

Research Article

Double Hopf Bifurcation with Strong Resonances in Delayed Systems with Nonlinearities

J. Xu¹ and K. W. Chung²

¹ School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 200092, China

² Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong

Correspondence should be addressed to J. Xu, xujian@tongji.edu.cn

Received 25 February 2009; Revised 6 September 2009; Accepted 11 October 2009

Recommended by Oleg Gendelman

An efficient method is proposed to study delay-induced strong resonant double Hopf bifurcation for nonlinear systems with time delay. As an illustration, the proposed method is employed to investigate the 1:2 double Hopf bifurcation in the van der Pol system with time delay. Dynamics arising from the bifurcation are classified qualitatively and expressed approximately in a closed form for either square or cubic nonlinearity. The results show that 1:2 resonance can lead to codimension-three and codimension-two bifurcations. The validity of analytical predictions is shown by their consistency with numerical simulations.

Copyright © 2009 J. Xu and K. W. Chung. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

To realistically model physical and biological systems with interactions, we generally need to take into account the influence of the past states. Such a system can be modeled by a set of delay differential equations (DDEs). Unlike a differential equation, the characteristic equation of a DDE has an infinite number of eigenvalues due to the time delay. This implies that an infinite number of modes may occur in the system governed by a DDE. It is possible that a double Hopf bifurcation is induced by the time delay if the characteristic equation at an equilibrium has eigenvalues with strict negative real parts except a pair of purely imaginary eigenvalues. The study of double Hopf bifurcation has recently attracted particular attention as the bifurcation exhibits rich dynamical behavior and creates complicated motions which reflect the intrinsic properties of the system. In our previous research [1–3], it was found that even a second-order controlled system with delayed feedback may undergo a double Hopf bifurcation when the time delay and feedback gain are varied. Such phenomenon was also observed by Reddy et al. [4], Balanov et al. [5], and Ma et al. [6]. These researches showed that the time delay in various systems has a qualitative effect on dynamics. Physically, energy

between two modes is exchanged when a double Hopf bifurcation occurs in the system under consideration. Therefore, a study of the effect of time delay on the dynamics of mathematical models gives insight into possible mechanism behind the observed behavior.

There are three cases: nonresonances, weak (high-order), and strong (low-order) resonances of the critical frequencies for a double Hopf bifurcation [7]. Let $\pm i\omega_-$ and $\pm i\omega_+$ be the two pairs of purely imaginary eigenvalues of the characteristic equation for a DDE and all other eigenvalues have negative real parts such that $\omega_+ \geq \omega_- > 0$, $k_1 : k_2 = \omega_- : \omega_+$ at the critical value of parameters. The double Hopf bifurcation is called nonresonance if $k_1 : k_2$ is irrational. If $k_1 : k_2$ is rational, the bifurcation is a strong (low-order) resonance when $k_1 = 1$ and $k_2 = 1, 2, 3$, or 4, and a weak (high-order) resonance otherwise. Many methods and techniques from the geometric theory of dynamical systems on ordinary differential equations may be extended to study the former two cases. However, the problem for the last case becomes complicated since a strong resonance may be a codimension-3 phenomenon which requires three parameters for its unfolding. In fact, one of two single modes induced by the 1 : 2 resonance may undergo saddle-node bifurcation, which leads to so-called mixed modes. Thus, the 1 : 2 resonance is treated as a degenerate double bifurcation or codimension-3 case when the saddle-node bifurcation point and double Hopf bifurcation one coincide in three-parameters space. It is well known that dynamics derived from strong resonance cannot be predicted from the theory of nonresonant double Hopf interactions. The investigation of delay-induced double Hopf bifurcation with strong resonances is both challenging and important. The first successful research for studying 1 : 2 strong resonant double Hopf bifurcation of a DDE was performed by Campbell and LeBlanc [8] in terms of the center manifold reduction (CMR). Using CMR for the study of resonant double Hopf bifurcation is tedious. Moreover, the unfolding parameters obtained usually do not carry any physical meaning. However, for the case of DDEs, the unfolding parameters are related to the quantities of a physical system such as time delay and feedback gains in controlled systems. For these issues, we propose an efficient but simple method in [9] to avoid the tedious computation encountered in CMR. The result will also bring out the physical significance of unfolding parameters in higher codimensional bifurcations.

In [9], we studied weak (high-order) resonant double Hopf bifurcations which arise from time delay and other physical parameters in a type of two first-order delay differential equations given by

$$\dot{Z}(t) = CZ(t) + DZ(t - \tau) + \epsilon F(Z(t), Z(t - \tau)), \quad (1.1)$$

where $Z(t) = (x(t), y(t))^T \in \mathbb{R}^2$, C and D are 2×2 real constant matrices such that $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ and $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$, F represents a nonlinear function in its variable with $F(0, 0) = 0$, ϵ is a *small* parameter representing strength of nonlinearities, and τ the time delay. In addition, we restricted our research to the following two cases in [9]:

- (a) $\det(D) = 0$, $c_{22}d_{11} - c_{21}d_{12} - c_{12}d_{21} + c_{11}d_{22} + c_{11}c_{22} - c_{12}c_{21} \neq 0$;
- (b) $\det(D) \neq 0$, $c_{11} = c_{22}$, $c_{12} = -c_{21}$, $d_{11} = d_{22}$, $d_{12} = d_{21} = 0$, $c_{11} > 0$, $c_{12} > 0$, $d_{11} \neq 0$.

We found a double Hopf bifurcation with 1 : 2 strong (or low-order) resonance occurring in (1.1). The strong resonant bifurcation was left open since it requires three parameters for the unfolding. In this paper, we extend our investigation to study this open problem.

