Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 841303, 17 pages
doi:10.1155/2009/841303

*Research Article*

# Matrix Measures in the Qualitative Analysis of Parametric Uncertain Systems

**Octavian Pastravanu and Mihaela-Hanako Matcovschi**

*Department of Automatic Control and Applied Informatics, Technical University "Gh. Asachi" of Iasi, 700050 Iasi, Romania*

Correspondence should be addressed to Mihaela-Hanako Matcovschi, mhanako@ac.tuiasi.ro

Received 7 December 2008; Revised 8 June 2009; Accepted 27 July 2009

Recommended by Tamas Kalmar-Nagy

The paper considers *parametric uncertain systems* of the form $\dot{x}(t) = Mx(t), M \in \mathcal{M}, \mathcal{M} \subset \mathbb{R}^{n\times n}$, where $\mathcal{M}$ is either a *convex hull*, or a *positive cone* of matrices, generated by the set of vertices $\mathcal{V} = \{M_1, M_2, \ldots, M_K\} \subset \mathbb{R}^{n\times n}$. Denote by $\mu_{\| \|}$ the matrix measure corresponding to a vector norm $\| \|$. When $\mathcal{M}$ is a convex hull, the condition $\mu_{\| \|}(M_k) \leq r < 0$, $1 \leq k \leq K$, is necessary and sufficient for the existence of *common strong Lyapunov functions* and *exponentially contractive invariant sets* with respect to the trajectories of the uncertain system. When $\mathcal{M}$ is a positive cone, the condition $\mu_{\| \|}(M_k) \leq 0$, $1 \leq k \leq K$, is necessary and sufficient for the existence of *common weak Lyapunov functions* and *constant invariant sets* with respect to the trajectories of the uncertain system. Both Lyapunov functions and invariant sets are described in terms of the vector norm $\| \|$ used for defining the matrix measure $\mu_{\| \|}$. Numerical examples illustrate the applicability of our results.



## 1. Introduction

First let us present the notations and nomenclature used in our paper.

For a square matrix $M \in \mathbb{R}^{n\times n}$, the *matrix norm* induced by a generic *vector norm* $\| \ \|$ is defined by $\|M\| = \sup_{y\in\mathbb{R}^n, y\neq 0}\|My\|/\|y\| = \max_{y\in\mathbb{R}^n, \|y\|=1}\|My\|$, and the *corresponding matrix measure* (also known as *logarithmic norm*) is given by $\mu_{\| \ \|}(M) = \lim_{\theta\downarrow 0}(\|I + \theta M\| - 1)/\theta$ ([1, page 41]). The spectrum of $M$ is denoted by $\sigma(M) = \{z \in \mathbb{C} \mid \det(sI - M) = 0\}$ and $\lambda_i(M) \in \sigma(M)$, $i = 1, \ldots, n$, represent its eigenvalues. If $M \in \mathbb{R}^{n\times n}$ is a symmetrical matrix, $M \prec 0$ ($M \succ 0$) means that matrix $M$ is negative (positive) definite. If $X \in \mathbb{R}^{n\times m}$, then $|X|$ represents the nonnegative matrix (for $m \geq 2$) or vector (for $m = 1$) defined by taking the absolute values of the entries of $X$. If $X, Y \in \mathbb{R}^{n\times m}$, then "$X \leq Y$", "$X < Y$" mean componentwise inequalities.

Matrix measures were used in the qualitative analysis of various types of differential systems, as briefly pointed out below, besides their applications in numerical analysis. Monograph ([1, pages 58-59]) derived upper and lower bounds for the norms of the solution vector and proposed stability criteria for time-variant linear systems. Further properties of matrix measures were revealed in [2]. Paper [3] provided bounds for the computer solution and the accumulated truncation error corresponding to the backward Euler method. The work in [4] gave a characterization of vector norms as Lyapunov functions for time-invariant linear systems. The work in [5] developed sufficient conditions for the stability of neural networks. The work in [6] explored contractive invariant sets of time-invariant linear systems. The work in [7] formulated sufficient conditions for the stability of interval systems. The work in [8] presented a necessary and sufficient condition for componentwise stability of time-invariant linear systems.

A compact survey on the history of matrix measures and the modern developments originating from this notion can be found in [9].

The current paper considers *parametric uncertain systems* of the form

$$\dot{x}(t) = Mx(t), \quad M \in \mathcal{M}, \ \mathcal{M} \subset \mathbb{R}^{n\times n}, \tag{1.1}$$

where $\mathcal{M}$ is either a *convex hull* of matrices,

$$\mathcal{M}_h = \left\{ M \in \mathbb{R}^{n\times n} \mid M = \sum_{k=1}^{K} \gamma_k M_k, \gamma_k \geq 0, \sum_{k=1}^{K} \gamma_k = 1 \right\}, \tag{1.2}$$

or a *positive cone* of matrices,

$$\mathcal{M}_c = \left\{ M \in \mathbb{R}^{n\times n} \mid M = \sum_{k=1}^{K} \gamma_k M_k, \gamma_k \geq 0, M \neq 0 \right\}, \tag{1.3}$$

generated by the set of vertex matrices $\mathcal{V} = \{M_1, M_2, \dots, M_K\} \subset \mathbb{R}^{n\times n}$.

In investigating the evolution of system (1.1) we assume that matrix $M$ is fixed, but arbitrarily taken from the matrix set $\mathcal{M}$ defined by (1.2) or (1.3). Thus, the parameters of system (1.1) are not time-varying. Consequently, once $M$ is arbitrarily selected from $\mathcal{M}$, the trajectory initialized in $x(t_0) = x_0$, namely, $x(t) = x(t; t_0, x_0) = e^{M(t-t_0)} x_0$, is defined for all $t \in \mathbb{R}_+$.

The literature of control engineering contains many papers that explore the stability robustness by considering systems of form (1.1), a great interest focusing on the case when the convex hull $\mathcal{M}_h$ is an *interval matrix* [7, 10–13].

For system (1.1) we define the following properties, in accordance with the definitions presented in [14–16] for a dynamical system.

*Definition 1.1.* (a) The uncertain system (1.1) is called *stable* if the equilibrium $\{0\}$ is stable, that is,

$$\forall \varepsilon > 0, \forall t_0 \in \mathbb{R}_+, \quad \exists \delta = \delta(\varepsilon) > 0 : \|x_0\| \leq \delta \Longrightarrow \|x(t; t_0, x_0)\| \leq \varepsilon, \quad \forall t \geq t_0 \tag{1.4}$$

for any solution of (1.1) corresponding to an $M \in \mathcal{M}$.

(b) The uncertain system (1.1) is called *exponentially stable* if the equilibrium $\{0\}$ is exponentially stable, that is,

$$\exists r < 0 : \forall \varepsilon > 0, \forall t_0 \in \mathbb{R}_+, \quad \exists \delta = \delta(\varepsilon) > 0 : \|x_0\| \leq \delta \Longrightarrow \|x(t; t_0, x_0)\| \leq \varepsilon e^{r(t-t_0)}, \quad t \geq t_0 \quad (1.5)$$

for any solution of (1.1) corresponding to an $M \in \mathcal{M}$.

*Remark 1.2.* Using the connection between linear system stability and matrix eigenvalue location (e.g. [15]), we have the following characterizations.

(a) The uncertain system (1.1) is stable if and only if

$$\forall M \in \mathcal{M} : \sigma(M) \subset \{s \in \mathbb{C} \mid \operatorname{Re} s \leq 0\} \text{ and if } \operatorname{Re} \lambda_{i_0}(M) = 0 \text{ then } \lambda_{i_0}(M) \text{ is simple.} \quad (1.6)$$

In this case, the matrix set $\mathcal{M}$ is said to be *quasistable*.

