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Research Article

Exact Solutions for a Third-Order KdV Equation with Variable Coefficients and Forcing Term

Alvaro H. Salas S[1,2] and Cesar A. Gómez S[3]

[1] *Department of Mathematics, Universidad de Caldas, Campus La Nubia, Manizales, Caldas Cll 65 # 26-10, A. A. 275, Colombia*
[2] *Department of Mathematics and Statistics, National University of Colombia, Manizales Cll 65, Palogrande Stadium, Colombia*
[3] *Department of Mathematics, National University of Colombia, Bogotá, Colombia*

Correspondence should be addressed to Alvaro H. Salas S, asalash2002@yahoo.com

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The general projective Riccati equation method and the Exp-function method are used to construct generalized soliton solutions and periodic solutions to special KdV equation with variable coefficients and forcing term.

1. Introduction

It is well known that in an early phase of the development of the solitons theory, there were already many applications in physics and engineering. In particular, traveling waves as solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1}$$

have been of some interest since 150 years. Some generalizations of this last equation have been studied recently. For instance, the equation

$$u_t + k_1 t^n u u_x + k_2 t^m u_{xxx} = 0, \tag{1.2}$$

where k_1, k_2 are arbitrary constants, which have applications in physics, has been analyzed in [1] from the point of view of its exact solutions. The search of explicit solutions to nonlinear partial differential equations (NLPDEs) using analytic methods is not an easy task. However, the use of computational methods facilitates this work. Some powerful computational methods such as the tanh method [2], the generalized tanh method [3, 4], the extended tanh method [5–10], the improved tanh-coth method [11–13], the Exp-function method [14–18], the modified Exp-function method [19], the Cole-Hopf transformation [20], the projective Riccati equation method (PREM) [21, 22], the generalized projective Riccati equations method [23–25], the extended hyperbolic function method [26], and many other methods have been developed in this direction. The PREM and the Exp-function method have been used in a satisfactory form to solve some NLPDEs [15–17, 24, 25, 27–30]. In this paper, we use this last two methods to obtain soliton and periodic solutions to the following special KdV equation with variable coefficients and forcing term:

$$u_t + \alpha(t) u u_x + k\alpha(t) u_{xxx} = F(t), \tag{1.3}$$

where $F(t)$ is an external forcing function varying with time t, k is a constant, and $\alpha = \alpha(t)$ is a function of t, $\alpha(t) \neq 0$. Equation (1.3) is a generalization of the following equation [15, 17, 31]:

$$u_t + \alpha u u_x + \beta u_{xxx} = F(t) \quad (\alpha, \beta = \text{const}), \tag{1.4}$$

which results from (1.3) by taking $\alpha = \text{const}$ and $k = \beta/\alpha$.

We suppose that the solution to (1.3) has the form

$$u(x,t) = \int F(t)dt + v(x,t). \tag{1.5}$$

Therefore, (1.3) reduces to

$$v_t + \alpha(t) v v_x + \alpha(t) \int F(t)dt\, v_x + k\alpha(t) v_{xxx} = 0. \tag{1.6}$$

Now we consider the transformation

$$v = V(\xi), \quad \xi = \lambda x + \int h(t)dt, \tag{1.7}$$

where λ is a constant and $h(t)$ is an unknown function of t to be determined later. Substituting (1.7) into (1.6), we obtain

$$h(t)V'(\xi) + \alpha(t)\lambda\big(V(\xi) + f(t)\big)V'(\xi) + k\lambda^3\alpha(t)V'''(\xi) = 0, \tag{1.8}$$

where

$$f = f(t) = \int F(t)dt. \tag{1.9}$$

2. The Exp-Function Method

Recently He and Wu [14] have introduced the Exp-function method to solve nonlinear equations. In particular, the Exp-function method is a useful tool for solving nonlinear equations with high nonlinearity. The method has been used in a satisfactory way by other authors to solve a great variety of nonlinear wave equations [14–19]. The Exp-function method is very simple and straightforward and is based on a priori assumption that traveling wave solutions to a nonlinear partial differential equation in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \tag{2.1}$$

can be found using the expression

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{n=-p}^{q} a_m \exp(m\xi)}, \tag{2.2}$$

where c, d, p, and q are positive integers which could be freely chosen; a_n and b_n are unknown constants to be determined. According to this, we suppose that solutions to (1.8) can be expressed in the form

$$v(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)} = \frac{a_{-c}\exp(-c\xi) + \cdots + a_d \exp(d\xi)}{b_{-p}\exp(-p\xi) + \cdots + b_q \exp(q\xi)}, \tag{2.3}$$

where c, d, p, and q are positive integers which are unknown to be determined later; a_n and b_m are unknown constants. We have the following two cases.

