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Research Article

Further Results Concerning Delay-Dependent H_∞ Control for Uncertain Discrete-Time Systems with Time-Varying Delay

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This paper addresses the problem of H_∞ control for uncertain discrete-time systems with time-varying delays. The system under consideration is subject to time-varying norm-bounded parameter uncertainties in both the state and controlled output. Attention is focused on the design of a memoryless state feedback controller, which guarantees that the resulting closed-loop system is asymptotically stable and reduces the effect of the disturbance input on the controlled output to a prescribed level irrespective of all the admissible uncertainties. By introducing some slack matrix variables, new delay-dependent conditions are presented in terms of linear matrix inequalities (LMIs). Numerical examples are provided to show the reduced conservatism and lower computational burden than the previous results.

1. Introduction

During the past decades, considerable attention has been paid to the problems of stability analysis and control synthesis of time-delay systems. Many methodologies have been proposed and a large number of results have been established (see, e.g., [1–4] and the references therein). All these results can be generally divided into two categories: delay-independent stability conditions [5, 6] and delay-dependent stability conditions [7–12]. The delay-independent stability condition does not take the delay size into consideration, and thus is often conservative especially for systems with small delays, while the delay-dependent stability condition makes fully use of the delay information and is usually less conservative than the delay-independent one.

Up to now, the most important approach to deal with delay in the states of the systems is the use of Lyapunov-Krasovskii functionals, which has been largely employed to obtain convex conditions mainly for continuous-time systems subjected to retarded states. However, discrete-time systems with state delay have received little attention. This mainly because that for precisely known discrete-time systems with constant delay, it is always possible to derive a delay-free system by state augmentation [10, 11]. Although, such an approach is valid for the system with constant delays, it fails to deal with time-varying delay case, which is more frequently encountered than the constant case in practice. Recent results on discrete time-delay systems can be found in [13] where delay-dependent stability criteria were considered using a sum inequality. In [14], stability conditions for discrete time-delay systems were presented, while less conservative results were given in [15] by using a more general Lyapunov-Krasovskii functional than that in [14]. In [16], the authors summarized the recent results concerning robust stabilization of discrete-time systems with state delay. Sufficient LMI conditions were presented checking the robust stability for a class of linear discrete-time systems with time-varying delay and polytopic uncertainties; robust state feedback gains with memory were also designed. These results were mainly with the stability analysis and state feedback controller design. Very few people have investigated the delay-dependent H_∞ control problem of discrete time-delay systems. In [17], the authors proposed an exponential output feedback H_∞ controller. Delay-dependent robust H_∞ control conditions for uncertain linear systems with lumped delays were given in [18], which were proved to be less conservative than some previous results. Also delay-dependent results were derived in [19] by combining a descriptor model transformation approach with Moon's bounding technique [9]. Very recently, in order to reduce the conservatism of the result in [19], a finite sum inequality approach was proposed in [20] and some less conservative H_∞ control condition was derived. Although the result in [20] is superior to that in [19], it is still a sufficient condition and has conservatism to some extent, which leaves open room for further improvement.

Naturally, one may say that whether we can employ the similar Lyapunov functional, fewer variables, and reduced complexity of the algorithm to obtain less conservatism than the existing results. In this paper, we will further study the robust H_∞ control problem for uncertain discrete-time system with time-varying delays. By introducing some slack matrix variables, new delay-dependent conditions for H_∞ control problem are proposed in terms of LMI form, while no model transformation and bounding technique are employed. It is also shown that the complexity of the algorithm is considerably reduced and the result in this paper is less conservative than that in [18–20]. Numerical examples are finally provided to demonstrate the effectiveness of the main results.

Notations

Throughout this paper, $\mathbb{R}^n$ represents the n-dimensional Euclidean space; $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices. For real symmetric matrices X and Y, the notation $X \geq Y$ (resp., $X > Y$) means that the matrix $X - Y$ is positive semidefinite (resp., positive definite). The superscript "T" denotes the transpose. I is an identity matrix with appropriate dimension. $\mathbb{Z}^+$ denotes the set of $\{0, 1, 2, \ldots\}$. $\mathcal{L}_2$ refers to the space of square summable infinite vector sequences. In symmetric block matrices, we use an asterisk "$*$" to represent a term that is induced by symmetry. Matrices, if not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Problem Formulation

Consider the following uncertain discrete-time systems with time-varying delay [20]:

$$\begin{aligned}
(\Sigma) : x(k+1) &= (A_0 + \Delta A_0(k))x(k) + (A_1 + \Delta A_1(k))x(k-d(k)) \\
&\quad + (B_1 + \Delta B_1(k))\omega(k) + (B_2 + \Delta B_2(k))u(k), \\
z(k) &= (C_0 + \Delta C_0(k))x(k) + (C_1 + \Delta C_1(k))x(k-d(k)) \\
&\quad + (D_{11} + \Delta D_{11}(k))\omega(k) + (D_{12} + \Delta D_{12}(k))u(k), \\
x(k) &= \phi(k), \quad \forall k \in \left[-\overline{h}, 0\right],
\end{aligned} \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $z(k) \in \mathbb{R}^p$ are the state, control input, and controlled output, respectively; $\omega(k) \in \mathbb{R}^q$ is the exogenous disturbance input, which belongs to $\mathcal{L}_2$. $\phi(k)$ is the initial condition; A_0, A_1, B_1, B_2, C_0, C_1, D_{11}, and D_{12} are known real constant matrices. The time-varying parameter uncertainties are norm-bounded and meet with

$$\begin{bmatrix} \Delta A_0(k) & \Delta A_1(k) & \Delta B_1(k) & \Delta B_2(k) \\ \Delta C_0(k) & \Delta C_1(k) & \Delta D_{11}(k) & \Delta D_{12}(k) \end{bmatrix} = \begin{bmatrix} \mathfrak{D}_1 \\ \mathfrak{D}_2 \end{bmatrix} F(k) \begin{bmatrix} E_1 & E_2 & E_3 & E_4 \end{bmatrix}, \tag{2.2}$$

where $F(k)$ is an unknown real time-varying matrix and satisfies the following bound condition:

$$F^T(k)F(k) \leq I. \tag{2.3}$$

$\mathfrak{D}_1, \mathfrak{D}_2$, and E_ℓ ($\ell = 1,2,3,4$) are known constant matrices of appropriate dimensions describing how the uncertainty $F(k)$ enters the nominal matrices of system (Σ). $d(k)$ denotes the time-varying delay satisfying

$$\underline{h} \leq d(k) \leq \overline{h}, \quad \forall k \in \mathbb{Z}^+, \tag{2.4}$$

where $\underline{h}$ and $\overline{h}$ are positive integer numbers.

Remark 2.1. The parameter uncertain structure in (2.2) and (2.3) has been widely used in the issues of robust control and filtering for uncertain systems; see, for example, [10, 21]. It comprises the "matching conditions" and many physical systems can be either exactly modeled in this manner or overbounded by (2.3).

Now, consider the following memoryless state feedback controller:

$$u(k) = Kx(k). \tag{2.5}$$

Applying this controller to system (Σ) results in the following closed-loop system:

$$
\begin{aligned}
(\Sigma_{\mathrm{cl}}) : x(k+1) &= \left(\overline{A}_0 + \mathfrak{D}_1 F(k)(E_1 + E_4 K)\right)x(k) + (A_1 + \Delta A_1(k))x(k-d(k)) \\
&\quad + (B_1 + \Delta B_1(k))\omega(k), \\
z(k) &= \left(\overline{C}_0 + \mathfrak{D}_2 F(k)(E_1 + E_4 K)\right)x(k) + (C_1 + \Delta C_1(k))x(k-d(k)) \\
&\quad + (D_{11} + \Delta D_{11}(k))\omega(k), \\
x(k) &= \phi(k), \quad \forall k \in \left[-\overline{h}, 0\right],
\end{aligned} \tag{2.6}
$$

where $\overline{A}_0 = A_0 + B_2 K, \overline{C}_0 = C_0 + D_{12} K$.