(b) The uncertain system (1.1) is exponentially stable if and only if

$$\exists r < 0 : \forall M \in \mathcal{M} : \sigma(M) \subset \{s \in \mathbb{C} \mid \operatorname{Re}\ s \leq r\}. \quad (1.7)$$

In this case, the matrix set $\mathcal{M}$ is said to be *Hurwitz stable*.

*Definition 1.3.* Consider the function

$$V : \mathbb{R}^n \longrightarrow \mathbb{R}_+, \qquad V(x) = \|x\|, \quad (1.8)$$

and its right Dini derivative, calculated along a solution $x(t)$ of (1.1):

$$\mathrm{D}^+ V(x(t)) = \lim_{\theta \downarrow 0} \frac{V(x(t+\theta)) - V(x(t))}{\theta}, \quad t \in \mathbb{R}_+. \quad (1.9)$$

(a) $V$ is called a *common strong Lyapunov function* for the uncertain system (1.1), *with the decreasing rate* $r < 0$, if for any solution $x(t)$ of (1.1) corresponding to an $M \in \mathcal{M}$, we have

$$\forall t \in \mathbb{R}_+ : \mathrm{D}^+ V(x(t)) \leq r V(x(t)). \quad (1.10)$$

(b) $V$ is called a *common weak Lyapunov function* for the uncertain system (1.1), if for any solution $x(t)$ of (1.1) corresponding to an $M \in \mathcal{M}$, we have

$$\forall t \in \mathbb{R}_+ : \mathrm{D}^+ V(x(t)) \leq 0. \quad (1.11)$$

*Definition 1.4.* The time-dependent set

$$X_r^{\varepsilon}(t;t_0) = \left\{x \in \mathbb{R}^n \mid \|x\| \le \varepsilon e^{r(t-t_0)}\right\}, \quad t, t_0 \in \mathbb{R}_+, t \ge t_0, \varepsilon > 0, r \le 0 \tag{1.12}$$

is called *invariant* with respect to the uncertain system (1.1) if for any solution $x(t)$ of (1.1) corresponding to an $M \in \mathcal{M}$, we have

$$\forall t_0 \in \mathbb{R}_+, \forall x_0 \in \mathbb{R}^n, \quad \|x_0\| \le \varepsilon \Longrightarrow \|x(t;t_0,x_0)\| \le \varepsilon e^{r(t-t_0)}, \quad \forall t \ge t_0, \tag{1.13}$$

meaning that any trajectory initiated inside the set $X_r^{\varepsilon}(t_0;t_0)$ will never leave $X_r^{\varepsilon}(t;t_0)$.

(a) A set of the form (1.12) with $r < 0$ is said to be *exponentially contractive*.

(b) A set of the form (1.12) with $r = 0$ is said to be *constant*.

This paper proves that matrix-measure-based inequalities applied to the vertices $M_k \in \mathcal{V}$, $1 \le k \le K$, provide necessary and sufficient conditions for the properties of the uncertain system (1.1) formulated by Definitions 1.3 and 1.4. The cases when the matrix set $\mathcal{M}$ is defined by the convex hull (1.2) and by the positive cone (1.3) are separately addressed. When $\| \ \|$ is a symmetric gauge function or an absolute vector norm and the vertices $M_k \in \mathcal{V}$, $1 \le k \le K$, satisfy some supplementary hypotheses, a unique test matrix $M^*$ can be found such that a single inequality using $\mu_{\| \ \|}(M^*)$ implies or is equivalent to the group of inequalities written for all vertices. Some numerical examples illustrate the applicability of the proposed theoretical framework.

Our results are extremely useful for refining the dynamics analysis of many classes of engineering processes modeled by linear differential systems with parametric uncertainties. Relying on necessary and sufficient conditions formulated in terms of matrix measures, we get more detailed information about the system trajectories than offered by the standard investigation of equilibrium stability.

## 2. Main Results

### 2.1. Uncertain System Defined by a Convex Hull of Matrices

**Theorem 2.1.** *Consider the uncertain system* (1.1) *with* $\mathcal{M} = \mathcal{M}_h$*, the convex hull defined by* (1.2)*, which, in the sequel, is referred to as the uncertain system* (1.1) *and* (1.2)*. Let* $\mu_{\| \ \|}$ *be the matrix measure corresponding to the vector norm* $\| \ \|$*, and* $r < 0$ *a constant. The following statements are equivalent.*

(i) *The vertices of the convex hull* $\mathcal{M}_h$ *fulfill the inequalities*

$$\forall M_k \in \mathcal{V} : \mu_{\| \ \|}(M_k) \le r. \tag{2.1}$$

(ii) *The function* $V$ *defined by* (1.8) *is a common strong Lyapunov function for the uncertain system* (1.1) *and* (1.2) *with the decreasing rate* $r$.

(iii) *For any* $\varepsilon > 0$*, the exponentially contractive set* $X_r^{\varepsilon}(t;t_0)$ *defined by* (1.12) *is invariant with respect to the uncertain system* (1.1) *and* (1.2).

*Proof.* We organize the proof in two parts. Part I proves the following results.

(R1) Inequalities (2.1) are equivalent to

$$\forall M \in \mathcal{M} : \mu_{\| \ \|}(M) \leq r. \tag{2.2}$$

(R2) Inequality (1.10) is equivalent to

$$\forall t_0 \in \mathbb{R}_+,\ \forall x_0 \in \mathbb{R}^n, \quad \|x(t;t_0,x_0)\| \leq e^{r(t-t_0)}\|x_0\|, \quad \forall t \geq t_0. \tag{2.3}$$

(R3) The matrix measure $\mu_{\| \ \|}$ fulfills the equality

$$\forall M \in \mathbb{R}^{n\times n} : \mu_{\| \ \|}(M) = \lim_{\theta\downarrow 0} \frac{\|e^{M\theta}\| - 1}{\theta}. \tag{2.4}$$

Part II uses (R1), (R2), and (R3) to show that (i), (ii), and (iii) are equivalent.

*Proof of Part I.* (R1) If (2.2) is true, then (2.1) is true, since $M_k \in \mathcal{M}_h$, for $k = 1,\ldots,K$. Conversely, if (2.1) is true, then, from the convexity of the matrix measure, we get

$$\forall M \in \mathcal{M}_h, \quad M = \sum_{k=1}^{K}\gamma_k M_k : \mu_{\| \ \|}(M) \leq \sum_{k=1}^{K}\gamma_k \mu_{\| \ \|}(M_k) \leq \sum_{k=1}^{K}(\gamma_k r) = \left(\sum_{k=1}^{K}\gamma_k\right) r = r. \tag{2.5}$$

(R2) If inequality (2.3) is true, then, for any solution $x(s)$ of (1.1) and (1.2) with initial condition set at $s_0 = t \geq 0$ as $x(s_0) = x_0$, we have

$$D^+V(x(s_0)) = \lim_{\theta\downarrow 0} \frac{\|x(s_0+\theta;s_0,x_0)\| - \|x_0\|}{\theta} \leq \left(\lim_{\theta\downarrow 0}\frac{e^{r\theta}-1}{\theta}\right)\|x_0\| = rV(x(s_0)). \tag{2.6}$$

Conversely, let $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$. If inequality (1.10) holds for $x(t) = x(t;t_0,x_0)$, consider the differential equation $\dot{y}(t) = ry(t)$ with the initial condition $y(t_0) = V(x(t_0)) = V(x_0)$. Then, according to [14, Theorem 4.2.11], $V(x(t)) \leq y(t) = e^{r(t-t_0)}y(t_0) = e^{r(t-t_0)}V(x_0)$, for all $t \geq t_0$.