2.1. Case 1: $p = c = 1$ *and* $d = q = 1$

In this case, the trial solution to (1.8) becomes

$$V(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{\exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{2.4}$$

Substituting (2.4) into (1.8) we obtain a polynomial equation in the variable $\eta = \exp(\xi)$. Equating to zero the coefficients of all powers of η yields a set of algebraic equations. Solving it with the aid of a computer, we get $a_0 = b_0(a_1 + 6k\lambda^2)$, $a_{-1} = (1/4)a_1 b_0^2$, $b_{-1} = b_0^2/4$, and $h(t) = -\lambda\alpha(t)(a_1 + f(t) + k\lambda^2)$, and, from (1.5), (1.7), and (2.3), one solution to (1.3) is given by

$$u(x,t) = \frac{a_1\left(b_0 + 2e^{\xi}\right)^2 + 24 b_0 k\lambda^2 e^{\xi}}{\left(b_0 + 2e^{\xi}\right)^2} + \int F(t)\,dt, \tag{2.5}$$

where

$$\xi = \xi(x,t) = \lambda x + \int h(t)dt = \lambda x - \int \lambda \alpha(t)\left(a_1 + k\lambda^2 + \int F(t)dt\right)dt, \tag{2.6}$$

and a_1, b_0, and λ are arbitrary real or complex numbers.

2.2. Case 2: $p = c = 2$ *and* $d = q = 2$

In this case, the trial solution to (1.8) becomes

$$V(\xi) = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi) + a_{-2} \exp(-2\xi)}{\exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi) + b_{-2} \exp(-2\xi)}. \tag{2.7}$$

As in the first case, we obtain an algebraic system. Solving it gives

$$\begin{aligned}
a_0 &= -\frac{-2a_1 b_1 (a_2 + 3k\lambda^2) + a_2 (a_2 b_1^2 + 6(b_1^2 - b_0)k\lambda^2) + a_1^2}{6k\lambda^2},\\
a_{-1} &= \frac{a_1 - a_2 b_1}{432k^3\lambda^6}\Big(-a_1 b_1 \left(27a_2 k\lambda^2 + 2a_2^2 + 72k^2\lambda^4\right)\\
&\quad + \left(a_2 + 6k\lambda^2\right)\left(12a_2 b_1^2 k\lambda^2 + a_2^2 b_1^2 + 72b_0 k^2\lambda^4\right) + a_1^2\left(a_2 + 9k\lambda^2\right)\Big),\\
a_{-2} &= \frac{1}{6912k^4\lambda^8} a_2 (a_1 - a_2 b_1)^2 \left(8a_2 b_1^2 k\lambda^2 - 2a_1 b_1 \left(a_2 + 4k\lambda^2\right) + a_2^2 b_1^2 + a_1^2 + 48b_0 k^2\lambda^4\right),\\
b_{-1} &= \frac{1}{432k^3\lambda^6}(a_1 - a_2 b_1)\left(9a_2 b_1^2 k\lambda^2 - a_1 b_1 \left(2a_2 + 9k\lambda^2\right) + a_2^2 b_1^2 + a_1^2 + 72b_0 k^2\lambda^4\right),\\
b_{-2} &= \frac{1}{6912k^4\lambda^8}(a_1 - a_2 b_1)^2 \left(8a_2 b_1^2 k\lambda^2 - 2a_1 b_1 \left(a_2 + 4k\lambda^2\right) + a_2^2 b_1^2 + a_1^2 + 48b_0 k^2\lambda^4\right),\\
h(t) &= -\lambda \alpha(t)(a_2 + f(t) + k\lambda^2).
\end{aligned} \tag{2.8}$$

From (1.5), (1.7), and (2.7) we may verify that to this set of values corresponds the solution

$$\begin{aligned}
u(x,t) = {} & \frac{1}{(a_1 - a_2 b_1 + 12k\lambda^2 e^{\xi})^2}\Big(a_1^2 a_2 + a_1\left(-2a_2^2 b_1 + 24a_2 k\lambda^2 e^{\xi} + 144k^2\lambda^4 e^{\xi}\right)\\
& \qquad + a_2\left(a_2^2 b_1^2 - 24a_2 b_1 k\lambda^2 e^{\xi} + 144k^2\lambda^4 e^{\xi}\left(e^{\xi} - b_1\right)\right)\Big)\\
& + \int F(t)dt,
\end{aligned} \tag{2.9}$$

where

$$\xi = \xi(x,t) = \lambda x + \int h(t)dt = \lambda x - \int \lambda\alpha(t)\left(a_2 + k\lambda^2 + \int F(t)dt\right)dt, \tag{2.10}$$

and a_1, a_2, b_1, and λ are arbitrary real or complex numbers.