The robust H_∞ control problem to be addressed in this paper can be formulated as developing a state feedback controller in the form of (2.5) such that

(1) the closed-loop system (Σ_{cl}) is robustly asymptotically stable when $\omega(k) = 0$, for all $k \geq 0$;

(2) the H_∞ performance $\|z\|_2 < \gamma\|\omega\|_2$ is guaranteed for all nonzero $\omega(k) \in \mathcal{L}_2$ and a prescribed $\gamma > 0$ under the zero-initial condition, for all admissible uncertainties and time-varying delays satisfying (2.2)–(2.4).

At the end of this section, let us introduce some important lemmas which will be used in the sequel.

Lemma 2.2 (Schur complement [22]). *Given constant matrices M, L, Q of appropriate dimensions, where M and Q are symmetric, then $Q > 0$ and $M + L^T Q^{-1} L < 0$ if and only if*

$$
\begin{bmatrix} M & L^T \\ L & -Q \end{bmatrix} < 0, \tag{2.7}
$$

or equivalently

$$
\begin{bmatrix} -Q & L \\ L^T & M \end{bmatrix} < 0. \tag{2.8}
$$

Lemma 2.3 (see [10]). *Let $\mathfrak{D}$, $\mathcal{E}$, and $\mathcal{F}$ be matrices with appropriate dimensions. Suppose $\mathcal{F}^T\mathcal{F} \leq I$, then for any scalar $\mu > 0$, there holds*

$$
\mathfrak{D}\mathcal{F}\mathcal{E} + \mathcal{E}^T\mathcal{F}^T\mathfrak{D}^T \leq \mu\mathfrak{D}\mathfrak{D}^T + \mu^{-1}\mathcal{E}^T\mathcal{E}. \tag{2.9}
$$

3. Main Results

In this section, some delay-dependent LMI-based conditions will be developed to solve the robust H_∞ control problem formulated in the previous section. First, we will consider the nominal system of system (Σ_{cl}) with $F(k) = 0$, for all $k > 0$, that is,

$$\begin{aligned}(\Sigma_{\mathrm{ncl}}) : x(k+1) &= \overline{A}_0 x(k) + A_1 x(k-d(k)) + B_1 \omega(k), \\ z(k) &= \overline{C}_0 x(k) + C_1 x(k-d(k)) + D_{11}\omega(k), \\ x(k) &= \phi(k), \quad \forall k \in \left[-\overline{h}, 0\right],\end{aligned} \tag{3.1}$$

where $\overline{A}_0 = A_0 + B_2 K, \overline{C}_0 = C_0 + D_{12} K$.

Theorem 3.1. *System (Σ_{ncl}) is asymptotically stable with a prescribed H_∞ disturbance attenuation level $\gamma > 0$, if there exist matrices $P > 0$, $Q > 0$, $R > 0$, W, and Y of appropriate dimensions such that*

$$\begin{bmatrix} \Theta_{11} & -Y + W^T & -\overline{h}Y & 0 & \overline{A}_0^T P & \left(\overline{A}_0 - I\right)^T \overline{h} R & \overline{C}_0^T \\ * & -W - W^T - Q & -\overline{h} W & 0 & A_1^T P & A_1^T \overline{h} R & C_1^T \\ * & * & -\overline{h} R & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & B_1^T P & B_1^T \overline{h} R & D_{11}^T \\ * & * & * & * & -P & 0 & 0 \\ * & * & * & * & * & -\overline{h} R & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{3.2}$$

where $\Theta_{11} = -P + Y + Y^T + (\overline{h} - \underline{h} + 1) Q$.

Proof. Let

$$y(\ell) = x(\ell+1) - x(\ell). \tag{3.3}$$

Then, it is easy to see that

$$x(k - d(k)) = x(k) - \sum_{\ell = k - d(k)}^{k-1} y(\ell). \tag{3.4}$$

Now, choose a Lyapunov-Krasovskii functional candidate for the time-delay system (Σ_{ncl}) as

$$V(k) = V_1(k) + V_2(k) + V_3(k), \tag{3.5}$$

where

$$
\begin{aligned}
V_1(k) &= x^T(k)Px(k),\\
V_2(k) &= \sum_{\ell=k-d(k)}^{k-1} x^T(\ell)Qx(\ell) + \sum_{j=-\overline{h}+2}^{-\underline{h}+1} \sum_{\ell=k+j-1}^{k-1} x^T(\ell)Qx(\ell),\\
V_3(k) &= \sum_{j=-\overline{h}+1}^{0} \sum_{\ell=k-1+j}^{k-1} y^T(\ell)Ry(\ell).
\end{aligned}
\tag{3.6}
$$

Taking the forward difference, we have

$$
\begin{aligned}
\Delta V_1(k) &= V_1(k+1) - V_1(k)\\
&= \left[\overline{A}_0 x(k) + A_1 x(k-d(k)) + B_1\omega(k)\right]^T P\left[\overline{A}_0 x(k) + A_1 x(k-d(k)) + B_1\omega(k)\right]\\
&\quad - x^T(k)Px(k)\\
&= x^T(k)\left[\overline{A}_0^T P\overline{A}_0 - P\right]x(k) + 2x^T(k)\overline{A}_0^T PA_1 x(k-d(k))\\
&\quad + 2x^T(k)\overline{A}_0^T PB_1\omega(k) + x^T(k-d(k))A_1^T PA_1 x(k-d(k))\\
&\quad + 2x^T(k-d(k))A_1^T PB_1\omega(k) + \omega^T(k)B_1^T PB_1\omega(k).
\end{aligned}
\tag{3.7}
$$

For any two matrices of appropriate dimensions Y and W, there holds

$$
\begin{aligned}
0 &= 2x^T(k)Y\sum_{\ell=k-d(k)}^{k-1} y(\ell) + 2x^T(k-d(k))W\sum_{\ell=k-d(k)}^{k-1} y(\ell)\\
&\quad - \left[2x^T(k)Y\sum_{\ell=k-d(k)}^{k-1} y(\ell) + 2x^T(k-d(k))W\sum_{\ell=k-d(k)}^{k-1} y(\ell)\right].
\end{aligned}
\tag{3.8}
$$