(R3) For $M \in \mathcal{M}_h$ and $\theta > 0$, we have $e^{M\theta} = I + \theta M + \theta O(\theta)$, with $\lim_{\theta\downarrow 0} O(\theta) = 0$. The triangle inequality $\|I+\theta M\| - \theta\|O(\theta)\| \leq \|I+\theta M+\theta O(\theta)\| \leq \|I+\theta M\| + \theta\|O(\theta)\|$ leads to $(\|I+\theta M\|-1)/\theta - \|O(\theta)\| \leq (\|e^{M\theta}\|-1)/\theta \leq (\|I+\theta M\|-1)/\theta + \|O(\theta)\|$. By taking $\lim_{\theta\downarrow 0}$, we finally obtain the equality (2.4).

*Proof of Part II.* (i)⇒(ii) For any solution $x(t)$ to (1.1) and (1.2) corresponding to an $M \in \mathcal{M}_h$, we get

$$\begin{aligned}\forall t \in \mathbb{R}_+ : \mathrm{D}^+ V(x(t)) = \mathrm{D}^+\| \, x(t)\| &= \lim_{\theta\downarrow 0} \frac{\| \, x(t+\theta)\| - \|x(t)\|}{\theta} \\ &= \lim_{\theta\downarrow 0} \frac{\left\|e^{M\theta}x(t)\right\| - \|x(t)\|}{\theta} \\ &\leq \left[\lim_{\theta\downarrow 0} \frac{\left\|e^{M\theta}\right\| - 1}{\theta}\right]\|x(t)\| \overset{(\mathrm{R3})}{=} \mu_{\| \ \|}(M)\|x(t)\| \overset{(\mathrm{R1})}{\leq} r\|x(t)\|.\end{aligned} \tag{2.7}$$

(ii)⇒(i) For all $t_0, \theta \in \mathbb{R}_+$, $\|e^{M\theta}\| = \sup_{x_0 \neq 0}\|e^{M\theta}x_0\|/\|x_0\| = \sup_{x_0\neq 0}\|x(t_0+\theta; t_0, x_0)\|/x_0 \overset{(\mathrm{R2})}{\leq} e^{r\theta}$. Hence, we have $\mu_{\| \ \|}(M) \overset{(\mathrm{R3})}{=} \lim_{\theta\downarrow 0}(\|e^{M\theta}\| - 1)/\theta \leq \lim_{\theta\downarrow 0}(e^{r\theta} - 1)/\theta = r$.

(ii)⇒(iii) By contradiction, assume that there exists $\varepsilon^* > 0$ such that the exponentially contractive set $X_r^{\varepsilon^*}(t; t_0)$ is not invariant with respect to the uncertain system (1.1) and (1.2). Then there exists a trajectory $\tilde{x}(t)$ of (1.1) and (1.2) for which condition (1.13) is violated, meaning that we can find $t^*, t^{**} \in \mathbb{R}_+$, $t^{**} > t^* \geq t_0$, so that $\|\tilde{x}(t^*)\| \leq \varepsilon^* e^{r(t^*-t_0)}$ and $\|\tilde{x}(t^{**})\| > \varepsilon^* e^{r(t^{**}-t_0)}$. This leads to $e^{r(t^{**}-t^*)}\|\tilde{x}(t^*)\| < \|\tilde{x}(t^{**})\|$, which contradicts (2.3). As a result, according to (R2), we contradict (ii).

(iii)⇒(ii) For arbitrary $t \geq t_0$, by taking $\varepsilon = \|x_0\|$ in (1.13), we get (2.3) that is equivalent to (1.10), via (R2). □

*Remark 2.2.* The equivalent conditions (i)–(iii) of Theorem 2.1 imply the exponential stability of the uncertain system (1.1) and (1.2). Indeed, if the flow invariance condition (1.13) from Definition 1.4 is satisfied, then condition (1.5) from Definition 1.1, for exponential stability, is satisfied with $\delta(\varepsilon) = \varepsilon$. Conversely, if (1.5) is true for a certain $\delta(\varepsilon) > \varepsilon$, but not for $\delta(\varepsilon) = \varepsilon$, then condition (1.13) is not met. In other words the uncertain system (1.1) and (1.2) may be exponentially stable without satisfying the equivalent conditions (i)–(iii) of Theorem 2.1.

*Remark 2.3.* Theorem 2.2 in [7] shows that condition (i) in Theorem 2.1 is *sufficient* for the Hurwitz stability of the convex hull of matrices defined by (1.2). According to Remark 2.2, the uncertain system (1.1) and (1.2) is exponentially stable. The fact that condition (i) in Theorem 2.1 is *necessary and sufficient* for stronger properties of the uncertain system (1.1) and (1.2) remained hidden for the investigations developed by [7].

*Remark 2.4.* Theorem 2.1 offers a high degree of generality for the qualitative analysis of uncertain system (1.1) and (1.2). Thus, from Theorem 2.1 particularized to the vector norm $\|x\|_2^H = \|Hx\|_2 = (x^T H^T H x)^{1/2}$, $x \in \mathbb{R}^{\mathrm{n}}$, (where $H$ is a nonsingular matrix) and the corresponding matrix measure $\mu_{\| \ \|_2^H}$, we get the following well-known characterization of the quadratic stability of uncertain system (1.1) and (1.2), $\exists\, Q \succ 0$ : for all $M_k \in \mho$ : $QM_k + M_k^T Q \prec 0$ (e.g., [16, page 213]). Indeed, according to ([1, page 41]) inequality (2.1) in Theorem 2.1 means $\lambda_{\max}((1/2)HM_kH^{-1} + (1/2)(HM_kH^{-1})^T) = \mu_{\| \ \|_2^H}(M_k) \leq r < 0$, which is equivalent to the condition $QM_k + M_k^T Q \prec 0$, with $Q = H^T H$, for all $M_k \in \mho$. In other words, Theorem 2.1 provides a comprehensive scenario that naturally accommodates results already available in particular forms for uncertain system (1.1) and (1.2).

### 2.2. Uncertain System Defined by a Positive Cone of Matrices

**Theorem 2.5.** *Consider the uncertain system* (1.1) *with* $\mathcal{M} = \mathcal{M}_c$*, the positive cone defined by* (1.3)*, which, in the sequel, is referred to as the uncertain system* (1.1) *and* (1.3)*. Let* $\mu_{\| \ \|}$ *be the matrix measure corresponding to the vector norm* $\| \ \|$*. The following statements are equivalent.*

(i) *The vertices of the positive cone* $\mathcal{M}_c$ *fulfill the inequalities*

$$\forall M_k \in \mathcal{V} : \mu_{\| \ \|}(M_k) \leq 0. \tag{2.8}$$

(ii) *The function* $V$ *defined by* (1.8) *is a common weak Lyapunov function for the uncertain system* (1.1) *and* (1.3)*.*

(iii) *For any* $\varepsilon > 0$*, the constant set* $X_0^{\varepsilon}(t; t_0)$ *defined by* (1.12) *is invariant with respect to the uncertain system* (1.1) *and* (1.3)*.*

*Proof.* It is similar to the proof of Theorem 2.1 where we take $r = 0$. □

*Remark 2.6.* The equivalent conditions (i)–(iii) of Theorem 2.5 imply the stability of the uncertain system (1.1) and (1.3). Indeed, if the flow invariance condition (1.13) from Definition 1.4 is satisfied, then condition (1.4) for stability from Definition 1.1 is satisfied with $\delta(\varepsilon) = \varepsilon$. Conversely, if (1.4) is true for a certain $\delta(\varepsilon) > \varepsilon$, but not for $\delta(\varepsilon) = \varepsilon$, then condition (1.13) is not met. In other words the uncertain system (1.1) and (1.3) may be stable without satisfying the equivalent conditions (i)–(iii) of Theorem 2.5.