3. General Projective Riccati Equation Method

The projective Riccati equation method was introduced initially in [21] and generalizations of this method have been used in a satisfactory way by several authors to solve nonlinear partial equations [22–25]. Using this last method [24, 25, 27–30], we seek solutions to (1.8) in the form

$$V(\xi) = a_0 + \sum_{i=1}^{m} \sigma^{i-1}(\xi)(a_i\sigma(\xi) + b_i\tau(\xi)), \tag{3.1}$$

where $a_0, a_1, b_1, \ldots$ are constants and $\sigma(\xi)$ and $\tau(\xi)$ satisfy the system

$$\sigma'(\xi) = \epsilon\sigma(\xi)\tau(\xi), \qquad \tau'(\xi) = R + \epsilon\tau^2(\xi) - \mu\sigma(\xi). \tag{3.2}$$

In (3.2), $\epsilon = \pm 1$ and R and μ are certain constants. These equations have following solutions.

Case 1. When $\epsilon = -1$ and $R \neq 0$,

$$\begin{aligned} \sigma_1(\xi) &= \frac{R\,\mathrm{sech}\left(\sqrt{R}\xi\right)}{\mu\,\mathrm{sech}\left(\sqrt{R}\xi\right) + 1}, \qquad \tau_1(\xi) = \frac{\sqrt{R}\tanh\left(\sqrt{R}\xi\right)}{\mu\,\mathrm{sech}\left(\sqrt{R}\xi\right) + 1}, \\ \sigma_2(\xi) &= \frac{R\,\mathrm{csch}\left(\sqrt{R}\xi\right)}{\mu\,\mathrm{csch}\left(\sqrt{R}\xi\right) + 1}, \qquad \tau_2(\xi) = \frac{\sqrt{R}\coth\left(\sqrt{R}\xi\right)}{\mu\,\mathrm{csch}\left(\sqrt{R}\xi\right) + 1}. \end{aligned} \tag{3.3}$$

Case 2. When $\epsilon = 1$ and $R \neq 0$,

$$\begin{aligned} \sigma_3(\xi) &= \frac{R\sec\left(\sqrt{R}\xi\right)}{\mu\sec\left(\sqrt{R}\xi\right) + 1}, \qquad \tau_3(\xi) = \frac{\sqrt{R}\tan\left(\sqrt{R}\xi\right)}{\mu\sec\left(\sqrt{R}\xi\right) + 1}, \\ \sigma_4(\xi) &= \frac{R\csc\left(\sqrt{R}\xi\right)}{\mu\csc\left(\sqrt{R}\xi\right) + 1}, \qquad \tau_4(\xi) = \frac{\sqrt{R}\cot\left(\sqrt{R}\xi\right)}{\mu\csc\left(\sqrt{R}\xi\right) + 1}. \end{aligned} \tag{3.4}$$

Case 3. When $R = \mu = 0$,

$$\sigma_5(\xi) = \frac{C}{\xi}, \qquad \tau_5(\xi) = \frac{1}{\epsilon\xi}. \tag{3.5}$$

In this last case, we seek solutions to (1.8) in the form

$$V(\xi) = \sum_{i=0}^{m} a_i \tau^i(\xi), \tag{3.6}$$

where $\tau'(\xi) = \tau^2(\xi)$.

For any pair $(\sigma(\xi), \tau(\xi))$ of functions given by (3.3) or (3.4) the following equation holds:

$$\tau^2(\xi) = -\epsilon\left[R - 2\mu\sigma(\xi) + \frac{\mu^2 - 1}{R}\sigma^2(\xi)\right]. \tag{3.7}$$

3.1. *Periodic and Soliton Solutions*

Periodic and soliton solutions are obtained when $R \neq 0$ and $\epsilon = \pm 1$ and it corresponds to the first two cases. Substituting (3.1), along with (3.2) and (3.7) into the left hand of (1.8) and collecting all terms with the same power in $\sigma^i(\xi)\tau^j(\xi)$, we get a polynomial in the variables $\sigma = \sigma(\xi)$ and $\tau = \tau(\xi)$. We may choose $m = 2$. Thus, we seek solutions to (1.8) in the form

$$V(\xi) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi) + \sigma(\xi)(a_2\sigma(\xi) + b_2\tau(\xi)). \tag{3.8}$$

We equate each coefficient of the polynomial to zero. This will give an overdetermined system of algebraic equations involving the parameters a_i, b_i $(i = 0, \dots, m)$, λ and μ, R, and the unknown function $h(t)$. Having determined these parameters, we may determine $V(\xi)$, and using (1.5) we obtain an exact solution $u(x, t)$ in a closed form. The corresponding system reads.

(i) $b_1^2 R^3 \alpha\lambda - b_1^2 R^3 \alpha\epsilon^2\lambda = 0$.

(ii) $6a_1kR\alpha\lambda^3\epsilon^4 - 6a_1kR\alpha\lambda^3\mu^2\epsilon^4 + 48a_2kR^2\alpha\lambda^3\mu\epsilon^4 - 3b_1b_2R\alpha\lambda\mu^2\epsilon^2 + 3b_1b_2R\alpha\lambda\epsilon^2 - 18a_2kR^2\alpha\lambda^3\mu\epsilon^2 + 4b_2^2R^2\alpha\lambda\mu\epsilon^2 + 3a_1a_2R^2\alpha\lambda\epsilon - b_2^2R^2\alpha\lambda\mu = 0$.