Substituting (3.4) and the previous equality into (3.7) gives

$$
\begin{aligned}
\Delta V_1(k) &= x^T(k)\left[\overline{A}_0^T P\overline{A}_0 - P\right]x(k) + 2x^T(k)\overline{A}_0^T PA_1\left[x(k) - \sum_{\ell=k-d(k)}^{k-1} y(\ell)\right]\\
&\quad + 2x^T(k)\overline{A}_0^T PB_1\omega(k) + x^T(k-d(k))A_1^T PA_1 x(k-d(k))\\
&\quad + 2x^T(k-d(k))A_1^T PB_1\omega(k) + \omega^T(k)B_1^T PB_1\omega(k)
\end{aligned}
$$

$$
\begin{aligned}
&= x^T(k)\left[\overline{A}_0^T P\overline{A}_0 - P + \overline{A}_0^T PA_1 + A_1^T P\overline{A}_0\right]x(k) + 2x^T(k)\left[Y - \overline{A}_0^T PA_1\right] \\
&\quad \times \sum_{\ell=k-d(k)}^{k-1} y(\ell) + 2x^T(k-d(k))W\sum_{\ell=k-d(k)}^{k-1} y(\ell) \\
&\quad - \left[2x^T(k)Y\sum_{\ell=k-d(k)}^{k-1} y(\ell) + 2x^T(k-d(k))W\sum_{\ell=k-d(k)}^{k-1} y(\ell)\right] \\
&\quad + 2x^T(k)\overline{A}_0^T PB_1\omega(k) + x^T(k-d(k))A_1^T PA_1 x(k-d(k)) \\
&\quad + 2x^T(k-d(k))A_1^T PB_1\omega(k) + \omega^T(k)B_1^T PB_1\omega(k) \\
&= x^T(k)\left[\overline{A}_0^T P\overline{A}_0 - P + \overline{A}_0^T PA_1 + A_1^T P\overline{A}_0\right]x(k) + 2x^T(k)\left[Y - \overline{A}_0^T PA_1\right] \\
&\quad \times [x(k) - x(k-d(k))] + 2x^T(k-d(k))W[x(k) - x(k-d(k))] \\
&\quad - \left[2x^T(k)Y\sum_{\ell=k-d(k)}^{k-1} y(\ell) + 2x^T(k-d(k))W\sum_{\ell=k-d(k)}^{k-1} y(\ell)\right] \\
&\quad + 2x^T(k)\overline{A}_0^T PB_1\omega(k) + x^T(k-d(k))A_1^T PA_1 x(k-d(k)) \\
&\quad + 2x^T(k-d(k))A_1^T PB_1\omega(k) + \omega^T(k)B_1^T PB_1\omega(k) \\
&= \frac{1}{d(k)}\sum_{\ell=k-d(k)}^{k-1}\Big\{x^T(k)\left[\overline{A}_0^T P\overline{A}_0 - P + Y + Y^T\right]x(k) \\
&\qquad\qquad + 2x^T(k)\left[\overline{A}_0^T PA_1 - Y + W^T\right]x(k-d(k)) \\
&\qquad\qquad + 2x^T(k)\overline{A}_0^T PB_1\omega(k) + 2x^T(k)[-d(k)Y]y(\ell) \\
&\qquad\qquad + x^T(k-d(k))\left[A_1^T PA_1 - W - W^T\right]x(k-d(k)) \\
&\qquad\qquad + 2x^T(k-d(k))A_1^T PB_1\omega(k) \\
&\qquad\qquad + 2x^T(k-d(k))[-d(k)W]y(\ell) + \omega^T(k)B_1^T PB_1\omega(k)\Big\}.
\end{aligned}
\tag{3.9}
$$

Similar to [17], we have

$$
\begin{aligned}
\Delta V_2(k) &= V_2(k+1) - V_2(k) \\
&\leq \left(\overline{h} - \underline{h} + 1\right)x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) \\
&= \frac{1}{d(k)}\sum_{\ell=k-d(k)}^{k-1}\left\{\left(\overline{h} - \underline{h} + 1\right)x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k))\right\}.
\end{aligned}
\tag{3.10}
$$

After some manipulations, we obtain

$$\Delta V_3(k) = V_3(k+1) - V_3(k) = y^T(k)\overline{h}Ry(k) - \sum_{\ell=k-\overline{h}}^{k-1} y^T(\ell)Ry(\ell). \tag{3.11}$$

Observe that

$$\begin{gathered}
-\sum_{\ell=k-\overline{h}}^{k-1} y^T(\ell)Ry(\ell) \leq -\sum_{\ell=k-d(k)}^{k-1} y^T(\ell)Ry(\ell), \\
y(k) = x(k+1) - x(k) = \left(\overline{A}_0 - I\right)x(k) + A_1x(k-d(k)) + B_1\omega(k).
\end{gathered} \tag{3.12}$$

This together with (3.11) gives

$$\begin{aligned}
\Delta V_3(k) &\leq y^T(k)\overline{h}Ry(k) - \sum_{\ell=k-d(k)}^{k-1} y^T(\ell)Ry(\ell) \\
&= \left[(\overline{A}_0 - I)x(k) + A_1x(k-d(k)) + B_1\omega(k)\right]^T\overline{h}R \\
&\quad \times \left[\left(\overline{A}_0 - I\right)x(k) + A_1x(k-d(k)) + B_1\omega(k)\right] - \sum_{\ell=k-d(k)}^{k-1} y^T(\ell)Ry(\ell) \\
&= \frac{1}{d(k)}\sum_{\ell=k-d(k)}^{k-1}\Big\{x^T(k)\left(\overline{A}_0 - I\right)^T\overline{h}R\left(\overline{A}_0 - I\right)x(k) - y^T(\ell)d(k)Ry(\ell) \\
&\qquad + 2x^T(k)\left(\overline{A}_0 - I\right)^T\overline{h}RA_1x(k-d(k)) + 2x^T(k)\left(\overline{A}_0 - I\right)^T\overline{h}RB_1\omega(k) \\
&\qquad + x^T(k-d(k))A_1^T\overline{h}RA_1x(k-d(k)) + 2x^T(k-d(k))A_1^T\overline{h}RB_1\omega(k) \\
&\qquad + \omega^T(k)B_1^T\overline{h}RB_1\omega(k)\Big\}.
\end{aligned} \tag{3.13}$$

Then, from (3.7)–(3.13), we have

$$\Delta V(k) = V(k+1) - V(k) = \frac{1}{d(k)}\sum_{\ell=k-d(k)}^{k-1} \eta^T(k,\ell)\Psi_1(d(k))\eta(k,\ell), \tag{3.14}$$

where

$$\eta(k,\ell) = \left[x^T(k) \;\; x^T(k-d(k)) \;\; y^T(\ell) \;\; \omega^T(k)\right]^T,$$
$$\Psi_1(d(k)) = \begin{bmatrix} \Psi_1(1,1) & \Psi_1(1,2) & -d(k)Y & \Psi_1(1,4) \\ * & \Psi_1(2,2) & -d(k)W & \Psi_1(2,4) \\ * & * & -d(k)R & 0 \\ * & * & * & \Psi_1(4,4) \end{bmatrix}, \tag{3.15}$$

with

$$\Psi_1(1,1) = -P + Y + Y^T + \left(\overline{h} - \underline{h} + 1\right)Q + \overline{A}_0^T P \overline{A}_0 + \left(\overline{A}_0 - I\right)^T \overline{h} R \left(\overline{A}_0 - I\right),$$
$$\Psi_1(1,2) = -Y + W^T + \overline{A}_0^T P A_1 + \left(\overline{A}_0 - I\right)^T \overline{h} R A_1,$$
$$\Psi_1(1,4) = \overline{A}_0^T P B_1 + \left(\overline{A}_0 - I\right)^T \overline{h} R B_1, \tag{3.16}$$
$$\Psi_1(2,2) = -W - W^T - Q + A_1^T P A_1 + A_1^T \overline{h} R A_1,$$
$$\Psi_1(2,4) = A_1^T P B_1 + A_1^T \overline{h} R B_1,$$
$$\Psi_1(4,4) = B_1^T P B_1 + B_1^T \overline{h} R B_1.$$

In the next, we will prove the conclusion from two aspects. First, we establish the asymptotic stability of system (Σ_{ncl}) with $\omega(k) = 0$ if (3.2) is satisfied. For this situation, (3.14) becomes

$$\Delta V(k) \leq \frac{1}{d(k)} \sum_{\ell=k-d(k)}^{k-1} \varphi^T(k,\ell) \Psi_2(d(k)) \varphi(k,\ell), \tag{3.17}$$

where

$$\varphi(k,\ell) = \left[x^T(k) \;\; x^T(k-d(k)) \;\; y^T(\ell)\right]^T,$$
$$\Psi_2(d(k)) = \begin{bmatrix} \Psi_1(1,1) & \Psi_1(1,2) & -d(k)Y \\ * & \Psi_1(2,2) & -d(k)W \\ * & * & -d(k)R \end{bmatrix}. \tag{3.18}$$

By Lemma 2.2, it can be verified that $\Delta V(k) < 0$ if (3.2) is true. Therefore, system (Σ_{ncl}) with $\omega(k) = 0$ is asymptotically stable according to the Lyapunov-Krasovskii stability theorem.