*Remark 2.7.* Theorem 2.3 in [7] claims that $\mu_{\| \ \|}(M_k) < 0$ for $k = 1, \ldots, K$ (i.e., condition (i) in Theorem 2.5 with strict inequalities) is *sufficient* for the Hurwitz stability of the positive cone of matrices $\mathcal{M}_c$ (1.3). However this is not true. Inequalities $\mu_{\| \ \|}(M_k) < 0$, $k = 1, \ldots, K$, imply $\sup_{M \in \mathcal{M}_c} \mu(M) = 0$, which, together with for all $M \in \mathcal{M}_c : \max_{i=1,\ldots,n}\{\mathrm{Re}\{\lambda_i(M)\} \leq \mu_{\| \ \|}(M)$, for example, [1], yield $\sup_{M \in \mathcal{M}_c} \max_{i=1,\ldots,n}\{\mathrm{Re}\,\lambda_i(M)\} \leq 0$. Thus, condition (1.7) for the Hurwitz stability of the positive cone of matrices $\mathcal{M}_c$ defined by (1.3) may be not satisfied. Although the hypothesis of [7, Theorem 2.3] is stronger than condition (i) in our Theorem 2.5, this hypothesis can guarantee only the stability (but not the exponential stability) of the uncertain system (1.1) and (1.3). Moreover, as already mentioned in Remark 2.3 for the matrix set $\mathcal{M}_h$ defined by (1.2), [7] does not discuss the necessity parts of the results.

## 3. Usage of a Single Test Matrix for Checking Condition (i) of Theorems 2.1 and 2.5

Condition (i) of both Theorems 2.1 and 2.5 represents inequalities of the form

$$\mu_{\| \ \|}(M_k) \leq r, \quad k = 1, \ldots, K, \tag{3.1}$$

which involve all the vertices $\mathcal{V} = \{M_1, M_2, \ldots, M_K\} \subset \mathbb{R}^{n \times n}$ of the matrix sets defined by (1.2) or (1.3), respectively. We are going to show that, in some particular cases, one can find a single test matrix $M^* \in \mathbb{R}^{n \times n}$ such that the satisfaction of inequality

$$\mu_{\| \ \|}(M^*) \leq r \tag{3.2}$$

guarantees the fulfillment of (3.1).

Given a real matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, let us define its *comparison matrix* $\overline{A} = (\overline{a}_{ij}) \in \mathbb{R}^{n\times n}$ by

$$\overline{a}_{ii} = a_{ii}, \quad i = 1, \ldots, n; \quad \overline{a}_{ij} = \left|a_{ij}\right|, \quad i, j = 1, \ldots, n, \; i \neq j. \tag{3.3}$$

**Proposition 3.1.** *(a) If the following hypotheses (H1), (H2) are satisfied, then inequality* (3.2) *is a sufficient condition for inequalities* (3.1).

(H1) *The vector norm $\|\ \|$ is a symmetric gauge function ([17, page 438]) (i.e., it is an absolute vector norm that is a permutation invariant function of the entries of its argument) and $\mu_{\|\ \|}$ is the corresponding matrix measure.*

(H2) *Matrix $M^* \in \mathbb{R}^{n\times n}$ satisfies the componentwise inequalities*

$$P_k{}^T \overline{M_k} P_k \leq M^*, \quad k = 1, \ldots, K, \tag{3.4}$$

*for some permutation matrices $P_k \in \mathbb{R}^{n\times n}$, $k = 1, \ldots, K$.*

*(b) If the above hypotheses (H1), (H2) are satisfied and there exists $M^{**} \in \mathcal{V}$ such that $\mu_{\|\ \|}(M^*) = \mu_{\|\ \|}(M^{**})$, then inequality* (3.2) *is a necessary and sufficient condition for inequalities* (3.1).

*Proof.* (a) We organize the proof in two parts. Part I proves the following results.

(R1) If $P \in \mathbb{R}^{n\times n}$ is a permutation matrix, then

$$\forall A \in \mathbb{R}^{n\times n} : \mu_{\|\ \|}\left(P^T A P\right) = \mu_{\|\ \|}(A). \tag{3.5}$$

(R2) Given $A \in \mathbb{R}^{n\times n}$, if the componentwise inequality

$$P^T \overline{A} P \leq M^* \tag{3.6}$$

is fulfilled for a permutation matrix $P \in \mathbb{R}^{n\times n}$, then

$$\mu_{\|\ \|}(A) \leq \mu_{\|\ \|}(M^*). \tag{3.7}$$

Part II uses (R2) to show that (3.2) implies (3.1).

*Proof of Part I.* (R1) From the definition of the matrix norm, there exists $x^* \in \mathbb{R}^{n\times n}$, $\|x^*\| = 1$, such that $\|A\| = \|Ax^*\|$. Since the considered vector norm $\|\ \|$ is permutation invariant, we have $\|x^*\| = \|P^T x^*\| = 1$ for a permutation matrix $P \in \mathbb{R}^{n\times n}$. This leads to $\|A\| = \|Ax^*\| = \|(AP)(P^T x^*)\| \leq \|AP\|\|P^T x^*\| = \|AP\|$. Let us prove that the strict inequality $\|A\| < \|AP\|$ does not hold. Assume, by contradiction, that $\|A\| < \|AP\|$. Then, there exists $y^* \in \mathbb{R}^{n\times n}$, $\|y^*\| = 1$, such that $\|APy^*\| = \|AP\| > \|\mathrm{A}\|$. Hence, $x^{**} = Py^* \in \mathbb{R}^{n\times n}$ with $\|x^{**}\| = \|Py^*\| = y^* = 1$ satisfies $\|A\| < \|Ax^{**}\|$ that contradicts the definition of $\|A\|$. Consequently, $\|A\| = \|AP\|$.

Similarly we prove that $\|AP\| = \|P^T AP\|$, yielding $\|A\| = \|P^T AP\|$. Thus, we get $\|I + \theta A\| = \|P^T(I+\theta A)P\| = \|I+\theta P^T AP\|$ and, consequently, $(\|I+\theta A\|-1)/\theta = (\|I+\theta P^T AP\|-1)/\theta$. By taking $\lim_{\theta\downarrow 0}$ we obtain equality (3.5).

(R2) First, we exploit the componentwise matrix inequality (3.6). For small $\theta > 0$, we get $0 \le I + \theta P^T \overline{A} P \le I + \theta M^*$ that leads to the following componentwise vector inequality $|(I + \theta P^T \overline{A} P)y| \le |(I + \theta P^T \overline{A} P)|y|| \le |(I + \theta M^*)|y||$, with $y \in \mathbb{R}^n$.

Since $\| \ \|$ is a symmetric gauge function, it is also an absolute vector norm, and, equivalently, a monotonic vector norm [17, Theorem 5.5.10]. Consequently, $\|(I+\theta P^T \overline{A} P)y\| \le \|(I + \theta P^T \overline{A} P)|y|\| \le \|(I + \theta M^*)|y|\|$, that implies $\|(I + \theta P^T \overline{A} P)y\| \le \|I + \theta M^*\| \, \||y|\| = \|I + \theta M^*\| \|y\|$. Thus, we get $\|(I + \theta P^T \overline{A} P)\| = \max_{\|y\|=1} \|(I + \theta P^T \overline{A} P)y\| \le \|I + \theta M^*\|$ and $(\|(I + \theta P^T \overline{A} P)\| - 1)/\theta \le (\|I + \theta M^*\| - 1)/\theta$. By taking $\lim_{\theta \downarrow 0}$ we obtain the inequality $\mu_{\| \ \|}(P^T \overline{A} P) \le \mu_{\| \ \|}(M^*)$.

Similarly, the componentwise matrix inequality $A \le \overline{A}$ leads to $\mu_{\| \ \|}(P^T A P) \le \mu_{\| \ \|}(P^T \overline{A} P)$. Finally, we have $\mu_{\| \ \|}(\mathrm{A}) = \mu_{\| \ \|}(P^T A P) \le \mu_{\| \ \|}(P^T \overline{A} P) \le \mu_{\| \ \|}(M^*)$.