(iii) $24a_2kR\alpha\lambda^3\epsilon^4 - 24a_2kR\alpha\lambda^3\mu^2\epsilon^4 - 2b_2^2R\alpha\lambda\mu^2\epsilon^2 + 2b_2^2R\alpha\lambda\epsilon^2 + 2a_2^2R^2\alpha\lambda\epsilon = 0$.

(iv) $6b_1k\alpha\lambda^3\mu^4\epsilon^5 + 6b_1k\alpha\lambda^3\epsilon^5 - 96b_2kR\alpha\lambda^3\mu^3\epsilon^5 - 12b_1k\alpha\lambda^3\mu^2\epsilon^5 + 96b_2kR\alpha\lambda^3\mu\epsilon^5 + 36b_2kR\alpha\lambda^3\mu^3\epsilon^3 - 36b_2kR\alpha\lambda^3\mu\epsilon^3 - 3a_2b_1R\alpha\lambda\mu^2\epsilon^2 - 3a_1b_2R\alpha\lambda\mu^2\epsilon^2 + 3a_2b_1R\alpha\lambda\epsilon^2 + 3a_1b_2R\alpha\lambda\epsilon^2 + 8a_2b_2R^2\alpha\lambda\mu\epsilon^2 - a_2b_2R^2\alpha\lambda\mu = 0$.

(v) $24b_2k\alpha\lambda^3\mu^4\epsilon^5 + 24b_2k\alpha\lambda^3\epsilon^5 - 48b_2k\alpha\lambda^3\mu^2\epsilon^5 - 4a_2b_2R\alpha\lambda\mu^2\epsilon^2 + 4a_2b_2R\alpha\lambda\epsilon^2 = 0$, $-6a_1kR^3\alpha\lambda^3\epsilon^4 + 5a_1kR^3\alpha\lambda^3\epsilon^2 - 3b_1b_2R^3\alpha\lambda\epsilon^2 + 2b_1^2R^2\alpha\lambda\mu\epsilon^2 + a_0a_1R^2\alpha\lambda\epsilon + a_1fR^2\alpha\lambda\epsilon + a_1R^2h\epsilon + 2b_1b_2R^3\alpha\lambda - b_1^2R^2\alpha\lambda\mu = 0$.

(vi) $-24a_2kR^3\alpha\lambda^3\epsilon^4 + 12a_1kR^2\alpha\lambda^3\mu\epsilon^4 + 16a_2kR^3\alpha\lambda^3\epsilon^2 - b_1^2R\alpha\lambda\mu^2\epsilon^2 - 2b_2^2R^3\alpha\lambda\epsilon^2 + b_1^2R\alpha\lambda\epsilon^2 - 6a_1kR^2\alpha\lambda^3\mu\epsilon^2 + 6b_1b_2R^2\alpha\lambda\mu\epsilon^2 + a_1^2R^2\alpha\lambda\epsilon + 2a_0a_2R^2\alpha\lambda\epsilon + 2a_2fR^2\alpha\lambda\epsilon + 2a_2R^2h\epsilon + b_2^2R^3\alpha\lambda - 2b_1b_2R^2\alpha\lambda\mu = 0$.

(vii) $6b_1kR^4\alpha\lambda^3\epsilon^5 - 8b_1kR^4\alpha\lambda^3\epsilon^3 - a_0b_1R^3\alpha\lambda\epsilon^2 - b_1fR^3\alpha\lambda\epsilon^2 - b_1R^3h\epsilon^2 + 2b_1kR^4\alpha\lambda^3\epsilon + a_0b_1R^3\alpha\lambda + b_1fR^3\alpha\lambda + b_1R^3h = 0$.

(viii) $24b_2kR^4\alpha\lambda^3\epsilon^5 - 24b_1kR^3\alpha\lambda^3\mu\epsilon^5 - 28b_2kR^4\alpha\lambda^3\epsilon^3 + 28b_1kR^3\alpha\lambda^3\mu\epsilon^3 - 2a_1b_1R^3\alpha\lambda\epsilon^2 - 2a_0b_2R^3\alpha\lambda\epsilon^2 - 2b_2fR^3\alpha\lambda\epsilon^2 + 2a_0b_1R^2\alpha\lambda\mu\epsilon^2 + 2b_1fR^2\alpha\lambda\mu\epsilon^2 - 2b_2R^3h\epsilon^2 + 2b_1R^2\mu h\epsilon^2 + 5b_2kR^4\alpha\lambda^3\epsilon - 5b_1kR^3\alpha\lambda^3\mu\epsilon + a_1b_1R^3\alpha\lambda + a_0b_2R^3\alpha\lambda + b_2fR^3\alpha\lambda - a_0b_1R^2\alpha\lambda\mu - b_1fR^2\alpha\lambda\mu + b_2R^3h - b_1R^2\mu h = 0$.

