = -\frac{\sqrt{(-4674\alpha + 8664)\sqrt{1-\alpha} - 144\gamma}}{57\alpha}, \tag{4.12}$$

where

$$\gamma = \sqrt{5(1-\alpha)(1309\alpha^2 - 2280\alpha + 1216)}. \tag{4.13}$$

Using (4.11)–(4.13) gives the values of the initial slope $A = y'(0)$ listed in Table 3. The formulas (4.11) and (4.12) suggest that the initial slope $A = y'(0)$ depends mainly on the parameter α, where $0 < \alpha < 1$. Table 3 exhibits the initial slopes $A = y'(0)$ for various values of α. Table 4 exhibits the values of $y(x)$ for $\alpha = 0.5$ for $x = 0.1$ to 1.0.

Table 4: [51, 57].

x	y^{kidder}	$y_{[2/2]}$	$y_{[3/3]}$
0.1	0.8816588283	0.8633060641	0.8979167028
0.2	0.7663076781	0.7301262261	0.7985228199
0.3	0.6565379995	0.6033054140	0.7041129703
0.4	0.5544024032	0.4848898717	0.6165037901
0.5	0.4613650295	0.3761603869	0.5370533796
0.6	0.3783109315	0.2777311628	0.4665625669
0.7	0.3055976546	0.1896843371	0.4062426033
0.8	0.2431325473	0.1117105165	0.3560801699
0.9	0.1904623681	0.04323673236	0.3179966614
1.0	0.1587689826	0.01646750847	0.2900255005

Example 4.3 (see [56]). Consider the following Thomas-Fermi (T-F) equation [6–13, 17, 31, 33, 34, 54] which arises in the mathematical modeling of various models in physics, astrophysics, solid state physics, nuclear charge in heavy atoms, and applied sciences:

$$y''(x) = \frac{y^{3/2}}{x^{1/2}}, \tag{4.14}$$

with boundary conditions

$$y(0) = 1, \qquad \lim_{x\to\infty} y(x) = 0. \tag{4.15}$$

By applying the convex homotopy,

$$y_0 + py_1 + p^2y_2 + \cdots = y_0(x) + p\iint_0^x \left(x^{-1/2}\left(y_0 + py_1 + p^2y_2 + \cdots\right)^{3/2}\right)dx dx. \tag{4.16}$$

Now, we apply a slight modification in the conventional initial value and take $y_0(x) = 1$, instead of $y_0(x) = 1 + Bx$, where $B = y'(0)$. By comparing the coefficient of like powers of p, the following approximants are obtained

$$\begin{aligned}
p^{(0)} &: y_0(x) = 1, \\
p^{(1)} &: y_1(x) = Bx + \frac{4}{3}x^{3/2}, \\
p^{(2)} &: y_2(x) = \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3, \\
p^{(3)} &: y_3(x) = \frac{3}{70}B^2x^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} + \frac{1}{3}x^3, \\
p^{(4)} &: y_4(x) = \frac{3}{70}B^2x^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} - \frac{1}{252}B^3x^{9/2} + \frac{1}{175}B^2x^5 \\
&\qquad + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2}.
\end{aligned} \tag{4.17}$$

Table 5: Padé approximants and initial slopes $y'(0)$ [56]

Padé approximants	Initial slope $y'(0)$	Error (%)
[2/2]	−1.211413729	23.71
[4/4]	−1.550525919	2.36
[7/7]	−1.586021037	12.9×10^{-2}
[8/8]	−1.588076820	3.66×10^{-4}
[10/10]	−1.588076779	3.64×10^{-4}

The series solution is given as

$$\begin{aligned} y(x) = {} & 1 + Bx + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2} \\ & - \frac{1}{252}B^3x^{9/2} + \frac{1}{175}B^2x^5 + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2} + \frac{1}{1056}B^4x^{11/2} + \frac{4}{1575}B^3x^6 \\ & + \frac{557}{100100}B^2x^{13/2} + \frac{4}{693}Bx^7 + \frac{101}{52650}x^{15/2} - \frac{3}{9152}B^5x^{13/2} - \frac{29}{24255}B^4x^7 \\ & - \frac{512}{351000}B^3x^{15/2} - \frac{46}{45045}B^2x^8 - \frac{113}{1178100}Bx^{17/2} + \frac{23}{473850}x^9 \ \cdots, \\ & \vdots \end{aligned} \tag{4.18}$$