Second, we show that subject to the zero initial condition, the discrete time-delay system (Σ_{ncl}) has a prescribed H_∞ disturbance attenuation level $\gamma > 0$, that is, $\|z\|_2 < \gamma\|\omega\|_2$ for all nonzero $\omega \in \mathcal{L}_2$. To this end, we introduce the following performance index:

$$J_N = \sum_{k=0}^{N}\left[z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k)\right], \tag{3.19}$$

where the scalar $N \in \mathbb{N}$. Noting the zero initial condition and (3.14), one can verify that

$$\begin{aligned} J_N &= \sum_{k=0}^{N}\left[z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) + \Delta V(k)\right] - V(N+1) \\ &\le \sum_{k=0}^{\infty}\left[z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) + \Delta V(k)\right] \\ &= \sum_{k=0}^{\infty}\sum_{\ell=k-d(k)}^{k-1}\frac{1}{d(k)}\eta^T(k,\ell)\Psi_3(d(k))\eta(k,\ell), \end{aligned} \tag{3.20}$$

where

$$\Psi_3(d(k)) = \begin{bmatrix} \Psi_3(1,1) & \Psi_3(1,2) & -d(k)Y & \Psi_3(1,4) \\ * & \Psi_3(2,2) & -d(k)W & \Psi_3(2,4) \\ * & * & -d(k)R & 0 \\ * & * & * & \Psi_3(4,4) \end{bmatrix}, \tag{3.21}$$

with

$$\begin{gathered} \Psi_3(1,1) = \Psi_1(1,1) + \overline{C}_0^T\overline{C}_0, \qquad \Psi_3(1,2) = \Psi_1(1,2) + \overline{C}_0^T C_1, \\ \Psi_3(1,4) = \Psi_1(1,4) + \overline{C}_0^T D_{11}, \qquad \Psi_3(2,2) = \Psi_1(2,2) + C_1^T C_1, \\ \Psi_3(2,4) = \Psi_1(2,4) + C_1^T D_{11}, \qquad \Psi_3(4,4) = \Psi_1(4,4) + D_{11}^T D_{11} - \gamma^2 I. \end{gathered} \tag{3.22}$$

Now, by Lemma 2.2, it follows from (3.2) that $\Psi_3(d(k)) < 0$, which together with (3.20) ensures that $J_N < 0$. This further implies that $\|z\|_2 < \gamma\|\omega\|_2$ holds under the zero initial condition. This complements the proof. □

Remark 3.2. In Theorem 3.1, two slack variables Y and W are introduced to reduce some conservatism in the existing delay-dependent conditions for the H_∞ control problem, while no bounding techniques for cross terms are involved. By doing so, we have provided a more flexible condition in (3.2). The advantage of these introduced variables can be seen from the numerical example later.

Remark 3.3. In [19], based on a descriptor system transformation method, a delay-dependent condition on the H_∞ control issue for system (Σ_{ncl}) was proposed. However, there is an additional constraint on the matrix A_1, that is, A_1 should be nonsingular. While, Theorem 3.1 in this paper gets rid of this constraint.

Very recently, for discrete time-delay system (Σ_{ncl}), a less conservative delay-dependent H_∞ condition was proposed in [20]. The rationale behind the method lies in providing *a finite sum inequality* as follows.

Lemma 3.4 (finite sum inequality [20, Lemma 1]). *For any matrices M_1, M_2, Z_{11}, Z_{12}, Z_{22}, $R \in \mathbb{R}^{n\times n}$, where $R = R^T \geq 0$, and Z_{13}, Z_{23}, $M_3 \in \mathbb{R}^{n\times q}$, $Z_{33} \in \mathbb{R}^{q\times q}$, the following inequality holds:*

$$-\sum_{\ell=k-d(k)}^{k-1} y^T(\ell)Ry(\ell) \leq \xi^T(k)\begin{bmatrix} v_{11} & v_{12} & M_3+\overline{h}Z_{13} \\ * & v_{22} & -M_3+\overline{h}Z_{23} \\ * & * & \overline{h}Z_{33}\end{bmatrix}\xi(k), \tag{3.23}$$

where

$$\xi(k) = \left[x^T(k) \;\; x^T(k-d(k)) \;\; \omega^T(k)\right]^T,$$
$$\begin{bmatrix} R & M_1 & M_2 & M_3 \\ * & Z_{11} & Z_{12} & Z_{13} \\ * & * & Z_{22} & Z_{23} \\ * & * & * & Z_{33}\end{bmatrix} \geq 0, \tag{3.24}$$

with

$$\begin{aligned} v_{11} &= M_1^T + M_1 + \overline{h}Z_{11}, \\ v_{12} &= -M_1^T + M_2 + \overline{h}Z_{12}, \\ v_{22} &= -M_2^T - M_2 + \overline{h}Z_{22}. \end{aligned} \tag{3.25}$$

By Theorem 3.1, we can obtain the following delay-dependent H_∞ disturbance attenuation condition, which has been reported in [20] recently.

Corollary 3.5 (see [20, Proposition 1]). *For a given $\gamma > 0$, system (Σ_{ncl}) is asymptotically stable with a prescribed H_∞ disturbance attenuation level γ for any time-varying delay satisfying* (2.4) *if there exist matrices $P > 0, Q > 0, R > 0, M_i, Z_{ij}$ $(i, j = 1, 2, 3)$ with appropriate dimensions such that* (3.24) *and the following inequality hold:*

$$\Omega_1 := \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \left(\overline{A}_0 - I\right)^T & \overline{h}\left(\overline{A}_0 - I\right)^T & \overline{C}_0^T \\ * & \Phi_{22} & \Phi_{23} & A_1^T & \overline{h}A_1^T & C_1^T \\ * & * & \Phi_{33} & B_1^T & \overline{h}B_1^T & D_{11}^T \\ * & * & * & -P^{-1} & 0 & 0 \\ * & * & * & * & -\overline{h}R^{-1} & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{3.26}$$

where

$$\begin{aligned}
\Phi_{11} &= P\left(\overline{A}_0 - I\right) + \left(\overline{A}_0 - I\right)^T P + M_1^T + M_1 + \left(\overline{h} - \underline{h} + 1\right)Q + \overline{h}Z_{11}, \\
\Phi_{12} &= PA_1 - M_1^T + M_2 + \overline{h}Z_{12}, \\
\Phi_{13} &= PB_1 + M_3 + \overline{h}Z_{13}, \\
\Phi_{22} &= -M_2^T - M_2 - Q + \overline{h}Z_{22}, \\
\Phi_{23} &= -M_3 + \overline{h}Z_{23}, \\
\Phi_{33} &= -\gamma^2 I + \overline{h}Z_{33}.
\end{aligned} \tag{3.27}$$

In the following, we will show that the result in [20] can be deduced from Theorem 3.1.