2. An Efficient Method for Studying Strong Resonant Double Hopf Bifurcations

We follow the idea of [9] in this section. It can be seen from (1.1) that $Z = 0$ is always an equilibrium point or trivial solution of the system. The characteristic equation at $Z = 0$ is given by

$$\det(\lambda I - C - D e^{-\lambda\tau}) = 0, \quad (2.1)$$

where I is the identity matrix. The characteristic equation (2.1) can be rewritten as

$$\lambda^2 - \lambda (c_1 + d_1 e^{-\lambda\tau}) + c_d e^{-\lambda\tau} + c_2 + \det(D) e^{-2\lambda\tau} = 0, \quad (2.2)$$

where

$$\begin{aligned} c_1 &= c_{11} + c_{22}, \\ d_1 &= d_{11} + d_{22}, \\ c_2 &= c_{11}c_{22} - c_{12}c_{21}, \\ c_d &= c_{22}d_{11} - c_{21}d_{12} - c_{12}d_{21} + c_{11}d_{22}. \end{aligned} \quad (2.3)$$

The roots of the characteristic equation (2.2) are usually called the eigenvalues of the equilibrium point of system (1.1). The stability of the trivial equilibrium point will change when the system under consideration has zero or a pair of purely imaginary eigenvalues. The former occurs when $\lambda = 0$ in (2.2) or $c_d + c_2 + \det(D) = 0$, which leads to a static bifurcation of the equilibrium point. The latter deals with the Hopf bifurcation such that the dynamical behavior of the system changes from a static stable state to a periodic motion or vice versa. The dynamics becomes complicated when the system has two pairs of purely imaginary eigenvalues at a critical value of the time delay. We will concentrate on such cases. For this, we let $c_d + c_2 + \det(D) \neq 0$. Thus, $\lambda = 0$ is not a root of the characteristic equation (2.2) in the present paper. Such assumption can be realized in engineering as long as one chooses a suitable feedback controller.

For case (a) with $\det(D) = 0$ but $c_d + c_2 \neq 0$, substituting $\lambda = a + i\omega$ into (2.2), and equating the real and imaginary parts to zero yield

$$\begin{aligned} a^2 - \omega^2 - ac_1 + c_2 - e^{-a\tau}\omega \sin(\tau\omega)d_1 + e^{-a\tau} \cos(\tau\omega)(c_d - ad_1) &= 0, \\ 2a\omega - \omega c_1 - e^{-a\tau}\omega \cos(\tau\omega) d_1 + e^{-a\tau} \sin(\tau\omega)(ad_1 - c_d) &= 0. \end{aligned} \quad (2.4)$$

One can obtain the explicit expressions for the critical stability boundaries by setting $a = 0$ in (2.4), given by

$$\begin{aligned} -\omega^2 + c_2 + c_d \cos(\tau\omega) - \omega d_1 \sin(\tau\omega) &= 0, \\ -\omega c_1 - c_d \sin(\tau\omega) - \omega d_1 \cos(\tau\omega) &= 0. \end{aligned} \quad (2.5)$$

Eliminating τ from (2.5), we have

$$\omega_{\pm} = \frac{\sqrt{d_1^2 - c_1^2 + 2c_2 \pm \sqrt{(d_1^2 - c_1^2 + 2c_2)^2 - 4(c_2^2 - c_d^2)}}}{\sqrt{2}}, \quad (2.6)$$

where $\omega_- < \omega_+$ and when the following conditions hold:

$$\begin{aligned} c_2^2 - c_d^2 &> 0, \\ (d_1^2 - c_1^2 + 2c_2)^2 &> 4(c_2^2 - c_d^2). \end{aligned} \quad (2.7)$$

Then, two families of surfaces, denoted by τ_- and τ_+ which correspond to ω_- and ω_+ , respectively, can be derived from (2.4) and are given by

$$\begin{aligned} \cos(\omega_- \tau_-) &= \frac{\omega_-^2 c_d - c_2 c_d - \omega_-^2 c_1 d_1}{c_d^2 + \omega_-^2 d_1^2}, \\ \cos(\omega_+ \tau_+) &= \frac{\omega_+^2 c_d - c_2 c_d - \omega_+^2 c_1 d_1}{c_d^2 + \omega_+^2 d_1^2}. \end{aligned} \quad (2.8)$$

It follows from (2.6) and (2.8) that ω_{\pm} and τ_{\pm} are functions of c_1 , d_1 , c_2 , and c_d given in (2.3). If there exist values of c_1 , d_1 , c_2 , and c_d such that

$$\tau_- = \tau_+, \quad (2.9)$$

$$\omega_- : \omega_+ = k_1 : k_2, \quad (2.10)$$

where $k_1 = 1$, $k_2 = 1, 2, 3$, or 4 , then such point is called the $k_1 : k_2$ strong or low-order resonant double Hopf bifurcation point in parameter space. Equations (2.9) and (2.10) not only determine the linearized system around the trivial equilibrium which has two pairs of purely imaginary eigenvalues $\pm i\omega_-$ and $\pm i\omega_+$, but also give a relation between ω_- and ω_+ . Equations (2.9) and (2.10) form the necessary conditions for the occurrence of a strong resonant double Hopf bifurcation point. Equation (2.10) yields

$$d_1^2 = c_1^2 - 2c_2 + \frac{k_1^2 + k_2^2}{k_1 k_2} \sqrt{c_2^2 - c_d^2}, \quad (2.11)$$

when condition (2.7) is satisfied. Substituting (2.11) into (2.6), one can obtain the frequencies in the simple expressions given by

$$\omega_- = \sqrt{\frac{k_1}{k_2} \sqrt{c_2^2 - c_d^2}}, \quad \omega_+ = \sqrt{\frac{k_2}{k_1} \sqrt{c_2^2 - c_d^2}}. \quad (2.12)$$

The other parameters can be determined by (2.9) or the following equation:

$$\begin{aligned} & \arccos\left(\frac{-(c_2 c_d k_2) + \sqrt{c_2^2 - c_d^2} (c_d - c_1 d_1) k_1}{c_d^2 k_2 + \sqrt{c_2^2 - c_d^2} d_1^2 k_1}\right) \\ &= \frac{k_1}{k_2} \arccos\left(\frac{-(c_2 c_d k_1) + \sqrt{c_2^2 - c_d^2} (c_d - c_1 d_1) k_2}{c_d^2 k_1 + \sqrt{c_2^2 - c_d^2} d_1^2 k_2}\right), \end{aligned} \quad (2.13)$$

where d_1 is given in (2.11). The corresponding value of the time delay at the strong resonant double Hopf bifurcation point, noted as τ_c , is given by

$$\begin{aligned} \tau_c &= \tau_- = \tau_+ \\ &= \sqrt{\frac{k_1}{k_2} \sqrt{c_2^2 - c_d^2}} \arccos\left(\frac{-(c_2 c_d k_1) + \sqrt{c_2^2 - c_d^2} (c_d - c_1 d_1) k_2}{c_d^2 k_1 + \sqrt{c_2^2 - c_d^2} d_1^2 k_2}\right). \end{aligned} \quad (2.14)$$