(ix) $-12b_1kR^2\alpha\lambda^3\epsilon^5 + 36b_1kR^2\alpha\lambda^3\mu^2\epsilon^5 - 96b_2kR^3\alpha\lambda^3\mu\epsilon^5 + 8b_1kR^2\alpha\lambda^3\epsilon^3 - 32b_1kR^2\alpha\lambda^3\mu^2\epsilon^3 + 92b_2kR^3\alpha\lambda^3\mu\epsilon^3 - a_0b_1R\alpha\lambda\mu^2\epsilon^2 - b_1fR\alpha\lambda\mu^2\epsilon^2 - 3a_2b_1R^3\alpha\lambda\epsilon^2 - 3a_1b_2R^3\alpha\lambda\epsilon^2 + a_0b_1R\alpha\lambda\epsilon^2 + b_1fR\alpha\lambda\epsilon^2 + 4a_1b_1R^2\alpha\lambda\mu\epsilon^2 + 4a_0b_2R^2\alpha\lambda\mu\epsilon^2 + 4b_2fR^2\alpha\lambda\mu\epsilon^2 - b_1R\mu^2h\epsilon^2 + b_1Rh\epsilon^2 + 4b_2R^2\mu h\epsilon^2 + 3b_1kR^2\alpha\lambda^3\mu^2\epsilon - 11b_2kR^3\alpha\lambda^3\mu\epsilon + a_2b_1R^3\alpha\lambda + a_1b_2R^3\alpha\lambda - a_1b_1R^2\alpha\lambda\mu - a_0b_2R^2\alpha\lambda\mu - b_2fR^2\alpha\lambda\mu - b_2R^2\mu h = 0$.

(x) $-48b_2kR^2\alpha\lambda^3\epsilon^5 - 24b_1kR\alpha\lambda^3\mu^3\epsilon^5 + 144b_2kR^2\alpha\lambda^3\mu^2\epsilon^5 + 24b_1kR\alpha\lambda^3\mu\epsilon^5 + 28b_2kR^2\alpha\lambda^3\epsilon^3 + 12b_1kR\alpha\lambda^3\mu^3\epsilon^3 - 100b_2kR^2\alpha\lambda^3\mu^2\epsilon^3 - 12b_1kR\alpha\lambda^3\mu\epsilon^3 - 2a_1b_1R\alpha\lambda\mu^2\epsilon^2 - 2a_0b_2R\alpha\lambda\mu^2\epsilon^2 - 2b_2fR\alpha\lambda\mu^2\epsilon^2 - 4a_2b_2R^3\alpha\lambda\epsilon^2 + 2a_1b_1R\alpha\lambda\epsilon^2 + 2a_0b_2R\alpha\lambda\epsilon^2 + 2b_2fR\alpha\lambda\epsilon^2 + 6a_2b_1R^2\alpha\lambda\mu\epsilon^2 + 6a_1b_2R^2\alpha\lambda\mu\epsilon^2 - 2b_2R\mu^2h\epsilon^2 + 2b_2Rh\epsilon^2 + 6b_2kR^2\alpha\lambda^3\mu^2\epsilon + a_2b_2R^3\alpha\lambda - a_2b_1R^2\alpha\lambda\mu - a_1b_2R^2\alpha\lambda\mu = 0$.

In the equations above, $f = f(t) = \int F(t)dt$, $\alpha = \alpha(t)$, and $h = h(t)$. Solving the previous system with the aid of a computer, we obtain many solutions to (1.3). These solutions may be obtained from (1.5) and are given by (3.9)–(3.19). In these formulas, $H(t) = \int h(t)\,dt$ and λ, a_0 are arbitrary parameters. In a formula containing $\sqrt{R}$, we suppose that $R > 0$ and if the expression $\sqrt{-R}$ appears, we choose $R < 0$. If a formula involves $\sqrt{-R}$ and $\sqrt{1-\mu^2}$ (see, e.g., (3.17)) we consider that $R < 0$ and $|\mu| \leq 1$.

First Group. $\epsilon = 1$:

(i) $a_2 = b_1 = b_2 = 0$, $a_1 = 6k\lambda^2$, $h(t) = -\lambda\alpha(t)(a_0 + \int F(t)dt - Rk\lambda^2)$, $\mu = -1$:

$$u_1(x,t) = \int F(t)dt + a_0 - \frac{6k\lambda^2 R}{1 - \sin\left(\sqrt{R}(\lambda x + H(t))\right)},$$

$$u_2(x,t) = \int F(t)dt + a_0 - \frac{6k\lambda^2 R}{1 - \cos\left(\sqrt{R}(\lambda x + H(t))\right)}, \tag{3.9}$$

$$u_3(x,t) = \int F(t)dt + a_0 - \frac{6k\lambda^2 R}{1 - \cosh\left(\sqrt{-R}(\lambda x + H(t))\right)}.$$