Setting $x^{1/2} = t$, the series solution is obtained as

$$\begin{aligned} y(t) = {} & 1 + Bt^2 + \frac{4}{3}t^3 + \frac{2}{5}Bt^5 + \frac{1}{3}t^6 + \frac{3}{70}B^2t^7 + \frac{2}{15}Bt^8 + \left(-\frac{1}{252}B^3 + \frac{2}{27}\right)t^9 + \frac{1}{175}B^2t^{10} \\ & + \left(\frac{1}{1056}B^4 + \frac{31}{1485}B\right)t^{11} + \left(\frac{4}{1575}B^3 + \frac{4}{405}\right)t^{12} + \left(-\frac{3}{9152}B^5 + \frac{557}{100100}B^2\right)t^{13} \\ & + \left(-\frac{29}{24255}B^4 + \frac{4}{693}B\right)t^{14} + \left(\frac{7}{499}B^6 - \frac{623}{351000}B^3 + \frac{101}{52650}\right)t^{15} \\ & + \left(\frac{68}{105105}B^4 - \frac{46}{45045}B^2\right)t^{16} + \left(-\frac{3}{43520}B^7 + \frac{153173}{116424000}B^4 - \frac{113}{1178100}B\right)t^{17} + \cdots . \end{aligned} \tag{4.19}$$

The diagonal Padé approximants can be applied [56] in order to study the mathematical behavior of the potential $y(x)$ and to determine the initial slope of the potential $y'(0)$.

Example 4.4 (see [42]). Consider the generalized variant of the Flierl-Petviashivili equation [37]

$$y'' + \frac{1}{x}y' - y^n - y^{n+1} = 0, \tag{4.20}$$

with boundary conditions

$$y(0) = \alpha, \qquad y'(0) = 0, \qquad y(\infty) = 0. \tag{4.21}$$

Using the transformation $u(x) = xy'(x)$, the generalized FP equation can be converted to the following first-order initial value problem:

$$u'(x) = x\left(\int_0^x \left(\frac{u(x)}{x}\right)^n + \left(\frac{u(x)}{x}\right)^{n+1} dx\right), \tag{4.22}$$

with initial conditions

$$u(0) = 0, \qquad u(0) = 0. \tag{4.23}$$

By applying the convex homotopy, we have

$$\begin{aligned} &u_0 + pu_1 + p^2u_2 + \cdots \\ &\quad = p\int_0^s \left(x\left(\int_0^s \left(\frac{1}{x}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\right)^n + \left(\frac{1}{x}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\right)^{n+1} dx\right)\right)dx. \end{aligned} \tag{4.24}$$

The series solution after four iterations is given by

$$\begin{aligned} u(x) = {} & \frac{\left(\alpha^n + \alpha^{n+1}\right)}{2}x^2 + \frac{\left(\alpha^n + \alpha^{n+1}\right)\left(n\alpha^n + (n+1)\alpha^{n+1}\right)}{16\ \alpha}x^4 \\ & + \frac{\left(\alpha^n + \alpha^{n+1}\right)\left(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2}\right)}{384\ \alpha^2}x^6 \\ & + \frac{\left(\alpha^n + \alpha^{n+1}\right)\left(n\left(18n^2 - 29n + 12\right)\alpha^{3n} + n\left(54n^2 - 33n + 7\right)\alpha^{3n+1} + \mathfrak{A}\right)}{18432\ \alpha^3}x^8 \\ & + \cdots, \end{aligned} \tag{4.25}$$

where $\mathfrak{A}$ denote $(18n^2 + 7n + 1)(3n\alpha^{3n+2} + (n+1)\alpha^{3n+3})$, and the inverse transformation will yield

$$\begin{aligned} y(x) = {} & \alpha + \frac{\left(\alpha^n + \alpha^{n+1}\right)}{4}x^2 + \frac{\left(\alpha^n + \alpha^{n+1}\right)\left(n\alpha^n + (n+1)\alpha^{n+1}\right)}{64\alpha}x^4 \\ & + \frac{\left(\alpha^n + \alpha^{n+1}\right)\left(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2}\right)}{2304\ \alpha^2}x^6 \\ & + \frac{\left(\alpha^n + \alpha^{n+1}\right)\left(n\left(18n^2 - 29n + 12\right)\alpha^{3n} + n\left(54n^2 - 33n + 7\right)\alpha^{3n+1} + \mathfrak{A}\right)}{147456\ \alpha^3}x^8 \\ & + \cdots, \end{aligned} \tag{4.26}$$

Table 6: Roots of the Padé approximants monopole [42] $\alpha, n = 1$.

Degree	Roots
[2/2]	−1.5
[4/4]	−2.50746
[6/6]	−2.390278
[8/8]	−2.392214

Table 7: Roots of the Padé approximants monopole [42] $\alpha, n = 3$.