For case (b) with $\det(D) \neq 0$ and $c_{11} = c_{22}$, $c_{12} = -c_{21}$, $d_{11} = d_{22}$, $d_{12} = d_{21} = 0$, $c_{11} > 0$, $c_{12} > 0$, $d_{11} < 0$, the characteristic equation (2.1) becomes

$$\lambda = c_{11} \pm i c_{12} + d_{11} e^{-\lambda \tau}. \quad (2.15)$$

By substituting $\lambda = a + i\omega$ into (2.15), we get

$$\begin{aligned} \omega &= \omega_{\pm} = c_{12} \pm \sqrt{d_{11}^2 e^{-2a\tau} - (a - c_{11})^2}, \\ a &= c_{11} - \frac{\omega - c_{21}}{\tan(\omega\tau)}. \end{aligned} \quad (2.16)$$

It follows from (2.16) that ω is real only when $d_{11}^2 \geq (a - c_{11})^2 e^{2a\tau}$. To obtain the critical boundary, set $a = 0$. This gives

$$\omega_- = c_{12} - \sqrt{d_{11}^2 - c_{11}^2}, \quad (2.17)$$

$$\omega_+ = c_{12} + \sqrt{d_{11}^2 - c_{11}^2},$$

$$\tau_- [j] = \frac{1}{\omega_-} \left(2j\pi - \cos^{-1} \left(-\frac{c_{11}}{d_{11}} \right) \right), \quad (2.18)$$

$$\tau_+ [j] = \frac{1}{\omega_+} \left(2j\pi + \cos^{-1} \left(-\frac{c_{11}}{d_{11}} \right) \right),$$

where $j = 1, 2, \dots$. The necessary conditions for the $k_1 : k_2$ occurrence of a strong resonant double Hopf bifurcation can be obtained by setting $\tau_-[j] = \tau_+[j]$ and $\omega_-/\omega_+ = k_1/k_2$ for $k_1 = 1$ and $k_2 = 1, 2, 3$ or 4 . This yields

$$(d_{11}) = -\frac{c_{11}}{\cos(2j\pi(k_2 - k_1)/(k_1 + k_2))}, \quad \tau_c = \frac{4jk_1\pi}{(k_1 + k_2)\omega_-}. \quad (2.19)$$

As mentioned above, a strong resonance may be a codimension-3 phenomenon which requires three parameters for its unfolding. For our purpose, the time delay is always chosen as a bifurcation parameter. The other two variable parameters come from the element of C and D , respectively, while the remaining parameters are kept fixed. Consequently, at a double Hopf point, these three variable parameters satisfy (2.11), (2.12) and (2.14) for case (a), and (2.17) and (2.19) for case (b). Correspondingly, denote C , D and τ by C_c , D_c and τ_c at a double Hopf point, respectively. To investigate the dynamical behavior in a neighborhood of the double Hopf point, we consider ϵC_ϵ , ϵD_ϵ , and $\epsilon \tau_\epsilon$ for a small ϵ , given by

$$\epsilon C_\epsilon = C - C_c, \quad \epsilon D_\epsilon = D - D_c, \quad \epsilon \tau_\epsilon = \tau - \tau_c. \quad (2.20)$$

Substituting (2.20) into (1.1) yields

$$\dot{Z} = C_c Z + D_c Z_{\tau_c} + \tilde{F}(Z, Z_{\tau_c}, Z_{\tau_c + \epsilon \tau_\epsilon}, \epsilon), \quad (2.21)$$

where $Z = Z(t)$, $Z_\tau = Z(t - \tau)$, and

$$\tilde{F}(\cdot) = D_c(Z_{\tau_c + \epsilon \tau_\epsilon} - Z_{\tau_c}) + \epsilon(C_\epsilon + D_\epsilon Z_{\tau_c + \epsilon \tau_\epsilon} + F(Z, Z_{\tau_c + \epsilon \tau_\epsilon})). \quad (2.22)$$

When $\epsilon = 0$, $\tilde{F} = 0$, and a resonant double Hopf bifurcation point occurs in the system under consideration. Thus, the zero-order approximate solution of (2.21) may be expressed as

$$Z_0(t) = \sum_{i=1}^2 \left[\begin{pmatrix} a_{k_i} \\ c_{k_i} \end{pmatrix} \cos(k_i \phi) + \begin{pmatrix} b_{k_i} \\ d_{k_i} \end{pmatrix} \sin(k_i \phi) \right], \quad (2.23)$$

which results in

$$Z_0(t - \tau_c) = \sum_{i=1}^2 \left[\begin{pmatrix} a_{k_i} \\ c_{k_i} \end{pmatrix} \cos(k_i \phi - k_i \omega \tau_c) + \begin{pmatrix} b_{k_i} \\ d_{k_i} \end{pmatrix} \sin(k_i \phi - k_i \omega \tau_c) \right], \quad (2.24)$$

where $\phi = \omega t$ and $\omega = \omega_-/k_1 = \omega_+/k_2$ is determined by (2.12) for case (a) and (2.17) for case (b). Substituting (2.23) and (2.24) into (2.21) when $\epsilon = 0$ and using the harmonic balance, one obtains that

$$M_{k_i} \begin{pmatrix} b_{k_i} \\ d_{k_i} \end{pmatrix} = N_{k_i} \begin{pmatrix} a_{k_i} \\ c_{k_i} \end{pmatrix}, \quad (2.25)$$

$$-M_{k_i} \begin{pmatrix} a_{k_i} \\ c_{k_i} \end{pmatrix} = N_{k_i} \begin{pmatrix} b_{k_i} \\ d_{k_i} \end{pmatrix}, \quad (2.26)$$

where $M_{k_i} = k_i \omega I + D_c \sin(k_i \omega \tau)$, $N_{k_i} = C_c + D_c \cos(k_i \omega \tau)$ and I is the 2×2 identity matrix. Equations (2.25) and (2.26) are in fact identical. Let $\begin{pmatrix} \tilde{a}_{k_i} \\ \tilde{b}_{k_i} \end{pmatrix} = (N_{k_i} / \det(N_{k_i})) \begin{pmatrix} a_{k_i} \\ b_{k_i} \end{pmatrix}$ and note that $\det(N_{k_i}) = \det(M_{k_i})$. It follows from (2.25) that the periodic solution of (2.21) for $\epsilon = 0$ is given by

$$Z_0(t) = \sum_{i=1}^2 \left(\tilde{N}_{k_i} \cos(k_i \phi) + \tilde{M}_{k_i} \sin(k_i \phi) \right) \begin{pmatrix} \tilde{a}_{k_i} \\ \tilde{b}_{k_i} \end{pmatrix}, \quad (2.27)$$

where $\tilde{N}_{k_i} = N_{k_i}^{-1} \det(N_{k_i})$, $\tilde{M}_{k_i} = M_{k_i}^{-1} \det(M_{k_i})$ for $i = 1, 2$.