Proof. It is easy to see that (3.26) is equivalent to

$$\begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \left(\overline{A}_0 - I\right)^T P & \left(\overline{A}_0 - I\right)^T \overline{h}R & \overline{C}_0^T \\
* & \Phi_{22} & \Phi_{23} & A_1^T P & A_1^T \overline{h}R & C_1^T \\
* & * & \Phi_{33} & B_1^T P & B_1^T \overline{h}R & D_{11}^T \\
* & * & * & -P & 0 & 0 \\
* & * & * & * & -\overline{h}R & 0 \\
* & * & * & * & * & -I
\end{bmatrix} < 0. \tag{3.28}$$

By (3.24) and using the Schur complement formula, we have

$$\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
* & Z_{22} & Z_{23} \\
* & * & Z_{33}
\end{bmatrix} - \begin{bmatrix} M_1^T \\ M_2^T \\ M_3^T \end{bmatrix} R^{-1} \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix} \geq 0. \tag{3.29}$$

This together with (3.28) implies

$$\begin{bmatrix}
\overline{\Phi}_{11} & \overline{\Phi}_{12} & \overline{\Phi}_{13} & \left(\overline{A}_0 - I\right)^T P & \left(\overline{A}_0 - I\right)^T \overline{h}R & \overline{C}_0^T & \overline{h}M_1^T \\
* & \overline{\Phi}_{22} & \overline{\Phi}_{23} & A_1^T P & A_1^T \overline{h}R & C_1^T & \overline{h}M_2^T \\
* & * & -\gamma^2 I & B_1^T P & B_1^T \overline{h}R & D_{11}^T & \overline{h}M_3^T \\
* & * & * & -P & 0 & 0 & 0 \\
* & * & * & * & -\overline{h}R & 0 & 0 \\
* & * & * & * & * & -I & 0 \\
* & * & * & * & * & 0 & -\overline{h}R
\end{bmatrix} < 0, \tag{3.30}$$

where

$$\begin{aligned}
\overline{\Phi}_{11} &= P\left(\overline{A}_0 - I\right) + \left(\overline{A}_0 - I\right)^T P + M_1^T + M_1 + \left(\overline{h} - \underline{h} + 1\right)Q, \\
\overline{\Phi}_{12} &= PA_1 - M_1^T + M_2, \\
\overline{\Phi}_{13} &= PB_1 + M_3, \\
\overline{\Phi}_{22} &= -M_2^T - M_2 - Q, \\
\overline{\Phi}_{23} &= -M_3.
\end{aligned} \tag{3.31}$$

Denote

$$L_1 = \begin{bmatrix} I & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}. \tag{3.32}$$

Pre- and postmultiplying (3.30) by L_1 and L_1^T, respectively, yields

$$\begin{bmatrix}
F & -M_1^T + M_2 & M_3 & \overline{A}_0^T P & \left(\overline{A}_0 - I\right)^T \overline{h}R & \overline{C}_0^T & \overline{h}M_1^T \\
* & -M_2^T - M_2 - Q & -M_3 & A_1^T P & A_1^T \overline{h}R & C_1^T & \overline{h}M_2^T \\
* & * & -\gamma^2 I & B_1^T P & B_1^T \overline{h}R & D_{11}^T & \overline{h}M_3^T \\
* & * & * & -P & 0 & 0 & 0 \\
* & * & * & * & -\overline{h}R & 0 & 0 \\
* & * & * & * & * & -I & 0 \\
* & * & * & * & * & 0 & -\overline{h}R
\end{bmatrix} < 0, \tag{3.33}$$

where $F = -P + M_1^T + M_1 + (\overline{h} - \underline{h} + 1)Q$.

Letting $M_1 = Y^T, M_2 = W^T, M_3 = 0$, we have

$$\begin{bmatrix} \Theta_{11} & -Y+W^T & 0 & \overline{A}_0^T P & \left(\overline{A}_0 - I\right)^T \overline{h}R & \overline{C}_0^T & \overline{h}Y \\ * & -W-W^T-Q & 0 & A_1^T P & A_1^T \overline{h}R & C_1^T & \overline{h}W \\ * & * & -\gamma^2 I & B_1^T P & B_1^T \overline{h}R & D_{11}^T & 0 \\ * & * & * & -P & 0 & 0 & 0 \\ * & * & * & * & -\overline{h}R & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & 0 & -\overline{h}R \end{bmatrix} < 0, \tag{3.34}$$

where Θ_{11} is defined in Theorem 3.1.

Denote

$$L_2 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}. \tag{3.35}$$

Pre- and postmultiplying (3.34) by L_2 and its transpose, respectively, yields (3.2). This completes the proof. □

Remark 3.6. From Corollary 3.5, it is noted that Theorem 3.1 in this paper is less conservative than Corollary 1 which was reported in [20]. It should be pointed out that neither model transformation (e.g., [23]) nor bounding technique (e.g., [19]) is employed here. Although it is proved that the finite sum inequality approach in [20] is better than other reported ones when dealing with delay-dependent stability analysis problem for discrete time-delay systems, it still gives relatively conservative results.

Remark 3.7. Compared with the delay-dependent H_∞ disturbance attenuation condition in [20], it is worth noting that one of the advantages in our paper is that the inequality in (3.2) involves significantly fewer variables than those in [20]. Specifically, in the case when $x(k) \in \mathbb{R}^n$, the number of the variables to be solved in (3.2) is $(n(7n+3))/2$, while in [20] the number of variables is $(13n^2+2q^2+6nq+3n)/2$. When $q = n$, that is, $\omega(k) \in \mathbb{R}^n$, the number of variables in [20] becomes $(21n(n+3))/2$, which is around 3 times more than those in Theorem 3.1. Therefore, from mathematical and practical points of view, our condition is more desirable than that in [20].

Now, we are in a position to solve the controller gain K from (3.2).

Define $\Pi_1 = \operatorname{diag}\{I, I, I, I, I, R^{-1}, I\}$. Multiplying (3.2) by Π_1^T and Π_1 on the left-hand side and the right-hand side, respectively, yields

$$\begin{bmatrix} \Theta_{11} & -Y+W^T & -\overline{h}Y & 0 & \overline{A}_0^T P & \overline{h}\left(\overline{A}_0-I\right)^T & \overline{C}_0^T \\ * & -W-W^T-Q & -\overline{h}W & 0 & A_1^T P & \overline{h}A_1^T & C_1^T \\ * & * & -\overline{h}R & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & B_1^T P & \overline{h}B_1^T & D_{11}^T \\ * & * & * & * & -P & 0 & 0 \\ * & * & * & * & * & -\overline{h}R^{-1} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0. \tag{3.36}$$

Let $P^{-1} = X$. Defining $\Pi_2 = \operatorname{diag}\{X, X, X, I, X, I, I\}$, then after performing congruence transformations on (3.36) by Π_2, we have

$$\begin{bmatrix} X\Theta_{11}X & X(-Y+W^T)X & -\overline{h}XYX & 0 & X\overline{A}_0^T & \overline{h}X\left(\overline{A}_0-I\right)^T & X\overline{C}_0^T \\ * & X(-W-W^T-Q)X & -\overline{h}XWX & 0 & XA_1^T & \overline{h}XA_1^T & XC_1^T \\ * & * & -\overline{h}XRX & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & B_1^T & \overline{h}B_1^T & D_{11}^T \\ * & * & * & * & -X & 0 & 0 \\ * & * & * & * & * & -\overline{h}R^{-1} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0. \tag{3.37}$$