(ii) $a_2 = b_1 = b_2 = 0$, $a_1 = -6k\lambda^2$, $h(t) = -\lambda\alpha(t)(a_0 + \int F(t)dt - Rk\lambda^2)$, $\mu = 1$:

$$u_4(x,t) = \int F(t)dt + a_0 - \frac{6k\lambda^2 R}{1 + \sin\left(\sqrt{R}(\lambda x + H(t))\right)},$$

$$u_5(x,t) = \int F(t)dt + a_0 - \frac{6k\lambda^2 R}{1 + \cos\left(\sqrt{R}(\lambda x + H(t))\right)},$$

$$u_6(x,t) = \int F(t)dt + a_0 - \frac{6k\lambda^2 R}{1+\cosh\left(\sqrt{-R}(\lambda x + H(t))\right)}. \tag{3.10}$$

(iii) $a_1 = -6k\lambda^2\mu$, $b_1 = 0$, $a_2 = 6k\lambda^2(\mu^2-1)/R$, $b_2 = 6k\lambda^2\sqrt{R(1-\mu^2)}/R$, $h(t) = -\lambda\alpha(t)(a_0 + \int F(t)dt - Rk\lambda^2)$, $\mu = \mu$:

$$\begin{aligned}
u_7(x,t) &= \int F(t)dt + a_0 \\
&\quad - \frac{6k\lambda^2 R\left(1+\mu\cos\left(\sqrt{R}(\lambda x + H(t))\right) - \sqrt{1-\mu^2}\sin\left(\sqrt{R}(\lambda x + H(t))\right)\right)}{\mu^2 + 2\mu\cos\left(\sqrt{R}(\lambda x + H(t))\right) + \cos^2\left(\sqrt{R}(\lambda x + H(t))\right)}, \\
u_8(x,t) &= \int F(t)dt + a_0 \\
&\quad - \frac{6k\lambda^2 R\left(1+\mu\cosh\left(\sqrt{-R}(\lambda x + H(t))\right) - \sqrt{\mu^2-1}\sinh\left(\sqrt{-R}(\lambda x + H(t))\right)\right)}{\left(\mu + \cosh\left(\sqrt{-R}(\lambda x + H(t))\right)\right)^2}, \\
u_9(x,t) &= \int F(t)dt + a_0 \\
&\quad - \frac{6\lambda^2 kR\left(1+\mu\sin\left(\sqrt{R}(\lambda x + H(t))\right) - \sqrt{1-\mu^2}\cos\left(\sqrt{R}(\lambda x + H(t))\right)\right)}{1+\mu^2 + 2\mu\sin\left(\sqrt{R}(\lambda x + H(t))\right) - \cos^2\left(\sqrt{R}(\lambda x + H(t))\right)}.
\end{aligned} \tag{3.11}$$

(iv) $a_1 = -6k\lambda^2\mu$, $b_1 = 0$, $a_2 = 6k\lambda^2(\mu^2-1)/R$, $b_2 = -6k\lambda^2\sqrt{R(1-\mu^2)}/R$, $h(t) = -\lambda\alpha(t)(a_0 + \int F(t)dt - Rk\lambda^2)$, $\mu = \mu$:

$$\begin{aligned}
u_{10}(x,t) &= \int F(t)dt + a_0 \\
&\quad - \frac{6k\lambda^2 R\left(1+\mu\cos\left(\sqrt{R}(\lambda x + H(t))\right) + \sqrt{1-\mu^2}\sin\left(\sqrt{R}(\lambda x + H(t))\right)\right)}{\mu^2 + 2\mu\cos\left(\sqrt{R}(\lambda x + H(t))\right) + \cos^2\left(\sqrt{R}(\lambda x + H(t))\right)}, \\
u_{11}(x,t) &= \int F(t)dt + a_0 \\
&\quad - \frac{6k\lambda^2 R\left(1+\mu\cosh\left(\sqrt{-R}(\lambda x + H(t))\right) + \sqrt{\mu^2-1}\sinh\left(\sqrt{-R}(\lambda x + H(t))\right)\right)}{\left(\mu + \cosh\left(\sqrt{-R}(\lambda x + H(t))\right)\right)^2}.
\end{aligned} \tag{3.12}$$

Figure 1 shows solution $u_{11}(x,t)$ for the choices $\mu = 1.3$, $R = -2.3$, $k = 2.3$, $\lambda = 1.9$, $a_0 = 1$, $F(t) = \sin(t)$, $\alpha(t) = t$, $-3 \le x \le 4$, and $-1 \le t \le 1.1$:

$$u_{12}(x,t) = \int F(t)dt + a_0 - \frac{6k\lambda^2 R\left(1 + \mu \sin\left(\sqrt{R}(\lambda x + H(t))\right) + \sqrt{1-\mu^2}\cos\left(\sqrt{R}(\lambda x + H(t))\right)\right)}{1 + \mu^2 + 2\mu \sin\left(\sqrt{R}(\lambda x + H(t))\right) - \cos^2\left(\sqrt{R}(\lambda x + H(t))\right)}. \tag{3.13}$$

Figure 2 shows solution $u_{12}(x,t)$ for the choices $\mu = 0.3$, $R = 2.3$, $k = 2.3$, $\lambda = 1.9$, $a_0 = 1$, $F(t) = \sin(t)$, $\alpha(t) = t$, $-3 \le x \le 4$, and $-1 \le t \le 1.1$.