In order to obtain an approximate solution of (2.21) when $\epsilon \neq 0$, we now consider a perturbation to (2.27), given by

$$Z(t) = \sum_{i=1}^2 \left(\tilde{N}_{k_i} \cos(k_i(\omega + \sigma(\epsilon))t) + \tilde{M}_{k_i} \sin(k_i(\omega + \sigma(\epsilon))t) \right) \begin{pmatrix} a_{k_i}(\epsilon) \\ b_{k_i}(\epsilon) \end{pmatrix}, \quad (2.28)$$

where $a_{k_i}(0) = \tilde{a}_{k_i}$, $b_{k_i}(0) = \tilde{b}_{k_i}$, and σ is a detuning parameter. The following theorem provides a method to determine $\begin{pmatrix} a_{k_i}(\epsilon) \\ b_{k_i}(\epsilon) \end{pmatrix}$ in (2.28).

Theorem 2.1. *If $W(t)$ is a periodic solution of the equation*

$$\dot{W}(t) = -C_c^T W(t) - D_c^T W(t + \tau_c), \quad (2.29)$$

with $W(t) = W(t + 2\pi/\omega)$, then

$$\int_{-\tau_c}^0 \left(D_c^T W(t + \tau_c) \right)^T \left(Z(t) - Z\left(t + \frac{2\pi}{\omega}\right) \right) dt - [W(0)]^T \left[Z\left(\frac{2\pi}{\omega}\right) - Z(0) \right] + \int_0^{2\pi/\omega} [W(t)]^T \tilde{F}(Z, Z_{\tau_c}, Z_{\tau_c + \epsilon \tau_c}, \epsilon) dt = 0. \quad (2.30)$$

Proof. Multiplying both sides of (2.21) by $[W(t)]^T$ and integrating with respect to t from zero to $2\pi/\omega$, one has

$$\begin{aligned} & \int_0^{2\pi/\omega} [W(t)]^T \dot{Z}(t) dt \\ &= \int_0^{2\pi/\omega} [W(t)]^T \left(C_c Z(t) + D_c Z(t - \tau_c) + \tilde{F}(Z, Z_{\tau_c}, Z_{\tau_c + \epsilon\tau_c}, \epsilon) \right) dt, \end{aligned} \quad (2.31)$$

where $2\pi/\omega$ is the period of $W(t)$. Equation (2.31) yields

$$\begin{aligned} & \int_0^{2\pi/\omega} \left(\dot{W}(t) + C_c^T W(t) + D_c^T W(t + \tau_c) \right)^T Z(t) dt - [W(0)]^T \left[Z\left(\frac{2\pi}{\omega}\right) - Z(0) \right] \\ &+ \int_{-\tau_c}^0 \left(D_c^T W(t + \tau_c) \right)^T \left(Z(t) - Z\left(t + \frac{2\pi}{\omega}\right) \right) dt \\ &+ \int_0^{2\pi/\omega} [W(t)]^T \tilde{F}(Z, Z_{\tau_c}, Z_{\tau_c + \epsilon\tau_c}, \epsilon) dt = 0. \end{aligned} \quad (2.32)$$

Equation (2.30) follows from (2.29) and (2.32) and the theorem is proved.

To apply the above theorem to determine $\begin{pmatrix} a_{k_i}(\epsilon) \\ b_{k_i}(\epsilon) \end{pmatrix}$, one must know the expression of $W(t)$ in (2.32). It follows from (2.27) that the periodic solution of (2.29) can be written as

$$W(t) = \sum_{i=1}^2 \left(\begin{pmatrix} -\tilde{N}_{k_i} \\ \tilde{M}_{k_i} \end{pmatrix}^T \cos(k_i \phi) + \begin{pmatrix} \tilde{M}_{k_i} \\ \tilde{N}_{k_i} \end{pmatrix}^T \sin(k_i \phi) \right) \begin{pmatrix} p_{k_i} \\ q_{k_i} \end{pmatrix}, \quad (2.33)$$

where p_{k_i} and q_{k_i} are independent constants. Substituting (2.28), and (2.33) into (2.30) and noting the independence of p_{k_i} and q_{k_i} yield four algebraic equations in $a_{k_i}(\epsilon)$, $b_{k_i}(\epsilon)$ and σ , which are called the amplitude modulation equations of (2.21). If there exist square terms in (2.21), then $r_1(\epsilon) = \sqrt{a_{k_1}^2(\epsilon) + b_{k_1}^2(\epsilon)}$ plays the role of master amplitude since $a_{k_2}(\epsilon)$, and $b_{k_2}(\epsilon)$ would not exist without $r_1(\epsilon)$ [10] for the 1 : 2 internal resonance where $k_1 = 1$ and $k_2 = 2$. Eliminating σ from the amplitude modulation equations, one can obtain three algebraic equations in $r_1(\epsilon)$, $a_2(\epsilon)$, and $b_2(\epsilon)$. \square

3. Application

As an application, we investigate the delay-induced 1 : 2 strong resonant double Hopf bifurcation in the van der Pol oscillator with delayed position feedback and show the efficiency and simplicity of the above method. The oscillator is governed by

$$\ddot{u} - (\alpha - \epsilon\gamma u^2)\dot{u} + u + \epsilon(\beta_1 + \beta_2 u)u^2 = A(u_\tau - u), \quad (3.1)$$

where $0 < \epsilon < 1$, α , γ , β_1 and β_2 are constants, $\beta_1 \geq 0$, $\beta_2 \geq 0$, $\gamma > 0$, $A < 0$, τ time delay, $u_\tau = u(t - \tau)$. System (3.1) can be rewritten in the form of (1.1), where $Z(t) = (x(t), y(t))^T = (u(t), \dot{u}(t))^T \in \mathbb{R}^2$ and

$$\begin{aligned} c_{11} &= 0, & c_{12} &= 1, & c_{21} &= -(1 + A), & c_{22} &= \alpha, \\ d_{11} &= 0, & d_{12} &= 0, & d_{21} &= A, & d_{22} &= 0. \end{aligned} \quad (3.2)$$