*Proof of Part II.* From (3.4), according to (R2) we get $\mu_{\| \ \|}(M_k) \le \mu_{\| \ \|}(M^*)$, $k = 1, \ldots, K$, which together with (3.2) lead to (3.1).

(b) The sufficiency is proved by (a). The necessity is ensured by the equality $\mu_{\| \ \|}(M^*) = \mu_{\| \ \|}(M^{**})$ and the inequality $\mu_{\| \ \|}(M^{**}) \le r$ (resulting from $M^{**} \in \mathcal{V}$). □

**Proposition 3.2.** *(a) If the following hypotheses (H1), (H2) are satisfied, then inequality* (3.2) *is a sufficient condition for inequalities* (3.1).

(H1) *The vector norm $\| \ \|$ is an absolute vector norm and $\mu_{\| \ \|}$ denotes the corresponding matrix measure.*

(H2) *Matrix $M^* \in \mathbb{R}^{n \times n}$ satisfies the componentwise inequalities*

$$\overline{M_k} \le M^*, \quad k = 1, \ldots, K. \tag{3.8}$$

*(b) If the above hypotheses (H1), (H2) are satisfied and there exists $M^{**} \in \mathcal{V}$ such that $\mu_{\| \ \|}(M^*) = \mu_{\| \ \|}(M^{**})$, then inequality* (3.2) *is a necessary and sufficient condition for inequalities* (3.1).

*Proof.* (a) We use the same technique as in the proof of Proposition 3.1 to show that, for a given $A \in \mathbb{R}^{n \times n}$ satisfying the componentwise inequality $A \le \overline{A} \le M^*$, the monotonicity of $\| \ \|$ implies $\mu_{\| \ \|}(A) \le \mu_{\| \ \|}(\overline{A}) \le \mu_{\| \ \|}(M^*)$.

(b) The proof of necessity is identical to Theorem 2.1. □

*Remark 3.3.* Proposition 3.2 allows one to show that the characterization of the componentwise exponential asymptotic stability (abbreviated CWEAS) of interval systems given by our previous work [12] represents a particular case of Theorem 2.1 applied for an absolute vector norm.

Indeed, assume that parametric uncertain system (1.1) and (1.2) is an interval system; that is, the convex hull of matrices has the particular form $\mathcal{M}_I = \{M \in \mathbb{R}^{n \times n} \mid M^- \le M \le M^+\}$. This system is said to be CWEAS if there exist $d_i > 0$, $i = 1, \ldots, n$, and $r < 0$ such that for all $t, t_0 \in \mathbb{R}_+$, $t \ge t_0$: $-d_i \le x_i(t_0) \le d_i \Rightarrow -d_i e^{r(t-t_0)} \le x_i(t) \le d_i e^{r(t-t_0)}$, $i = 1, \ldots, n$, where $x_i(t_0)$, $x_i(t)$ denote the components of the initial condition $x(t_0)$ and of the corresponding solution $x(t)$, respectively. According to [12] the interval system is CWEAS if and only if $\widetilde{M} d \le r d$, where $d = [d_1 \cdots d_n]^T \in \mathbb{R}^n$ and the matrix $\widetilde{M} = (\widetilde{m}_{ij})$ is built from the entries of the matrices $M^- = (m_{ij}^-)$ and $M^+ = (m_{ij}^+)$ by $\widetilde{m}_{ii} = m_{ii}^+, i = 1, \ldots, n$, and $\widetilde{m}_{ij} = \max\{|m_{ij}^-|, |m_{ij}^+|\}$, $i \neq j$, $i, j = 1, \ldots, n$. On the other hand, Theorem 2.1 characterizes CWEAS if applied for the vector norm $\|x\|_\infty^D = \|D\,x\|_\infty = \max_{i=1,\ldots,n}\{x_i/d_i\}$, with $D = \mathrm{diag}\{1/d_1,\ldots,1/d_n\}$. At the same time, we can use Proposition 3.2(b), since (3.8) is satisfied with $M^* = \widetilde{M}$, $\|\ \|_\infty^D$ is an absolute vector norm, and there exist $M^{**} = (m_{ij}^{**})$ belonging to the set of vertices of $\mathcal{M}_I$ such that $m_{ii}^{**} = \widetilde{m}_{ii}$, $i = 1,\ldots,n$, $|m_{ij}^{**}| = \widetilde{m}_{ij}$, $i \neq j$, $i,j = 1,\ldots,n$, which implies $\mu_{\|\ \|_\infty^D}(\widetilde{M}) = \max_{i=1,\ldots,n}\{\widetilde{m}_{ii} + \sum_{j=1,j\neq i}^{n}\widetilde{m}_{ij}(d_j/d_i)\} = \max_{i=1,\ldots,n}\{m_{ii}^{**} + \sum_{j=1,j\neq i}^{n}|m_{ij}^{**}|(d_j/d_i)\} = \mu_{\|\ \|_\infty^D}(M^{**})$. Thus $\mu_{\|\ \|_\infty^D}(\widetilde{M}) \leq r$ is a necessary and sufficient condition for the CWEAS of the interval system. Finally we notice that $\mu_{\|\ \|_\infty^D}(\widetilde{M}) \leq r$ is equivalent to $\widetilde{m}_{ii}(d_i/d_i) + \sum_{j=1,j\neq i}^{n}\widetilde{m}_{ij}(d_j/d_i) \leq r$, $i = 1,\ldots,n$, showing that the CWEAS characterization $\widetilde{M}d \leq rd$ derived in [12] for interval systems is incorporated into the current approach to parametric uncertain systems.

*Remark 3.4.* Propositions 3.1 and 3.2 can be stated in a more general form, by using, instead of a single test matrix $M^*$, several test matrices $M_1^*, M_2^*, \ldots, M_L^*$, with $L$ being significantly smaller than $K$. Each $M_\ell^*$, $\ell = 1,\ldots,L$, will have to satisfy inequality (3.4) in Proposition 3.1 or inequality (3.8) in Proposition 3.2, for some vertex matrices in $\mathcal{V}_\ell \subseteq \mathcal{V}$, such that $\bigcup_{\ell=1}^{L}\mathcal{V}_\ell = \mathcal{V}$.

## 4. Illustrative Examples

This section illustrates the applicability of our results to three examples. Examples 4.1 and 4.2 refer to case studies presented by literature of control engineering, in [18, 19], respectively. Example 4.3 aims to develop a relevant intuitive support for invariant sets with respect to the dynamics of a mechanical system with two uncertain parameters.

*Example 4.1.* Let us consider the set of matrices [18]:

$$\mathcal{V} = \{M_1, M_2, M_3\} \subset \mathbb{R}^{3\times 3}, \qquad M_1 = \begin{bmatrix} -3.461 & 0.951 & -0.410 \\ -0.480 & -2.725 & -0.17225 \\ 2.903 & -2.504 & -1.014 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} -3.690 & 0.136 & -1.144 \\ -0.648 & -2.437 & -0.273 \\ 2.314 & -0.282 & -0.0734 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} -4.800 & -4.574 & -0.324 \\ -0.386 & -6.355 & -0.189 \\ 3.866 & 3.611 & -2.046 \end{bmatrix}. \tag{4.1}$$

Paper [18] shows that matrices $M_1$, $M_2$, and $M_3$ have the following common quadratic Lyapunov function:

$$V_{\mathrm{CQLF}}(x) = x^T Q x, \quad Q = \begin{bmatrix} 12.6 & -5.70 & 5.70 \\ -5.70 & 7.50 & -2.40 \\ 5.70 & -2.40 & 3.12 \end{bmatrix}, \quad Q \succ 0, \tag{4.2}$$

since $QM_k + M_k^T Q \prec 0$, $k = \overline{1,3}$.