Setting $\widetilde{Y} = XYX$, $\widetilde{Q} = XQX$, $\widetilde{W} = XWX$, $\widetilde{K} = KX$, $\widetilde{R} = R^{-1}$, we obtain

$$\begin{bmatrix} \widetilde{\Theta}_{11} & -\widetilde{Y}+\widetilde{W}^T & -\overline{h}\widetilde{Y} & 0 & \widetilde{\Theta}_{12} & \widetilde{\Theta}_{13} & \widetilde{\Theta}_{14} \\ * & -\widetilde{W}-\widetilde{W}^T-\widetilde{Q} & -\overline{h}\widetilde{W} & 0 & XA_1^T & \overline{h}XA_1^T & XC_1^T \\ * & * & -\overline{h}X\widetilde{R}^{-1}X & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & B_1^T & \overline{h}B_1^T & D_{11}^T \\ * & * & * & * & -X & 0 & 0 \\ * & * & * & * & * & -\overline{h}\widetilde{R} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{3.38}$$

where

$$\begin{gathered} \widetilde{\Theta}_{11} = -X + \widetilde{Y} + \widetilde{Y}^T + \left(\overline{h} - \underline{h} + 1\right)\widetilde{Q}, \qquad \widetilde{\Theta}_{12} = XA_0^T + \widetilde{K}^T B_2^T, \\ \widetilde{\Theta}_{13} = \overline{h}XA_0^T + \overline{h}\widetilde{K}^T B_2^T - \overline{h}X, \qquad \widetilde{\Theta}_{14} = XC_0^T + \widetilde{K}^T D_{12}^T. \end{gathered} \tag{3.39}$$

It is clear that (3.38) is a nonlinear matrix inequality in the matrix variables X, $\tilde{Q}$, $\tilde{R}$, $\tilde{Y}$, $\widetilde{W}$, and $\tilde{K}$, due to the existence of the nonlinear term $-\bar{h}X\tilde{R}^{-1}X$. In order to solve the desired controller K, we will propose three methods in the sequel.

Let $R = P$, that is, take a particular Lyapunov-Krasovskii functional in (3.5). Then, the following result holds naturally.

Theorem 3.8. *System* (Σ_{ncl}) *is asymptotically stable with a prescribed* H_∞ *disturbance attenuation level* $\gamma > 0$*, if there exist matrices* $X > 0$*,* $\tilde{Q} > 0$*,* $\tilde{Y}$*,* $\widetilde{W}$*, and* $\tilde{K}$ *of appropriate dimensions such that the following LMI holds:*

$$\Xi_1 = \begin{bmatrix} \tilde{\Theta}_{11} & -\tilde{Y} + \widetilde{W}^T & -\bar{h}\tilde{Y} & 0 & \tilde{\Theta}_{12} & \tilde{\Theta}_{13} & \tilde{\Theta}_{14} \\ * & -\widetilde{W} - \widetilde{W}^T - \tilde{Q} & -\bar{h}\widetilde{W} & 0 & XA_1^T & \bar{h}XA_1^T & XC_1^T \\ * & * & -\bar{h}X & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & B_1^T & \bar{h}B_1^T & D_{11}^T \\ * & * & * & * & -X & 0 & 0 \\ * & * & * & * & * & -\bar{h}X & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0. \tag{3.40}$$

Moreover, a robustly stabilizing state feedback controller is given by (2.5) *with* $K = \tilde{K}X^{-1}$.

Remark 3.9. Theorem 3.8 provides a simple method in solving the controller gain K by introducing a special Lyapunov-Krasovskii functional. Although it has some good merits, it may bring some conservatism due to the restriction of $R = P$.

Note that

$$\left(\tilde{R} - X\right)^T \tilde{R}^{-1}\left(\tilde{R} - X\right) \geq 0, \tag{3.41}$$

which implies

$$-\bar{h}X\tilde{R}^{-1}X \leq \bar{h}\tilde{R} - 2\bar{h}X. \tag{3.42}$$

From (3.38) and (3.42), the following theorem follows immediately.

Theorem 3.10. *System* (Σ_{ncl}) *is asymptotically stable with a prescribed* H_∞ *disturbance attenuation level* $\gamma > 0$*, if there exist matrices* $X > 0$, $\widetilde{Q} > 0$, $\widetilde{R} > 0$, $\widetilde{Y}$, $\widetilde{W}$, *and* $\widetilde{K}$ *of appropriate dimensions such that the following LMI holds:*

$$\Xi_2 = \begin{bmatrix} \widetilde{\Theta}_{11} & -\widetilde{Y} + \widetilde{W}^T & -\overline{h}\widetilde{Y} & 0 & \widetilde{\Theta}_{12} & \widetilde{\Theta}_{13} & \widetilde{\Theta}_{14} \\ * & -\widetilde{W} - \widetilde{W}^T - \widetilde{Q} & -\overline{h}\widetilde{W} & 0 & XA_1^T & \overline{h}XA_1^T & XC_1^T \\ * & * & \overline{h}\widetilde{R} - 2\overline{h}X & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & B_1^T & \overline{h}B_1^T & D_{11}^T \\ * & * & * & * & -X & 0 & 0 \\ * & * & * & * & * & -\overline{h}\widetilde{R} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0. \tag{3.43}$$

Moreover, a robustly stabilizing state feedback controller is given by (2.5) *with* $K = \widetilde{K}X^{-1}$.

Remark 3.11. It is clear that there also exists conservatism because of the replacement $-\overline{h}X\widetilde{R}^{-1}X$ with $\overline{h}\widetilde{R} - 2\overline{h}X$.

In the sequel, we will resort to the cone complementary linearization method [24] to further reduce the conservatism. Introduce a new matrix variable $S > 0$, which satisfies

$$X\widetilde{R}^{-1}X \geq S. \tag{3.44}$$

It is easily seen that inequality (3.44) is more general than that in (3.42). Note that (3.44) is equivalent to

$$\begin{bmatrix} S^{-1} & X^{-1} \\ * & \widetilde{R}^{-1} \end{bmatrix} \geq 0. \tag{3.45}$$

Letting $\mathcal{S} = S^{-1}$, $\mathcal{X} = X^{-1}$, $\mathcal{R} = \widetilde{R}^{-1}$, we obtain the following theorem.

Theorem 3.12. *System* (Σ_{ncl}) *is asymptotically stable with a prescribed* H_∞ *disturbance attenuation level* $\gamma > 0$, *if there exist matrices* $X > 0$, $\widetilde{Q} > 0$, $\widetilde{R} > 0$, $\mathcal{R} > 0$, $S > 0$, $\mathcal{S} > 0$, $\mathcal{X} > 0$, $\widetilde{Y}$, $\widetilde{W}$, *and* $\widetilde{K}$ *of appropriate dimensions such that*

$$\Xi_3 = \begin{bmatrix} \widetilde{\Theta}_{11} & -\widetilde{Y} + \widetilde{W}^T & -\overline{h}\widetilde{Y} & 0 & \widetilde{\Theta}_{12} & \widetilde{\Theta}_{13} & \widetilde{\Theta}_{14} \\ * & -\widetilde{W} - \widetilde{W}^T - \widetilde{Q} & -\overline{h}\widetilde{W} & 0 & XA_1^T & \overline{h}XA_1^T & XC_1^T \\ * & * & -\overline{h}S & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & B_1^T & \overline{h}B_1^T & D_{11}^T \\ * & * & * & * & -X & 0 & 0 \\ * & * & * & * & * & -\overline{h}\widetilde{R} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{3.46}$$

$$\begin{bmatrix} \mathcal{S} & \mathcal{X} \\ * & \mathcal{R} \end{bmatrix} \geq 0, \tag{3.47}$$

$$S\mathcal{S} = I, \qquad X\mathcal{X} = I, \qquad \mathcal{R}\widetilde{R} = I. \tag{3.48}$$

Moreover, a robustly stabilizing state feedback controller is given by (2.5) *with* $K = \widetilde{K}X^{-1}$.