Substituting (3.2) into (2.3) yields

$$c_1 = \alpha, \quad d_1 = 0, \quad c_2 = 1 + A, \quad c_d = -A. \quad (3.3)$$

Thus, (3.1) belongs to case (a) of Section 1 since $\det(D) = 0$ and $c_d + c_2 = 1 \neq 0$. To investigate the 1 : 2 strong resonant double Hopf bifurcation, we choose α in C , A in D and the delay τ as the three variable parameters and keep the other parameters fixed. It follows from the analysis in the above section that α , A , and τ satisfy (2.11), (2.12), and (2.14) at the 1 : 2 resonant double Hopf bifurcation point for $k_1 = 1$ and $k_2 = 2$, where $\alpha_c = 0$, $A_c = -3/8$, $\tau_c = 2\pi$. Correspondingly, $\omega_- = 1/2$ and $\omega_+ = 1$. Any values of A , τ and α in the neighborhood of (A_c, τ_c, α_c) can be expressed as $A = A_c + \epsilon A_\epsilon$, $\tau = \tau_c + \epsilon \tau_\epsilon$ and $\alpha = \alpha_c + \epsilon \alpha_\epsilon$. Equation (3.1) can be expressed in the form of (2.21), where $C_c = \begin{pmatrix} 0 & 1 \\ -(A_c + \omega_0^2) & \alpha_c \end{pmatrix}$, $D_c = \begin{pmatrix} 0 & 0 \\ A_c & 0 \end{pmatrix}$, $\tilde{F} = \begin{pmatrix} 0 \\ A_c(x_{\tau_c + \epsilon \tau_\epsilon} - x_{\tau_c}) + \epsilon(A_\epsilon(x_{\tau_c + \epsilon \tau_\epsilon} - x) + \alpha_\epsilon y - x^2(\beta_1 + \beta_2 x + \gamma y)) \end{pmatrix}$. By simple computation, one has $\tilde{N}_{k_1} = \begin{pmatrix} 0 & -1 \\ 1/4 & 0 \end{pmatrix}$, $\tilde{M}_{k_1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $\tilde{N}_{k_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\tilde{M}_{k_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It follows from (2.33) that the solution of (2.29) is given by

$$W(t) = \begin{pmatrix} \frac{1}{2}p_1 \sin\left(\frac{t}{2}\right) - \frac{1}{4}q_1 \cos\left(\frac{t}{2}\right) + p_2 \sin(t) - q_2 \cos(t) \\ p_1 \cos\left(\frac{t}{2}\right) + \frac{1}{2}q_1 \sin\left(\frac{t}{2}\right) + p_2 \cos(t) + q_2 \sin(t) \end{pmatrix}. \quad (3.4)$$

Noting that there is a square term in (3.1) when $\beta_1 \neq 0$ and from discussion at the end of the last section, one may express an approximate solution of (3.1) at $O(\epsilon)$ as

$$Z(t) = \begin{pmatrix} r_1 \cos(\varphi t) + a_2 \cos(2\varphi t) - b_2 \sin(2\varphi t) \\ -\frac{1}{2}r_1 \sin(\varphi t) - b_2 \cos(2\varphi t) - a_2 \sin(2\varphi t) \end{pmatrix}, \quad (3.5)$$

where $\varphi = 1/2 + \epsilon\sigma$. Substituting (3.4) and (3.5) into (2.30) and noting that p_1, p_2, q_1 and q_2 are independent variables yield the following four algebraic equations in r_1, a_2, b_2 , and $\epsilon\sigma$ with three independent bifurcation parameters $\epsilon A_\epsilon, \epsilon\tau_\epsilon$, and $\epsilon\alpha_\epsilon$, given by

$$\begin{aligned}
r_1 \left(8\epsilon A_\epsilon + 4a_2\beta_1\epsilon + 3 \left(r_1^2 + 2 \left(a_2^2 + b_2^2 \right) \right) \epsilon\beta_2 - 4\epsilon\sigma \right) &= 0, \\
r_1 \left(2\epsilon\gamma r_1^2 - 8\epsilon\alpha_\epsilon + 16b_2\epsilon\beta_1 + 4 \left(a_2^2 + b_2^2 \right) \epsilon\gamma + 12\pi\epsilon\sigma + 3\epsilon\tau_\epsilon \right) &= 0, \\
6\epsilon\beta_2 \left(a_2 \left(a_2^2 + b_2^2 \right) \right) + 4r_1^2 \left(\epsilon\beta_1 + 3a_2\epsilon\beta_2 \right) - 2b_2^3\epsilon\gamma - 32a_2\epsilon\sigma \\
+ b_2 \left(8\epsilon\alpha_\epsilon - 2 \left(a_2^2 + 2r_1^2 \right) \epsilon\gamma + 3 \left(4\pi\epsilon\sigma + \epsilon\tau_\epsilon \right) \right) &= 0, \\
-2\epsilon\gamma a_2^3 - 6b_2\epsilon\beta_2 a_2^2 - 6b_2 \left(b_2^2 + 2r_1^2 \right) \epsilon\beta_2 + 32b_2\epsilon\sigma \\
+ \left(8\epsilon\alpha_\epsilon - 2 \left(b_2^2 + 2r_1^2 \right) \epsilon\gamma + 3 \left(4\pi\epsilon\sigma + \epsilon\tau_\epsilon \right) \right) a_2 &= 0.
\end{aligned} \tag{3.6}$$

Eliminating $\epsilon\sigma$ from (3.6), one has

$$\begin{aligned}
r_1 \left(24\pi\epsilon A_\epsilon - 8\epsilon\alpha_\epsilon + 3\epsilon\tau_\epsilon + 12\pi\epsilon a_2\beta_1 + 16\epsilon b_2\beta_1 \right. \\
+ \left. \left(2a_2^2 + 2b_2^2 + r_1^2 \right) \left(2\gamma\epsilon + 9\pi\epsilon\beta_2 \right) \right) &= 0, \\
-24\pi\epsilon A_\epsilon b_2 - 3\epsilon b_2\pi_\epsilon + 2\epsilon\gamma \left(b_2^3 + 4b_2r_1^2 \right) - 8\epsilon b_2\alpha_\epsilon \\
+ \epsilon\beta_2 \left(42a_2^3 - 18b_2^3\pi - 9\pi b_2r_1^2 \right) + 2a_2^2 \left(16\epsilon\beta_1 + b_2 \left(\gamma\epsilon - 9\pi\epsilon\beta_2 \right) \right) \\
+ 2a_2 \left(32\epsilon A_\epsilon - 6\pi\epsilon b_2\beta_1 + 21b_2^2\epsilon\beta_2 + 6\epsilon\beta_2r_1^2 \right) &= 0, \\
6\epsilon\beta_2 b_2 \left(7b_2^2 + 2r_1^2 \right) + 6a_2^2 \left(2\pi\epsilon\beta_1 + 7b_2\epsilon\beta_2 \right) \\
+ 2a_2^3 \left(9\pi\epsilon\beta_2 - \epsilon\gamma \right) + 8 \left(3\pi a_2 + 8b_2 \right) \epsilon A_\epsilon \\
+ a_2 \left(-4\gamma\epsilon r_1^2 + 8\epsilon\alpha_\epsilon + 32\epsilon b_2\beta_1 + 9\pi\epsilon r_1^2\beta_2 + 2b_2^2 \left(9\pi\epsilon\beta_2 - \epsilon\gamma \right) + 3\epsilon\tau_\epsilon \right) &= 0.
\end{aligned} \tag{3.7}$$