Second Group. $\epsilon = -1$*:*

(v) $a_1 = b_1 = 0$, $a_2 = 6k\lambda^2/R$, $b_2 = -6k\lambda^2/\sqrt{-R}$, $h(t) = -\lambda\alpha(t)(a_0 + \int F(t)dt + Rk\lambda^2)$, $\mu = 0$:

$$u_{13}(x,t) = \int F(t)dt + a_0 + \frac{6k\lambda^2 R\left(1 + \sin\left(\sqrt{-R}(\lambda x + H(t))\right)\right)}{\cos^2\left(\sqrt{-R}(\lambda x + H(t))\right)}. \tag{3.14}$$

(vi) $a_1 = b_1 = 0$, $a_2 = 6k\lambda^2/R$, $b_2 = 6k\lambda^2/\sqrt{-R}$, $h(t) = -\lambda\alpha(t)(a_0 + \int F(t)dt + Rk\lambda^2)$, $\mu = 0$:

$$u_{14}(x,t) = \int F(t)dt + a_0 + \frac{6k\lambda^2 R\left(1 - \sin\left(\sqrt{-R}(\lambda x + H(t))\right)\right)}{\cos^2\left(\sqrt{-R}(\lambda x + H(t))\right)}. \tag{3.15}$$

(vii) $a_1 = 6k\lambda^2\mu$, $a_2 = 6k\lambda^2(1-\mu^2)/R$, $b_1 = 0$, $b_2 = 6k\lambda^2\sqrt{R(\mu^2-1)}/R$, $h(t) = -\lambda\alpha(t)(a_0 + \int F(t)dt + Rk\lambda^2)$, $\mu = \mu$:

$$u_{15}(x,t) = \int F(t)dt + a_0 + \frac{6k\lambda^2\mu R}{\mu + \cosh\left(\sqrt{R}(\lambda x + H(t))\right)} + \frac{6k\lambda^2 R\left(1 - \mu^2 + \sqrt{\mu^2-1}\sinh\left(\sqrt{R}(\lambda x + H(t))\right)\right)}{\left(\mu + \cosh\left(\sqrt{R}(\lambda x + H(t))\right)\right)^2}, \tag{3.16}$$

$$u_{16}(x,t) = \int F(t)dt + a_0 + \frac{6k\lambda^2\mu R}{\mu + \cos\left(\sqrt{-R}(\lambda x + H(t))\right)} + \frac{6k\lambda^2 R\left(1 - \mu^2 + \sqrt{1-\mu^2}\sin\left(\sqrt{-R}(\lambda x + H(t))\right)\right)}{\mu^2 + 2\mu\cos\left(\sqrt{-R}(\lambda x + H(t))\right) + \cos^2\left(\sqrt{-R}(\lambda x + H(t))\right)}. \tag{3.17}$$

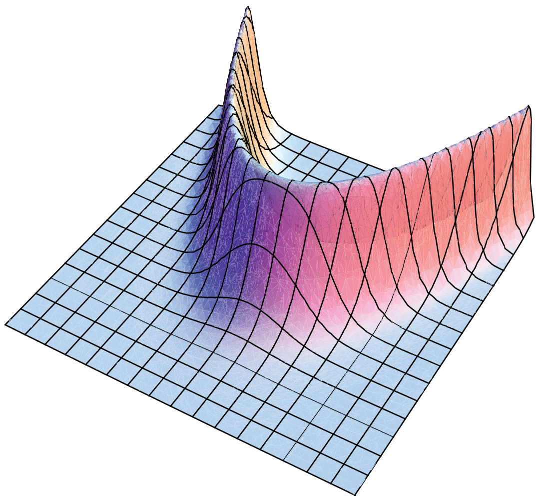

Figure 1: Graphic of solution $u_{11}(x,t)$, $-3 \le x \le 4$ and $-1 \le t \le 1.1$.