We define the convex hull of matrices $\mathcal{M}_h$ having the set of vertices $\mathcal{V}$ (4.1), that is,

$$\mathcal{M}_h = \left\{ M \in \mathbb{R}^{3\times 3} \mid M = \sum_{k=1}^{3} \gamma_k M_k,\ \gamma_k \geq 0, \sum_{k=1}^{3} \gamma_k = 1 \right\}, \quad M_k \in \mathcal{V}, \tag{4.3}$$

and the positive cone of matrices $\mathcal{M}_c$ having the set of vertices $\mathcal{V}$ (4.1), that is,

$$\mathcal{M}_c = \left\{ M \in \mathbb{R}^{3\times 3} \mid M = \sum_{k=1}^{3} \gamma_k M_k,\ \gamma_k \geq 0, M \neq 0 \right\}, \quad M_k \in \mathcal{V}. \tag{4.4}$$

In $\mathbb{R}^3$ we define the vector norm

$$\|x\|_2^H = \|Hx\|_2, \quad \text{with } H = \begin{bmatrix} 3.2079 & -0.9173 & 1.2116 \\ -0.9173 & 2.5580 & -0.3393 \\ 1.2116 & -0.3393 & 1.2397 \end{bmatrix}, \tag{4.5}$$

where $H^2 = H^T H = Q$, in accordance with Remark 2.4, and consider the corresponding matrix measure $\mu_{\| \|_2^H}(M) = \mu_{\| \|_2}(HMH^{-1})$. For the vertex-matrices in $\mathcal{V}$ (4.1) simple computations give $\mu_{\| \|_2^H}(M_1) = -1.5915$, $\mu_{\| \|_2^H}(M_2) = -0.5271$, and $\mu_{\| \|_2^H}(M_3) = -0.1999$.

(i) Theorem 2.1 applied to the qualitative analysis of uncertain system (1.1) and (4.3) reveals the following properties.

(a) The function

$$V : \mathbb{R}^3 \longrightarrow \mathbb{R}_+, \qquad V(x) = \|x\|_2^H \tag{4.6}$$

is a common strong Lyapunov function for the uncertain system (1.1) and (4.3) with the decreasing rate $r = -0.1999$.

(b) Any exponentially contractive set $X^{\varepsilon}_{-0.1999}(t; t_0)$ of the form

$$X^{\varepsilon}_{-0.1999}(t; t_0) = \left\{ x \in \mathbb{R}^3 \mid \|x\|_2^H \leq \varepsilon e^{-0.1999\ (t-t_0)} \right\}, \quad t, t_0 \in \mathbb{R}_+, t \geq t_0, \varepsilon > 0 \tag{4.7}$$

is invariant with respect to the uncertain system (1.1) and (4.3).

(ii) Theorem 2.5 applied to the qualitative analysis of uncertain system (1.1) and (4.4) reveals the following properties.

(a) The function $V$ defined by (4.5) is a common weak Lyapunov function for the uncertain system (1.1) and (4.4).

(b) Any constant set of the form

$$X^{\varepsilon}_{0}(t; t_0) = \left\{ x \in \mathbb{R}^3 \mid \|x\|_2^H \leq \varepsilon \right\}, \quad t, t_0 \in \mathbb{R}_+, t \geq t_0, \varepsilon > 0 \tag{4.8}$$

is invariant with respect to the uncertain system (1.1) and (4.4).

*Example 4.2.* Let us consider the interval matrix [19]:

$$\mathcal{M}_I = \left\{ M \in \mathbb{R}^{2\times 2} \mid M = M_0 + \triangle, |\triangle| \le R \right\}, \quad M_0 = \begin{bmatrix} -3.8 & 1.6 \\ 0.6 & -4.2 \end{bmatrix}, \quad R = 0.17 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{4.9}$$

Obviously, the set $\mathcal{M}_I$ can be regarded as a convex hull with $K = 2^4$ vertices

$$\mathcal{V} = \{ M_k = M_0 + \triangle_k, k = 1, \ldots, 16 \}, \quad \text{where } \triangle_k = 0.17 \begin{bmatrix} \pm 1 & \pm 1 \\ \pm 1 & \pm 1 \end{bmatrix}. \tag{4.10}$$

The comparison matrices $\overline{M_k}$, $k = 1, \ldots, 16$, of the vertices of $\mathcal{V}$ (4.10) are built in accordance with (3.3) yielding $\overline{M_k} = M_k$, $k = 1, \ldots, 16$ (since all $M_k$ are essentially nonnegative matrices). The dominant vertex $M^* = M_0 + R = \left[\begin{smallmatrix} -3.63 & 1.77 \\ 0.77 & -4.03 \end{smallmatrix}\right] \in \mathcal{V}$ satisfies inequalities (3.8), meaning that $M^*$ also satisfies inequalities (3.4) with all the permutation matrices equal to the unity matrix, $P_k = I$. Therefore we can apply both Propositions 3.1 and 3.2.

First, we apply Proposition 3.1 for the usual Hölder norms $\| \ \|_p$, with $p \in \{1, 2, \infty\}$, which are symmetric gauge functions. We calculate the matrix measures $r_p = \mu_{\| \ \|_p}(M^*)$ for $p \in \{1, 2, \infty\}$ and obtain the values $r_1 = -2.26$, $r_2 = -2.5443$, $r_\infty = -1.86$. Consequently all the vertex matrices in $\mathcal{V}$ (4.10) satisfy the inequalities

$$\mu_{\| \ \|_p}(M_k) \le r_p, \quad k = 1, \ldots, 16, \ p \in \{1, 2, \infty\}. \tag{4.11}$$

Thus, for the qualitative analysis of uncertain system (1.1) and (4.9) we can employ Theorem 2.1 that reveals the following properties.

(a) The function

$$V : \mathbb{R}^2 \longrightarrow \mathbb{R}_+, \qquad V(x) = \|x\|_p, \quad p \in \{1, 2, \infty\} \tag{4.12}$$

is a common strong Lyapunov function for the uncertain system (1.1) and (4.9) with the decreasing rate $r_p$, $p \in \{1, 2, \infty\}$.

(b) Any exponentially contractive set of the form

$$X^{\varepsilon}_{r_p}(t; t_0) = \left\{ x \in \mathbb{R}^2 \mid \|x\|_p \le \varepsilon e^{r_p(t - t_0)} \right\}, \quad t, t_0 \in \mathbb{R}_+, t \ge t_0, \varepsilon > 0, p \in \{1, 2, \infty\} \tag{4.13}$$

is invariant with respect to the uncertain system (1.1) and (4.9).

Next, we show that Proposition 3.2 allows refining the properties discussed above of the uncertain system (1.1) and (4.9). The refinement will consist in finding common strong Lyapunov functions and exponentially contractive sets with faster decreasing rates than presented above for $p \in \{1, 2, \infty\}$.

The dominant vertex $M^*$ is an essentially positive matrix (all off-diagonal entries are positive) and we can use the Perron Theorem, in accordance with [20]. Denote by $\lambda_{\max}(M^*) = -2.6456$ the Perron eigenvalue. From the left and right Perron eigenvectors of $M^*$ we can construct the diagonal matrices $D_1 = \mathrm{diag}\{0.7882, 1\}$, and $D_2 = \mathrm{diag}\{0.6596, 1\}$, $D_\infty = \mathrm{diag}\{0.5562, 1\}$, such that, for the vector norms defined in $\mathbb{R}^2$ by $\|x\|_p^{D_p} = \|D_p x\|_p$ we have $\mu_{\|\ \|_p^{D_p}}(M^*) = \lambda_{\max}(M^*)$, $p \in \{1, 2, \infty\}$. These vector norms are absolute without being permutation invariant; hence they are not symmetric gauge functions. Nonetheless, for these norms we may apply Proposition 3.2 with $r = \lambda_{\max}(M^*) = -2.6456$ proving that the vertex matrices in $\mathcal{V}$ (4.10) satisfy the inequalities

$$\mu_{\|\ \|_p^{D_p}}(M_k) \le r, \quad k = 1, \ldots, 16, \ p \in \{1, 2, \infty\}. \tag{4.14}$$

Thus, for the qualitative analysis of uncertain system (1.1) and (4.9) we can employ Theorem 2.1 that reveals the following properties.