For $\beta_2 = 0$, it follows from (3.7) that $(0, 0, 0)$, denoted as (r_{10}, a_{20}, b_{20}) , is always a root. Besides, up to three other roots (in the positive quadrant), denoted as (r_{11}, r_{20}) , (r_{10}, r_{21}) , and (r_{12}, r_{22}) ,

can be obtained analytically with $r_2 = (a_2^2 + b_2^2)^{1/2}$ from (3.7) They are given by

$$\begin{aligned}
 r_{11} &= \sqrt{\frac{8\alpha_\epsilon - 3\tau_\epsilon - 24A_\epsilon\pi}{2\gamma}}, \\
 r_{21} &= \sqrt{\frac{8\alpha_\epsilon + 3\tau_\epsilon}{2\gamma}}, \\
 r_{12} &= \sqrt{\frac{8l - 128\beta_1^3 + 3\beta_1\gamma(8\alpha_\epsilon - 9\tau_\epsilon)}{18\beta_1\gamma^2}}, \\
 r_{22} &= \sqrt{\frac{-16l + 256\beta_1^3 + 3\beta_1\gamma(8\alpha_\epsilon - 9\tau_\epsilon)}{18\beta_1\gamma^2}},
 \end{aligned} \tag{3.8}$$

where $l = \sqrt{256\beta_1^6 - 144A_\epsilon^2\beta_1^2\gamma^2 + 6\beta_1^4\gamma(8\alpha_\epsilon - 9\tau_\epsilon)}$.

It can be seen from (3.7) and (3.8) that $r_{1(\cdot)}$, $r_{2(\cdot)}$, and σ determine the feature of motions of the system (3.1) when the strong resonant Hopf point at (A_c, α_c, τ_c) is perturbed by α_ϵ , A_ϵ , and τ_ϵ for given values of β_1 and γ . Such motion can be amplitude death when (r_{10}, a_{20}, b_{20}) is stable, single-mode periodic when (r_{11}, r_{20}) or (r_{10}, r_{21}) is a stable solution, or double-periodic when there is a nonzero stable solution in (r_{12}, r_{22}) . Therefore, it is necessary to classify the solutions of the algebraic equation (3.7) in the neighborhood of the strong double Hopf point (A_c, α_c, τ_c) . To this end, we represent the critical surfaces in the (A, α, τ) -space which partition the space into several regions where the number and the stability of the solutions are distinct. In Figure 1, S_1 , S_2 , and S_3 are three different surfaces in the (A, α, τ) -space, defined by

$$\begin{aligned}
 S_1 &= ((A, \alpha, \tau) \mid r_{11} = 0, \sigma \neq 0), \\
 S_2 &= ((A, \alpha, \tau) \mid r_{21} = 0, \sigma \neq 0), \\
 S_3 &= ((A, \alpha, \tau) \mid r_{12} = 0, r_{22} = 0, \sigma \neq 0).
 \end{aligned} \tag{3.9}$$

For given values of β_1 and γ , the stability of the solutions on these regions may be obtained by observing the eigenvalues of the Jacobian matrix of (3.8) at the corresponding solution. In Figure 1, a three-dimensional view of the asymptotic boundaries is shown for $\beta_1 = \gamma = 0.1$. The space is separated into six regions noted as (I)–(VI). There are a stable trivial solution $(r_{10}, r_{20}) = (0, 0)$ and an unstable periodic solution (r_{10}, r_{21}) in region (I) which is an amplitude death region. With (A, α, τ) changing to region (II), the trivial solution loses its stability. Two unstable solutions at $(0, 0)$ and $(r_{11}, 0)$ exist in region (III). When (A, α, τ) enters into region (IV), there are three unstable solutions given by $(0, 0)$, $(r_{11}, 0)$ and $(0, r_{21})$. There are a stable nontrivial solution (r_{12}, r_{22}) and three unstable solutions $(0, 0)$, $(0, r_{21})$ and $(r_{11}, 0)$ in region (V). With (A, α, τ) varying to region (VI), the nontrivial solution (r_{12}, r_{22}) disappears and the solution $(r_{11}, 0)$ becomes stable, which results in a stable single-mode or periodic solution. It follows from (3.5) and (3.8) that the stable single-mode or periodic solution of (3.1) is expressed as $u(t) = \sqrt{(8\alpha_\epsilon - 3\tau_\epsilon - 24A_\epsilon\pi)/2\gamma} \cos((1/2 + e\sigma)t)$ with a frequency $1/2 + e\sigma$ for a small e . Similarly, the mixed-mode solution of (3.1) is given by $u(t) = r_{12} \cos((1/2 + e\sigma)t) + r_{22} \cos(2(1/2 + e\sigma)t + \theta)$ when the parameters under consideration are located in region (V), where r_{12} and r_{22} satisfy (3.8) and θ is a constant.

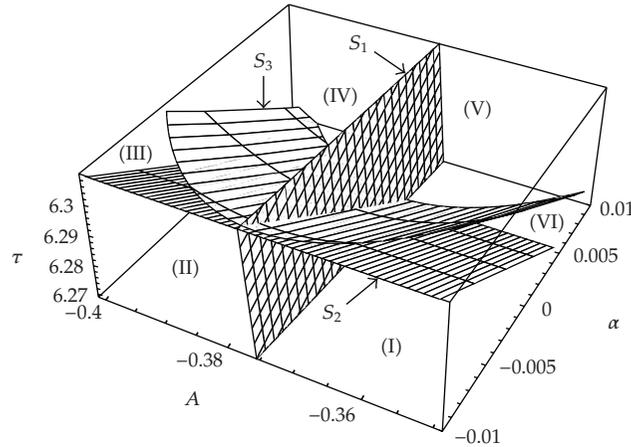


Figure 1: Critical surfaces of (3.1) for 1 : 2 strong resonance in the neighborhood of (A_c, α_c, τ_c) , where $\beta_1 = 0.1$, $\gamma = 0.1$, and $\beta_2 = 0$.