Remark 3.13. As one can see that the inequality conditions in Theorem 3.12 are not strict LMI conditions due to the equation constraints in (3.48). However, by resorting to the cone complementary linearization method in [24] and the optimization solver in [25], the nonconvex feasibility problem formulated by (3.46), (3.47), and (3.48) can be transformed into the following nonlinear minimization problem subject to LMIs:

$$\begin{gathered} \text{minimize} \quad \mathrm{Tr}\left(S\mathcal{S} + X\mathcal{X} + \mathcal{R}\widetilde{R}\right) \\ \text{subject to} \quad (3.46) \text{ and } (3.47), \\ \begin{bmatrix} \mathcal{S} & I \\ I & S \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \mathcal{X} & I \\ I & X \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \mathcal{R} & I \\ I & \widetilde{R} \end{bmatrix} \geq 0. \end{gathered} \tag{3.49}$$

According to the cone complementarity problem (CCP) in [24], if the solution of the above minimization problem is $6n$, we can say from Theorem 3.12 that system (Σ_{ncl}) is asymptotically stable with a prescribed H_∞ disturbance attenuation level $\gamma > 0$ via the controller (2.5) with $K = \widetilde{K}X^{-1}$. Although it is very difficult to always find the global optimal solution, the proposed nonlinear minimizatiion problem is easier to solve than the original nonconvex feasibility problem. Based on the linearization method in [24], we can solve the above nonlinear minimization problem using an iterative algorithm presented in the following.

Algorithm 3.14. We have the following steps.

Step 1. Choose a sufficiently initial $\gamma_{\text{ini}} > 0$ such that (3.46), (3.47), and (3.49) are feasible. Set $\gamma_{s0} = \gamma_{\text{ini}}$.

Step 2. Find a feasible set $(X^0, \mathcal{X}^0, S^0, \mathcal{S}^0, \mathcal{R}^0, \widetilde{R}^0, \widetilde{Y}^0, \widetilde{W}^0, \widetilde{K}^0, \widetilde{Q}^0, \varepsilon^0)$ satisfying (3.46), (3.47), and (3.49). Set $k = 0$.

Step 3. Solve the following LMI problem:

$$\begin{aligned}&\text{minimize} \quad \operatorname{Tr}\left(S^k\mathcal{S} + X^k\mathcal{X} + \mathcal{R}^k\widetilde{R} + S\mathcal{S}^k + X\mathcal{X}^k + \mathcal{R}\widetilde{R}^k\right)\\ &\text{subject to} \quad (3.46),\ (3.47),\ \text{and}\ (3.49).\end{aligned} \tag{3.50}$$

Set $S^{k+1} = S$, $X^{k+1} = X$, $\mathcal{R}^{k+1} = \mathcal{R}$, $\mathcal{S}^{k+1} = \mathcal{S}$, $\mathcal{X}^{k+1} = \mathcal{X}$, $\widetilde{R}^{k+1} = \widetilde{R}$.

Step 4. If matrix (3.46) is satisfied and

$$\left|\operatorname{Tr}\left(S^k\mathcal{S} + X^k\mathcal{X} + \mathcal{R}^k\widetilde{R} + S\mathcal{S}^k + X\mathcal{X}^k + \mathcal{R}\widetilde{R}^k\right) - 6n\right| < \delta \tag{3.51}$$

for some sufficient small scalar $\delta > 0$, then decrease γ_{ini} to some extent and set $\gamma_{s0} = \gamma_{\text{ini}}$ and go to Step 2. If one of the conditions in (3.47) and (3.51) is not satisfied within a specified number of iterations, then exit. Otherwise, set $k = k + 1$ and go to Step 3.

Now, we are in a position to present the delay-dependent robust conditions concerning H_∞ control of system (Σ) with uncertainties based on Theorems 3.8, 3.10, and 3.12, respectively. By Lemma 2.3, we can easily have the following results.

Theorem 3.15. *System* (Σ) *is asymptotically stable with a prescribed* H_∞ *disturbance attenuation level* $\gamma > 0$, *if there exist a scalar* $\varepsilon > 0$, *matrices* $X > 0$, $\widetilde{Q} > 0$, $\widetilde{Y}$, $\widetilde{W}$, *and* $\widetilde{K}$ *of appropriate dimensions such that the following LMI holds:*

$$\begin{bmatrix} \Xi_1 & \varepsilon\mathfrak{D} & \mathcal{E}^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0, \tag{3.52}$$

where Ξ_1 *is defined in* (3.40), *and*

$$\mathfrak{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathfrak{D}_1^T & \overline{h}\mathfrak{D}_1^T & \mathfrak{D}_2^T \end{bmatrix}^T, \tag{3.53}$$

$$\mathcal{E} = \begin{bmatrix} E_1X + E_4\widetilde{K} & E_2X & 0 & E_3 & 0 & 0 & 0 \end{bmatrix}. \tag{3.54}$$

Moreover, a robustly stabilizing state feedback controller is given by (2.5) *with* $K = \widetilde{K}X^{-1}$.

Theorem 3.16. *System* (Σ) *is asymptotically stable with a prescribed* H_∞ *disturbance attenuation level* $\gamma > 0$*, if there exist a scalar* $\varepsilon > 0$*, matrices* $X > 0$*,* $\widetilde{Q} > 0$*,* $\widetilde{R} > 0$*,* $\widetilde{Y}$*,* $\widetilde{W}$*, and* $\widetilde{K}$ *of appropriate dimensions such that the following LMI holds:*

$$\begin{bmatrix} \Xi_2 & \varepsilon\mathfrak{D} & \mathcal{E}^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0, \tag{3.55}$$

where Ξ_2*,* $\mathfrak{D}$*,* $\mathcal{E}$ *are defined in* (3.43), (3.53), *and* (3.54), *respectively. A robustly stabilizing state feedback controller is given by* (2.5) *with* $K = \widetilde{K}X^{-1}$.

Theorem 3.17. *System* (Σ) *is asymptotically stable with a prescribed* H_∞ *disturbance attenuation level* $\gamma > 0$*, if there exist a scalar* $\varepsilon > 0$*, matrices* $X > 0$*,* $\widetilde{Q} > 0$*,* $\widetilde{R} > 0$*,* $\mathcal{R} > 0$*,* $S > 0$*,* $\mathcal{S} > 0$*,* $\mathcal{X} > 0$*,* $\widetilde{Y}$*,* $\widetilde{W}$*, and* $\widetilde{K}$ *of appropriate dimensions such that* (3.47), (3.49) *and the following LMI holds:*

$$\begin{bmatrix} \Xi_3 & \varepsilon\mathfrak{D} & \mathcal{E}^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0, \tag{3.56}$$

where Ξ_3*,* $\mathfrak{D}$*,* $\mathcal{E}$ *are defined in* (3.46), (3.53), *and* (3.54), *respectively.* $K = \widetilde{K}X^{-1}$ *is the corresponding controller which is derived.*

4. Examples

In this section, two examples are used to demonstrate the effectiveness of the proposed methods.