(a) The function

$$V : \mathbb{R}^2 \longrightarrow \mathbb{R}_+, \qquad V(x) = \left\|D_p x\right\|_p, \quad p \in \{1, 2, \infty\} \tag{4.15}$$

is a common strong Lyapunov function for the uncertain system (1.1) and (4.9) with the decreasing rate $r = -2.6456$.

(b) Any exponentially contractive set of the form

$$X^{\varepsilon}_{-2.6456}(t; t_0) = \left\{x \in \mathbb{R}^2 \mid \left\|D_p x\right\|_p \le \varepsilon e^{-2.6456(t-t_0)}\right\}, \quad t, t_0 \in \mathbb{R}_+, t \ge t_0, \varepsilon > 0, p \in \{1, 2, \infty\} \tag{4.16}$$

is invariant with respect to the uncertain system (1.1) and (4.9).

Note that all the conclusions regarding the qualitative analysis of the uncertain system (1.1) and (4.9) remain valid in the case when we consider the modified interval matrix

$$\mathcal{M}'_I = \left\{M \in \mathbb{R}^{2\times 2} \mid M^- \le M \le M^+\right\}, \quad M^+ = \begin{bmatrix} -3.63 & 1.77 \\ 0.77 & -4.03 \end{bmatrix}, \quad M^- = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a \le -3.63,\ |b| \le 1.77,\ |c| \le 0.77,\ d \le -4.03, \tag{4.17}$$

which has the same dominant vertex $M^*$ as the original interval matrix $\mathcal{M}_I$ (4.9).

*Example 4.3.* Let us consider the translation of the mechanical system in Figure 1. A coupling device CD (with negligible mass) connects, in parallel, the following components: a cart (with mass $m$) in series with a damper (with viscous friction coefficient $\gamma_1$) and a spring (with spring constant $k$) in series with a damper (with viscous friction coefficient $\gamma_2$).

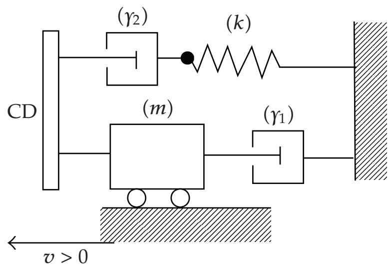



**Figure 1:** The mechanical system used in Example 4.3.

The system dynamics in form (1.1) is described by

$$\begin{bmatrix} \dot{F}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k}{\gamma_2} & k \\ -\dfrac{1}{m} & -\dfrac{\gamma_1}{m} \end{bmatrix} \begin{bmatrix} F(t) \\ v(t) \end{bmatrix}, \quad (4.18)$$

where the state variables are the spring force $F(t)$ and the cart velocity $v(t)$. We consider $F > 0$ when the spring is elongated and $F < 0$ when it is compressed as well as $v > 0$ when the cart moves to the left and $v < 0$ when it moves to the right.

The viscous friction coefficients have unique values, namely, $\gamma_1 = 2\,\mathrm{Ns/mm}$, $\gamma_2 = 0.5\,\mathrm{Ns/mm}$, whereas the cart mass and the spring constant have uncertain values belonging to the intervals $1.5\,\mathrm{kg} \le m \le 2\,\mathrm{kg}$, $3\,\mathrm{N/mm} \le k \le 4\,\mathrm{N/mm}$. Therefore we introduce the notation $M(m;k) = \left[\begin{smallmatrix} -2k & k \\ -1/m & -2/m \end{smallmatrix}\right]$ that allows describing the set of system matrices $\mathcal{M} = \{M(m;k) \mid 1.5 \le m \le 2, 3 \le k \le 4\}$ as the convex hull of form (1.2) defined by the vertices

$$\begin{gathered} \mathcal{V} = \{M_1, M_2, M_3, M_4\}, \quad \text{where } M_1 = M(1.5;3), \\ M_2 = M(1.5;4), \quad M_3 = M(2;3), \quad M_4 = M(2;4). \end{gathered} \quad (4.19)$$

For the initial conditions $F(t_0) = F_0$, $v(t_0) = v_0$, we analyze the free response of the system. We want to see if there exists $r < 0$ such that

$$\forall F_0, |F_0| \le 3\,\mathrm{N}, \forall v_0, \quad |v_0| \le 4\,\mathrm{mm/s} \Longrightarrow |F(t)| \le 3e^{r(t-t_0)}, \quad |v(t)| \le 4e^{r(t-t_0)}, \quad \forall t \ge t_0. \quad (4.20)$$

The problem can be approached in terms of Theorem 2.1, by considering in $\mathbb{R}^2$ the vector norm $\|x\|_\infty^D = \|Dx\|_\infty$, with $D = \mathrm{diag}\{1/3, 1/4\}$, and the exponentially contractive set

$$X_r^1(t;t_0) = \left\{ x \in \mathbb{R}^2 \mid \|x\|_\infty^D \le e^{r(t-t_0)} \right\}, \quad t, t_0 \in \mathbb{R}_+, \ t \ge t_0. \quad (4.21)$$

Obviously, condition (4.20) is equivalent with the invariance of the set (4.21) with respect to the uncertain system (4.18). By calculating the matrix measures $\mu_{\|\ \|_\infty^D}(M_k) = \mu_{\|\ \|_\infty}(DM_kD^{-1})$ for the vertices $M_k$, $k = 1, \ldots, 4$, in $\mathcal{V}$ (4.19), we show that condition (2.1) in Theorem 2.1 is satisfied for $r = -0.625\ \mathrm{s}^{-1}$. Hence, the set (4.21) is invariant with respect to

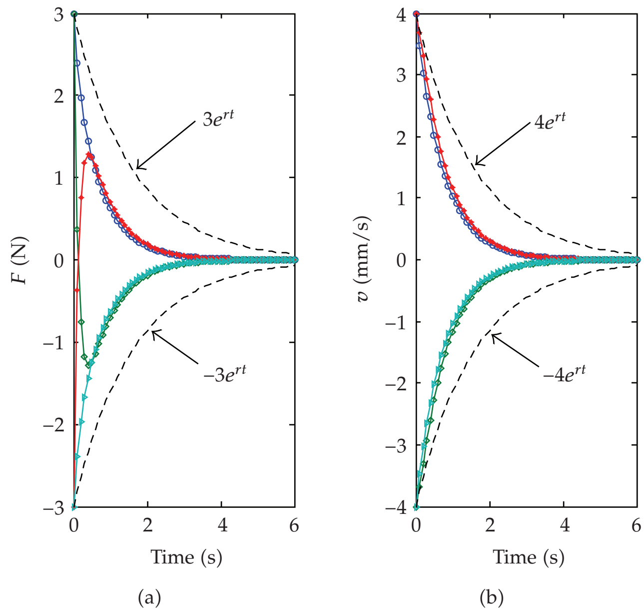



**Figure 2:** Time-evolution of the state-space trajectories corresponding to four initial conditions.

the uncertain system, and condition (4.20) is fulfilled regardless of the concrete values of $m$, $k$, $1.5\,\text{kg} \le m \le 2\,\text{kg}$, $3\,\text{N/mm} \le k \le 4\,\text{N/mm}$.