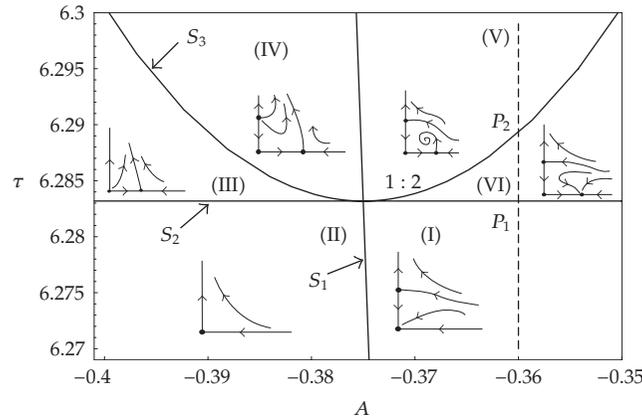


Figure 2: Cross-section of Figure 1 at $\alpha = 0$, where the dashed line represents $A = -0.36$ along which the analytical bifurcation diagram is shown in Figure 3. P_1 and P_2 are the intersection points of line $A = -0.36$ with surfaces S_2 and S_3 , respectively.

It is seen that the mixed-mode solution consists of two frequencies, that is, $1/2 + \epsilon\sigma$ and $2(1/2 + \epsilon\sigma)$. This implies that the mixed-mode solution is a frequency-doubling or period-doubling solution. Thus, the parameters cross through S_3 from region (VI) to region (V), the solution of (3.1) undergoes a period-doubling bifurcation. Different from the nonresonant case [11], the critical boundaries have strong curvatures when the solution of the single-mode amplitude equation undergoes a saddle-node bifurcation, which corresponds to a period-doubling solution occurring in (3.1). Thus, the parameter α has strong influence on the critical boundaries. This confirms that the bifurcation at (A_c, α_c, τ_c) is codimension three.

To have a clearer picture of the analytical prediction of Figure 1, we consider the cross-section at $\alpha = 0$ which is shown in Figure 2, in which the number and the stability of the analytical solutions in regions (I)–(VI) are also displayed by r_1 versus r_2 . It is seen in Figure 2 that there are a stable trivial, periodic and doubling periodic solutions in regions (I), (VI) and (V), respectively. No stable solution exists in regions (II), (III) and (IV). To study

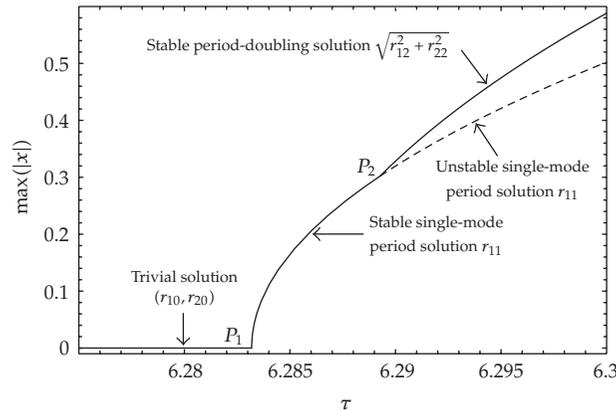


Figure 3: Analytical bifurcation diagram along the dashed line $A = -0.36$ in Figure 2.

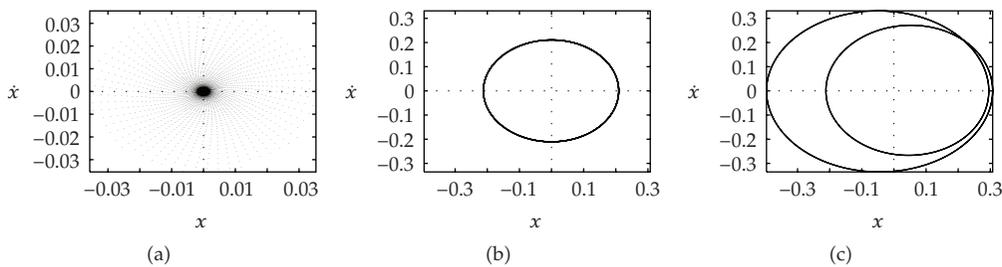


Figure 4: Numerical simulation for (3.1) along the vertical line $A = -0.36$ in Figure 2 with (a) $\tau = 6.28$, (b) $\tau = 6.285$, and (c) $\tau = 6.295$, showing the stable trivial, periodic and period-doubling solutions. Parameters considered belong to regions (I), (VI), and (V).

the delay-induced bifurcations analytically, we fix the value of A at $A = -0.36$ and vary τ along this vertical dashed line (see Figure 2). The corresponding bifurcation diagram of the analytical solutions derived from (3.5) and (3.8) for $\sigma \neq 0$ in terms of $\text{Max}(|x|) = \sqrt{r_1^2 + r_2^2}$ and τ is shown in Figure 3.

Figure 3 shows that no vibration occurs in the system when the delay is smaller than the value at P_1 , that is, $\tau = \tau_c = 2\pi$. The system undergoes a Hopf bifurcation at P_1 and a stable branch of periodic solution r_{11} in (3.8) bifurcates from the trivial solution at P_1 . With the delay increasing, the periodic solution becomes unstable and a period-doubling bifurcation occurs at P_2 with a maximum amplitude $\sqrt{r_{12}^2 + r_{22}^2}$.

Now, numerical simulation is employed to qualitatively examine the validity of the present method. The Runge-Kutta scheme is adopted to produce the numerical results of (3.1) for (α, A, τ) in regions displayed in Figure 1. To compare with our analytical predictions mentioned above, we keep β_2, β_1 , and γ the same as those in Figures 1–3. First, we verify the analytical results shown in Figure 3 by keeping $A = -0.36$, and varying τ . The values of τ are chosen to be 6.28, 6.285, and 6.295, respectively, as shown in Figures 4(a), 4(b) and 4(c). Figure 4 shows that there are a stable trivial, periodic and period-doubling solutions when τ is located in regions (I), (VI) and (V), respectively. The numerical simulation also shows no stable solutions in the other regions. Therefore, the analytical prediction is in good agreement with the numerical result, which implies that solution (3.5) from the present method is

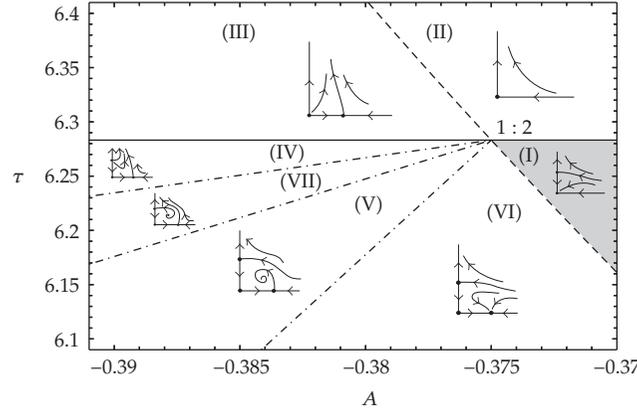


Figure 5: Critical boundaries of (3.1) when $\epsilon = 0.1$, $\beta_1 = 0$, $\beta_2 = 4.0$, and $\gamma = 1.0$ for 1 : 2 weak resonant double Hopf bifurcation where solid lines, dashed, and dot-dashed represent boundaries, and amplitude death region is in grey.

valid qualitatively. In addition, the period-doubling bifurcation is a direct consequence of the proximity to the 1 : 2 resonance. The analytical results are also in agreement with those in [8].