Degree	Roots
[2/2]	−2.0
[4/4]	−2.0
[6/6]	−2.0
[8/8]	−2.0

Table 8: Roots of the Padé approximants monopole [42] α.

Degree	Roots
[2/2]	0.0
[4/4]	−.2197575908
[6/6]	−1.1918424398
[8/8]	−1.848997181

Table 9: Roots [42] of the Padé approximants [8/8] monopole for several values of n.

n	[8/8] roots	n	[8/8] roots
1	−2.392213866	7	−1.000708285
2	−2.0	8	−1.000601615
3	−1.848997181	9	−1.000523005
4	−1.286025892	10	−1.000462636
5	−1.001101141	11	−1.000262137
6	−1.000861533	$n \to \infty$	−1.0

where $\mathfrak{A}$ denote $(18n^2 + 7n + 1)(3n\alpha^{3n+2} + (n+1)\alpha^{3n+3})$. Diagonal Padé approximants can be applied [42] to find the roots of the FP monopole α for $n \geq 1$.

Table 9 shows that the roots of the monopole α converge to −1 as n increases.

Example 4.5 (see [58, 59]). Consider the two-dimensional nonlinear inhomogeneous initial boundary value problem for the integro-differential equation related to the Blasius problem

$$y''(x) = \alpha - \frac{1}{2}\int_0^x y(t)y''(t)dt, \quad -\infty < x < 0, \tag{4.27}$$

with boundary conditions

$$\begin{gathered} y(0) = 0, \qquad y'(0) = 1, \\ \lim_{x \to \infty} y'(x) = 0, \end{gathered} \tag{4.28}$$

Table 10: Padé approximants and numerical value of α [53].

Padé approximant	α
[2/2]	0.5778502691
[3/3]	0.5163977793
[4/4]	0.5227030798

where the constant α is positive and defined by

$$y''(0) = \alpha, \quad \alpha > 0. \tag{4.29}$$

By applying the convex homotopy, we have

$$y_0 + py_1 + \cdots = y_0(x) - p\iint_0^x \left(\int_0^x (y_0 + py_1 + \cdots) \left(\frac{d^2 y_0}{dx^2} + p\frac{d^2 y_1}{dx^2} + \cdots \right) \right) dx dx. \tag{4.30}$$

Proceeding as before, the series solution is given as

$$\begin{aligned} y(x) = {} & x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \frac{1}{960}\alpha x^6 + \frac{11}{20160}\alpha^2 x^7 + \left(\frac{11}{161280}\alpha^3 + \frac{1}{960}\alpha \right) x^8 \\ & - \frac{43}{967680}\alpha^2 x^9 + \left(\frac{1}{52960}\alpha - \frac{5}{387072}\alpha^3 \right) x^{10} + \left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4 \right) x^{11} \\ & + \left(-\frac{1}{16220160}\alpha + \frac{1}{7257792}\alpha^3 \right) x^{12} + \cdots, \end{aligned} \tag{4.31}$$

and consequently

$$\begin{aligned} y'(x) = {} & 1 + \alpha x - \frac{1}{12}\alpha x^3 - \frac{1}{48}\alpha^2 x^4 + \frac{1}{160}\alpha x^5 + \frac{11}{2880}\alpha^2 x^6 \left(\frac{11}{20160}\alpha^3 - \frac{1}{2688}\alpha \right) x^7 \\ & - \frac{43}{107520}\alpha^2 x^8 + 10 \left(\frac{1}{552960}\alpha - \frac{5}{387072}\alpha^3 \right) x^9 + 11 \left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4 \right) x^{10} \\ & + 12 \left(-\frac{1}{16220160}\alpha + \frac{1}{725760}\alpha^3 \right) x^{11} + \cdots . \end{aligned} \tag{4.32}$$

Now, we apply the diagonal Padé approximants to determine a numerical value for the constant α by using the given condition. Padé approximant of $y'(x)$ usually converges on the entire real axis [58, 59]. Moreover, $y'(x)$ is free of singularities on the real axis. Substituting the boundary conditions $y'(-\infty) = 0$ in each Padé approximant which vanishes if the coefficient of x with the highest power in the numerator vanishes. By solving the resulting polynomials of these coefficients, we obtain the values of α listed in Table 10 [58, 59].

5. Conclusion

In this paper, we applied a reliable combination of He's polynomials and the diagonal Padé approximants for obtaining approximate solutions of various singular and nonsingular boundary value problems of diversified physical nature. The proposed algorithm is employed without using linearization, discretization, transformation, or restrictive assumptions. The fact that the suggested technique solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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