The graphical plots in Figures 2 and 3 present the simulation results for a system belonging to the considered family, that corresponds to the concrete values $m = 1.75$ kg, $k = 3.5$ N/mm. We take four distinct initial conditions given by the combinations of $F_0 = \pm 3$, $v_0 = \pm 4$ at $t_0 = 0$. Figure 2 exhibits the evolution of $F(t)$ and $v(t)$, as 2D plots (function values versus time). The dotted lines mark the bounds $\pm 3e^{rt}$, $\pm 4e^{rt}$ as used in condition (4.20), with $t_0 = 0$. Figure 3 offers a 3D representation of the exponentially contractive set $X_r^1(t;0)$ defined by (4.21), with $t_0 = 0$, as well as a state-space portrait, presenting the same four trajectories as in Figure 2.

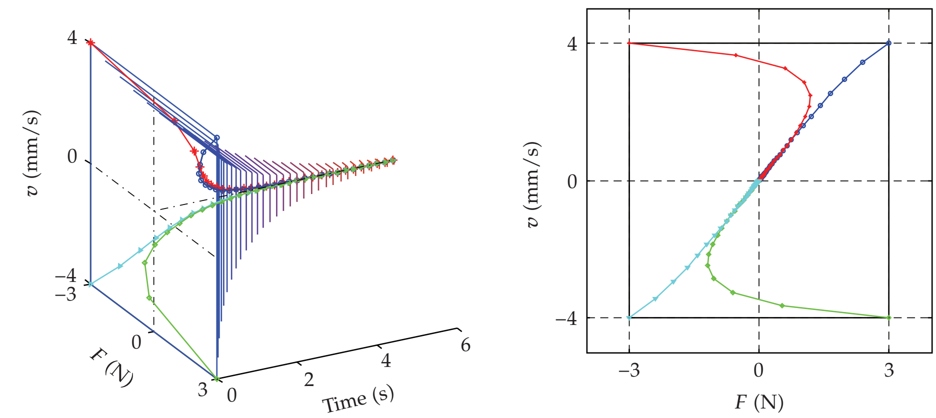
As a general remark, it is worth mentioning that the problem considered above is far from triviality. If, instead of condition (4.20), we use the more general form

$$\forall F_0, |F_0| \le F^*, \forall v_0, \quad |v_0| \le v^* \Longrightarrow |F(t)| \le F^* e^{r^*(t-t_0)}, \quad |v(t)| \le v^* e^{r^*(t-t_0)}, \quad \forall t \ge t_0, \tag{4.22}$$

then Theorem 2.1 shows that (4.22) can be satisfied if and only if $\gamma_2 = 0.5 < (F^*/v^*) < \gamma_1 = 2$; if this condition is fulfilled, then (4.22) is satisfied for

$$r^* = \max\left[3\left(\frac{v^*}{F^*} - 2\right)k_1,\ 0.5\left(\frac{F^*}{v^*} - 2\right)\right]. \tag{4.23}$$

The request $\gamma_2 < \gamma_1$ has a simple motivation even from the operation of the system. Assume that $\gamma_1 < \gamma_2$ and $F_0 = F^*$, $v_0 > 0$. Immediately after $t_0 > 0$, the elongation of the spring will increase (since the damper with $\gamma_2$ moves slower than the damper with $\gamma_1$). Thus, at the first moments after $t_0 > 0$, we will have $F(t) > F^*$ and condition (4.22) is violated.




(a) (b)

**Figure 3:** (a) 3D representation of the exponentially contractive set $X_r^1(t;0)$. (b) State-space portrait. The same four trajectories as in Figure 2.

## 5. Conclusions

Many engineering processes can be modeled by linear differential systems with uncertain parameters. Our paper considers two important classes of such models, namely, those defined by convex hulls of matrices and by positive cones of matrices. We provide new results for the qualitative analysis which are able to characterize, by necessary and sufficient conditions, the existence of common Lyapunov functions and of invariant sets. These conditions are formulated in terms of matrix measures that are evaluated for the vertices of the convex hull or positive cone describing the system uncertainties. Although matrix measures are stronger instruments than the eigenvalue location, their usage as necessary and sufficient conditions is explained by the fact that set invariance is a stronger property than stability. We also discuss some particular cases when the matrix-measure-based test can be applied to a single matrix, instead of all vertices. The usage of the theoretical concepts and results is illustrated by three examples that outline both computational and physical aspects.

## Acknowledgment

The authors are grateful for the support of CNMP Grant 12100/1.10.2008 - SICONA.

## References


[1] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath and Company, Boston, Mass, USA, 1965.

[2] T. Ström, "Minimization of norms and logarithmic norms by diagonal similarities," *Computing*, vol. 10, no. 1-2, pp. 1–7, 1972.

[3] C. Desoer and H. Haneda, "The measure of a matrix as a tool to analyze computer algorithms for circuit analysis," *IEEE Transactions on Circuits Theory*, vol. 19, no. 5, pp. 480–486, 1972.

[4] H. Kiendl, J. Adamy, and P. Stelzner, "Vector norms as Lyapunov functions for linear systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 839–842, 1992.

[5] Y. Fang and T. G. Kincaid, "Stability analysis of dynamical neural networks," *IEEE Transactions on Neural Networks*, vol. 7, no. 4, pp. 996–1006, 1996.

[6] L. Gruyitch, J.-P. Richard, P. Borne, and J.-C. Gentina, *Stability Domains*, vol. 1 of *Nonlinear Systems in Aviation, Aerospace, Aeronautics, and Astronautics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.
[7] Z. Zahreddine, "Matrix measure and application to stability of matrices and interval dynamical systems," *International Journal of Mathematics and Mathematical Sciences*, no. 2, pp. 75–85, 2003.
[8] O. Pastravanu and M. Voicu, "On the componentwise stability of linear systems," *International Journal of Robust and Nonlinear Control*, vol. 15, no. 1, pp. 15–23, 2005.
[9] G. Söderlind, "The logarithmic norm. History and modern theory," *BIT Numerical Mathematics*, vol. 46, no. 3, pp. 631–652, 2006.
[10] J. Chen, "Sufficient conditions on stability of interval matrices: connections and new results," *IEEE Transactions on Automatic Control*, vol. 37, no. 4, pp. 541–544, 1992.
[11] M. E. Sezer and D. D. Šiljak, "On stability of interval matrices," *IEEE Transactions on Automatic Control*, vol. 39, no. 2, pp. 368–371, 1994.
[12] O. Pastravanu and M. Voicu, "Necessary and sufficient conditions for componentwise stability of interval matrix systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 6, pp. 1016–1021, 2004.
[13] T. Alamo, R. Tempo, D. R. Ramírez, and E. F. Camacho, "A new vertex result for robustness problems with interval matrix uncertainty," *Systems & Control Letters*, vol. 57, no. 6, pp. 474–481, 2008.
[14] A. Michel, K. Wang, and B. Hu, *Qualitative Theory of Dynamical Systems. The Role of Stability Preserving Mappings*, vol. 239 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2nd edition, 2001.
[15] X. Liao, L. Q. Wang, and P. Yu, *Stability of Dynamical Systems*, Elsevier, Amsterdam, The Netherlands, 2007.
[16] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*, Systems & Control: Foundations & Applications, Birkhäuser, Boston, Mass, USA, 2008.
[17] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1990.
[18] Y. Mori, T. Mori, and Y. Kuroe, "On a class of linear constant systems which have a common quadratic Lyapunov function," in *Proceedings of the 37th IEEE Conference on Decision & Control*, vol. 3, pp. 2808–2809, Tampa, Fla, USA, 1998.
[19] L. Kolev and S. Petrakieva, "Assessing the stability of linear time-invariant continuous interval dynamic systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 393–397, 2005.
[20] M. H. Matcovschi and O. Pastravanu, "Perron-Frobenius theorem and invariant sets in linear systems dynamics," in *Proceedings of the 15th IEEE Mediterranean Conference on Control and Automation (MED '07)*, Athens, Greece, 2007.