(viii) $a_1 = 6k\lambda^2\mu$, $a_2 = -6k\lambda^2(1-\mu^2)/R$, $b_1 = 0$, $b_2 = -6k\lambda^2\sqrt{R(\mu^2-1)}/R$, $h(t) = -\lambda\alpha(t)(a_0 + \int F(t)dt + Rk\lambda^2)$, $\mu = \mu$:

$$u_{17}(x,t) = \int F(t)dt + a_0 + \frac{6k\lambda^2\mu R}{\mu + \cosh\left(\sqrt{R}(\lambda x + H(t))\right)} + \frac{6k\lambda^2 R\left(1-\mu^2-\sqrt{\mu^2-1}\sinh\left(\sqrt{R}(\lambda x + H(t))\right)\right)}{\left(\mu + \cosh\left(\sqrt{R}(\lambda x + H(t))\right)\right)^2}, \tag{3.18}$$

$$u_{18}(x,t) = \int F(t)dt + a_0 + \frac{6k\lambda^2\mu R}{\mu + \cos\left(\sqrt{-R}(\lambda x + H(t))\right)} + \frac{6k\lambda^2 R\left(1-\mu^2-\sqrt{1-\mu^2}\sin\left(\sqrt{-R}(\lambda x + H(t))\right)\right)}{\mu^2 + 2\mu\cos\left(\sqrt{-R}(\lambda x + H(t))\right) + \cos^2\left(\sqrt{-R}(\lambda x + H(t))\right)}. \tag{3.19}$$

3.2. Rational Solutions

We seek rational solutions to (1.8) ($R = \mu = 0$) in the form given by (3.6) with $m = 2$,

$$V(\xi) = \sum_{i=0}^{2} a_i \tau^i(\xi), \tag{3.20}$$

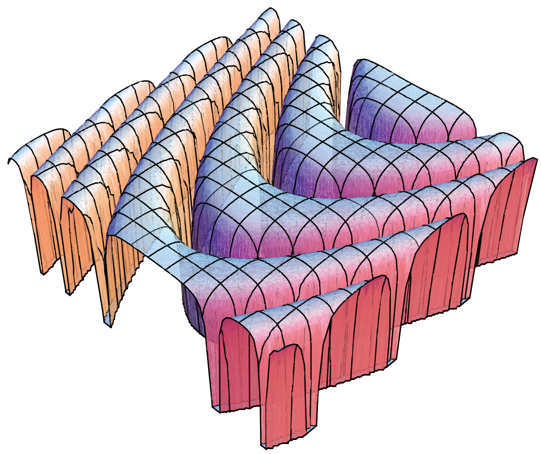

Figure 2: Graphic of solution $u_{12}(x,t)$, $-3 \le x \le 4$ and $-1 \le t \le 1.1$.

where

$$\tau'(\xi) = \tau^2(\xi). \tag{3.21}$$

Note that this last equation has the general solution

$$\tau(\xi) = -\frac{1}{\xi + C}. \tag{3.22}$$

We now substitute (3.20) into (1.8), and using the relation $\tau'(\xi) = \tau^2(\xi)$ we obtain an equation whose left-hand side is a polynomial in the variable $\tau = \tau(\xi)$. We equate each coefficient of this polynomial to zero and we get the following algebraic system.

(i) $h(t)a_1 + \alpha(t)\lambda(\int F(t)dt)a_1 + \alpha(t)\lambda a_0 a_1 = 0$.

(ii) $\alpha(t)\lambda a_1^2 + 2h(t)a_2 + 2\alpha(t)\lambda(\int F(t)dt)a_2 + 2\alpha(t)\lambda a_0 a_2 = 0$.

(iii) $6ka_1\lambda^3 + 3a_1a_2\lambda = 0$.

(iv) $24ka_2\lambda^3 + 2a_2^2\lambda = 0$.

Solving this system gives $a_0 = a_0$, $a_1 = 0$, $a_2 = -12k\lambda^2$ and

$$h(t) = -\lambda\alpha(t)\left(\int F(t)dt + a_0\right). \tag{3.23}$$

A rational solution of (1.8) is given by

$$V(\xi) = a_0 - \frac{12k\lambda^2}{(\xi + C)^2}. \tag{3.24}$$

According to (1.5), (1.6), and (1.7) we obtain the following rational solution to (1.3):

$$u_{19}(x,t) = \int F(t)dt + a_0 - \frac{12k\lambda^2}{\left(\lambda x - \int (\lambda\alpha(t)(\int F(t)dt + a_0))dt + C\right)^2}. \tag{3.25}$$

4. Conclusions

In this paper, by using the projective Riccati equation and the Exp-function methods, with the help of a symbolic computation engine, we obtain exact solutions for a generalized KdV equation with forcing term (1.3). The methods certainly works well for a large class of very interesting nonlinear equations. The main advantage of these methods is their capability of greatly reducing the size of computational work compared to existing techniques such as the pseudospectral method, the inverse scattering method, Hirota's bilinear method, and the truncated Painlevé expansion. The Exp-function method gives us more general solutions with some free parameters than the projective Riccati equation method. It also has other interesting applications. For instance, the Exp-function method may be applied not only to differential equations but also to differential-difference equations or Stochastic equations [32–35].

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