4. Discussion and Conclusion

We have proposed an analytical method to consider various delay-induced solutions derived from double Hopf bifurcation with strong resonance for a type of two first-order delay differential equations. It should be noted that the present method can also be applied to the weak and nonresonant cases. For example, one cannot solve any nontrivial solutions with $\sigma \neq 0$ from (3.6) and (3.7) when $\beta_1 = 0$ and $\beta_2 \neq 0$ for 1 : 2 resonance. This implies that there is no solution in the form of (3.5). However, the solution can be expressed in the following form:

$$Z(t) = \begin{pmatrix} r_1 \cos(\varphi_1 t + \theta_1) + r_2 \cos(\varphi_2 t + \theta_2) \\ -\frac{1}{2} r_1 \sin(\varphi_1 t + \theta_1) - r_2 \sin(\varphi_2 t + \theta_2) \end{pmatrix}, \quad (4.1)$$

where $\varphi_1 = (1/2 + \epsilon\sigma_1)$ and $\varphi_2 = (1 + \epsilon\sigma_2)$ and the stabilities of the solutions are shown in Figure 5, which are similar to that in [11]. Thus, the Hopf-Hopf interactions with 1 : 2 resonance become weak.

The results provided in the present paper can be also understood further by comparing those given by Campbell and Leblanc [8], where they use a center manifold reduction and normal form analysis to deduce a four-dimensional invariant (center) manifold, the flow on which is a good approximation for the long-term behavior of solutions to the original differential equation. Correspondingly, the truncated amplitude equation on the center manifold is presented in $O(2)$ providing that square terms are not degenerate. Such truncated amplitude equation is also given in the present work, as expressed in (3.7) when the cubic terms are neglected, or $\beta_2 = \gamma = 0$. Thus, the bifurcation diagrams of the corresponding

truncated normal form in $O(2)$ are shown in Figure 2. When the square terms in the truncated normal form are degenerate, one has to consider the truncated normal form in $O(3)$. The corresponding amplitude equation can be also obtained correctly in (3.7) without the square terms or $\beta_1 = 0$ in (3.1). For this case, the dynamical behavior near the double Hopf point is distinct topologically, as shown in Figure 5. Consequently, one can distinguish two cases for strong and weak resonances.

Finally, a brief conclusion is represented to end this paper. An efficient method is proposed to study delay-induced strong resonant double Hopf bifurcation for nonlinear systems with time delay. The method is first described and leads to the main theorem. As an application, the proposed method is employed to investigate the double Hopf bifurcation with 1 : 2 strong resonance in the van der Pol system with time delay. Dynamics arising from the bifurcation are classified qualitatively and expressed approximately in a closed form for either square or cubic nonlinearity. The results show that 1 : 2 resonance can lead to codimension-three and codimension-two bifurcations. The validity of analytical predictions is illustrated by their consistency with numerical simulations. As a result, a simple but efficient method is developed for the classification of unfolding of higher codimensional bifurcations in DDEs. The present research brings out the physical meaning and significance of unfolding parameters in higher codimensional bifurcations.

Acknowledgments

This work is supported by Hong Kong Research Grants Council under CERG Grant (Cityu 1007/05E) and Key Project of National Nature and Science Foundation of China under Grant no. 10532050. The senior author also thanks the support of the National Outstanding Young Funds of China under Grant no. 10625211 and the Program of Shanghai Subject Chief Scientist under Grant no. 08XD14044.

References

- [1] J. Xu and K. W. Chung, "Effects of time delayed position feedback on a van der Pol-Duffing oscillator," *Physica D*, vol. 180, no. 1-2, pp. 17–39, 2003.
- [2] J. Xu and P. Yu, "Delay-induced bifurcations in a nonautonomous system with delayed velocity feedbacks," *International Journal of Bifurcation and Chaos*, vol. 14, no. 8, pp. 2777–2798, 2004.
- [3] P. Yu, Y. Yuan, and J. Xu, "Study of double Hopf bifurcation and chaos for an oscillator with time delayed feedback," *Communications in Nonlinear Science and Numerical Simulation*, vol. 7, no. 1-2, pp. 69–91, 2002.
- [4] D. V. R. Reddy, A. Sen, and G. L. Johnston, "Dynamics of a limit cycle oscillator under time delayed linear and nonlinear feedbacks," *Physica D*, vol. 144, no. 3-4, pp. 335–357, 2000.
- [5] A. G. Balanov, N. B. Janson, and E. Schöll, "Delayed feedback control of chaos: bifurcation analysis," *Physical Review E*, vol. 71, no. 1, Article ID 016222, 9 pages, 2005.
- [6] S. Q. Ma, Q. Lu, and Z. S. Feng, "Double Hopf bifurcation for van der Pol-Duffing oscillator with parametric delay feedback control," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 993–1007, 2008.
- [7] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, vol. 112 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2nd edition, 1998.
- [8] S. A. Campbell and V. G. LeBlanc, "Resonant Hopf-Hopf interactions in delay differential equations," *Journal of Dynamics and Differential Equations*, vol. 10, no. 2, pp. 327–346, 1998.
- [9] J. Xu, K.-W. Chung, and C.-L. Chan, "An efficient method for studying weak resonant double Hopf bifurcation in nonlinear systems with delayed feedbacks," *SIAM Journal on Applied Dynamical Systems*, vol. 6, no. 1, pp. 29–60, 2007.

- [10] A. Luongo, A. Digidio, and A. Paolone, "On the proper form of the amplitude modulation equations for resonant systems," *Nonlinear Dynamics*, vol. 27, no. 3, pp. 237–254, 2002.
- [11] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, vol. 42 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.