Example 4.1. Consider the following discrete-time systems with time-varying delay:

$$\begin{aligned} x(k+1) &= (A_0 + \mathfrak{D}_1 F(k)E_1)x(k) + (A_1 + \mathfrak{D}_1 F(k)E_2)x(k-d(k)) + B_1\omega(k) + B_2 u(k), \\ z(k) &= C_0 x(k) + D_{12}u(k), \\ x(k) &= 0, \quad \forall k \le 0, \end{aligned} \tag{4.1}$$

where

$$\begin{gathered} A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1.01 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -0.02 & -0.005 \\ 0 & -0.01 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ B_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \qquad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D_{12} = 0.1, \qquad \mathfrak{D}_1 = 0.2I, \\ E_1 = E_2 = 0.01I, \qquad F^T(k)F(k) \le I, \end{gathered} \tag{4.2}$$

and $d(k)$ is a delay satisfying (2.4).

Table 1: The achieved minimum H_∞ performances γ in this paper and corresponding controller gain K for $\overline{h} = \underline{h} = 64$.

Method	γ	K
Theorem 3.15	94.0	$[-1.7345 \quad -2.1540]$
Theorem 3.16	51.1	$[-4.0969 \quad -3.4550]$
Theorem 3.17	11.2	$[10.6422 \quad -74.5353]$

Table 2: The iterations and corresponding controller gain K for $\overline{h} = \underline{h} = 64$ under different cases of γ.

Method	γ	K	Iterations
[20]	15.5	$[-46.4416 \quad -68.1845]$	235
Theorem 3.17	15.5	$[-4.4617 \quad -14.5328]$	47
[20]	16	$[-44.9680 \quad -62.3831]$	223
Theorem 3.17	16	$[-4.8684 \quad -12.6606]$	40
[20]	17	$[-39.0994 \quad -53.8247]$	186
Theorem 3.17	17	$[-5.0961 \quad -11.3692]$	39
[20]	18	$[-36.1621 \quad -48.5035]$	171
Theorem 3.17	18	$[-5.2827 \quad -10.4921]$	40
[20]	20	$[-31.9456 \quad -41.2442]$	154
Theorem 3.17	20	$[-5.4217 \quad -8.6190]$	35

In the following, different cases of $d(k)$ are involved.

Case 1. Delay $d(k)$ is time invariant.

First, suppose $\overline{h} = \underline{h} = 64$. For this situation, we will compare the results in Theorems 3.15, 3.16, and 3.17. For this reason, we calculate the minimum value of γ for which system (4.1) is robustly stabilizable via state feedback (2.5). The obtained results are listed in Table 1, from which we can see that the conditions in Theorem 3.17 are less conservative than those in Theorems 3.15 and 3.16. Therefore, we will only compare the results in Theorem 3.17 with those in the previous literatures in the sequel.

Second, when $\overline{h} = \underline{h} = 64$, Zhang and Han [20] also calculated the achieved minimum H_∞ performances γ, the corresponding controller gain K, and the iterations. Here, in order to show much less conservative results (or lower computational burden) of Theorem 3.17 than [20], we give Table 2. Noting from this table, we conclude that in order to achieve the same disturbance attenuation level, Theorem 3.17 needs significantly less iterations and smaller gain. From this table, we also have verified Remark 3.9.

Third, in [19, 20], the authors also calculated the achieved minimum H_∞ performance for $\overline{h} = 64$, respectively. However, according to Theorem 3.17, much less H_∞ performance is obtained, which is listed in Table 3. From this table, one can see that Theorem 3.17 in this paper provides much less H_∞ performances.

Now, we are in a position to calculate the maximum delay bound $\overline{h}$, which can guarantee that system (4.1) is robustly stabilizable via state feedback (2.5). The details are given in Table 4.

Case 2. Delay $d(k)$ is time varying.

Table 3: The achieved minimum H_∞ performances γ and corresponding controller gain K for $\overline{h} = \underline{h} = 64$.

Method	γ	K
[19]	180.07	$[-1.7345 \quad -2.1540]$
[20]	15.5	$[-46.4416 \quad -68.1845]$
Theorem 3.17	11.2	$[10.6422 \quad -74.5353]$

Table 4: The maximum delay bound $\overline{h}$ for system (4.1) under Case 1.

Method	$\overline{h}$	K
[19]	41	$[-0.6311 \quad -2.3615]$
[26]	67	Unprovided
[20]	70	$[-93.2010 \quad -71.2670]$
Theorem 3.17	180	$[-8.9659 \quad -67.0971]$

Table 5: The maximum delay bound $\overline{h}$ for system (4.1) under Case 2.

Method	$\overline{h}$	K (corresponding to the minimum γ)
[19]	43	$[-6.7766 \quad -20.5924]$
[20]	48	Unprovided
Theorem 3.17	52	$[-136.9648 \quad -49.5596]$

Under this case, Fridman and Shaked [19] concluded that system (4.1) can be stabilized for all $\overline{h} \le 43$. In [20], it is obtained that system (4.1) is robustly stabilizable for $\overline{h} \le 48$. However, by Theorem 3.17, we have that system (4.1) is robustly stabilizable for $\overline{h} \le 52$. The details are shown in Table 5.

In [20], Zhang and Han also gave the minimum H_∞ performances γ and corresponding controller gain K for a set of $\underline{h}$ when $\overline{h} = 48$. Here, we also present a table to demonstrate the lower computational complexity and smaller controller gains than those in [20], which are listed in Table 6.

Example 4.2. Consider the discrete-time systems (4.1) (this example was first presented in [27]) with matrices

$$A_0 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad D_{12} = 1, \qquad \mathfrak{D}_1 = 0, \tag{4.3}$$
$$E_1 = E_2 = 0.$$

From Table 7, we can see that the condition in our paper can obtain a smaller H_∞ performance γ than [18, 27] for this example.

Table 6: The corresponding controller gain K for different $\underline{h}$ and γ when $\overline{h} = 48$.

Method	$\underline{h}$	γ	K
[20]	1	65	$[-24.8606 \quad -74.4157]$
Theorem 3.17	1	65	$[-2.2015 \quad -11.1964]$
[20]	8	50	$[-24.9197 \quad -76.6840]$
Theorem 3.17	8	50	$[-1.9750 \quad -14.275]$
[20]	18	40	$[-22.2636 \quad -68.8382]$
Theorem 3.17	18	40	$[-1.9376 \quad -15.3371]$
[20]	28	30	$[-18.0179 \quad -76.6840]$
Theorem 3.17	28	30	$[-1.3337 \quad -17.0357]$
[20]	38	20	$[-14.5769 \quad -50.1003]$
Theorem 3.17	38	20	$[1.4013 \quad -23.7523]$
[20]	43	18	$[-8.6551 \quad -36.0802]$
Theorem 3.17	43	18	$[1.1377 \quad -16.8044]$

Table 7: The achieved minimum H_∞ performances γ and corresponding controller gain K.

Method	γ	K
[27]	0.1166	$[-1.1689 \quad -1.0000]$
[18]	0.3500	$[-1.2430 \quad -0.9977]$
Theorem 3.17	0.1126	$[-1.8077 \quad -1.0621]$

5. Conclusions

The problem of H_∞ control for uncertain discrete-time systems with time-varying delay has been studied. By introducing slack matrix variables, delay-dependent LMI based conditions have been developed to design a stable state feedback controller, which ensures the asymptotic stability of the resulting closed-loop system and guarantees a prescribed disturbance attenuation level irrespective of all the admissible uncertainties. Numerical examples have been provided to demonstrate the effectiveness and applicability of the proposed approach.

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