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Research Article

Optimal Inventory Policy Involving Ordering Cost Reduction, Back-Order Discounts, and Variable Lead Time Demand by Minimax Criterion

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This paper allows the backorder rate as a control variable to widen applications of a continuous review inventory model. Moreover, we also consider the backorder rate that is proposed by combining Ouyang and Chuang (2001) (or Lee (2005)) with Pan and Hsiao (2001) to present a new form. Thus, the backorder rate is dependent on the amount of shortages and backorder price discounts. Besides, we also treat the ordering cost as a decision variable. Hence, we develop an algorithmic procedure to find the optimal inventory policy by minimax criterion. Finally, a numerical example is also given to illustrate the results.

1. Introduction

In most of literatures dealing with inventory problems, either in deterministic or probabilistic model, lead time is viewed as a prescribed constant or a stochastic variable, and is not subject to control (see, e.g., Naddor [1], Liberatore [2], Magson [3], Kim and Park [4], Silver and Peterson [5], Foote et al. [6], Azoury and Brill [7] and Chiu [8]). However, as pointed out by Tersine [9], lead time usually comprises several components, such as setup time, process time, wait time, move time, and queue time. In many practical situations, lead time can be reduced using an added crash cost; in other words, it is controllable. By shortening the lead time, we can lower the safety stock, reduce the stock-out loss and improve the service level to the customer so as to increase the competitive edge in business. Firstly, Liao and Shyu [10] presented a probabilistic model in which the order quantity is predetermined and the lead time is the only decision variable. Secondly, Ben-Daya and

Raouf [11] have also extended the model of Liao and Shyu [10] by considering both the lead time and the order quantity as decision variables and the situation of shortage is neglected. Subsequently, Ouyang et al. [12] considered an inventory model with a mixture of backorders and lost sales to generalized Ben-Daya and Raouf's [11] model, where the backorder rate is fixed.

In this study, we consider to allow the backorder rate as a control variable. Under most market behaviors, we can often observe that many products of famous brands or fashionable commodities may lead to a situation in which customers prefer their demands to be backordered while shortages occur. Certainly, if the quantity of shortages is accumulated to a degree that exceeds the waiting patience of customers, some may refuse the backorder case. However, the supplier can offer a price discount on the stock-out item in order to secure more backorders. In the real market as unsatisfied demands occur, the longer the length of lead time is, the larger the amount of shortages is, the smaller the proportion of customers can wait, and hence the smaller the backorder rate would be. But, the larger the backorder discount is, the larger the backorder rate would be. Thus, the backorder rate is dependent on the amount of shortages and backorder price discounts. Therefore, we also consider the backorder rate that is proposed by combining Ouyang and Chuang [13] (or Lee [14]) with Pan and Hsiao [15] to present a new form. In addition, there are many authors that (Porteus [16–18], Billington [19], Nasri et al. [20], Kim et al. [21], Paknejad et al. [22], Sarker and Coates [23], Ouyang et al. [24, 25], Moon and Choi [26], Chuang et al. [27], Lin and Hou [28], and Chang et al. [29]) have investigated the effects of investing in reducing ordering cost. Hence, we treat the ordering cost as a decision variable in this study.

Because the demand of different customers is not identical in the lead time, we cannot only use a single distribution (such as [13]) to describe the demand of the lead time. It is more reasonable that mixture distribution is applied to describe the lead time demand than single distribution is used. Besides, in many practical situations, the probability distributional information of lead time demand is often quite limited. Since Lee et al. [30] consider that the lead time demand follows a mixture of normal distributions, we relax the assumption about the form of the mixture of distribution functions of the lead time demand. Therefore, we consider that any mixture of distribution functions (d.f.s); say $F_* = pF_1 + (1-p)F_2$, of the lead time demand has only known finite first and second moments (and hence, mean and standard deviations are also known and finite) but we make no assumption on the distribution form of F_*. That is, F_1 and F_2 of F_* belong to the class Ω of all single d.f.s' with finite mean and standard deviation. Our goal is to solve a mixture inventory model by using the minimax criterion. This is, the minimax criterion (such as Wu et al. [31]) for our model is to find the most unfavorable d.f.s F_1 and F_2 in F_* for each decision variable and then to minimize over the decision variables. Finally, one numerical example is also given to illustrate that when $p = 0$ or 1, the model considers only one kind of customers' demand; when $0 < p < 1$, the model considers two kinds of customers' demand for the fixed backorder parameters ε and δ. It implies that the minimum expected total annual costs of two kinds of customers' demand are larger than the minimum expected total annual cost of one kind of customers' demand. Thus, the minimum expected total annual cost increases as the distance between p and 0 (or 1) increases for the fixed backorder parameters ε and δ. Hence, if the true distribution of the lead time demand is a mixture of normal distributions, we use a single distribution (such as [13]) to substitute the true distribution of the lead time demand then the minimum expected total annual cost will be underestimated.

2. Model Formulation

To establish the mathematical model, the notation and assumptions employed throughout the paper are as follows:

A: odering cost per order,

D: average demand per year,

h: inventory holding cost per item per year,

L: length of lead time,

Q: order quantity,

r: reorder point,

X: lead time demand with the mixtures of distribution function,

β: fraction of the demand backordered during the stock-out period, $\beta \in [0, 1]$,

π_0 : gross marginal profit per unit,

π_x: back-order price discount offered by the supplier per unit, $0 \leq \pi_x \leq \pi_0$,

δ, ε: back-order parameters, $0 \leq \delta \leq 1, 0 \leq \varepsilon \leq \infty$,

p: the weight of the component distributions, $0 \leq p \leq 1$,

x^+: maximum value of x and 0, that is, $x^+ = \max\{x, 0\}$,

x^-: maximum value of $-x$ and 0, that is, $x^- = \max\{-x, 0\}$:

$$I_{(0<x<r)} = \begin{cases} 1, & 0 < x < r, \\ 0, & \text{o.w.}, \end{cases} \tag{2.1}$$

$B(r) = E(X - r)^+$: the expected shortage quantity at the end of cycle,

q: the allowable stock-out probability during L,

k: the safety factor which satisfies $P(X > r) = q$,

μ^*: the mean of lead time demand with the mixture of distributions,

σ^*: the standard deviation of lead time demand with the mixture of distributions,

A_0: original ordering cost,

$I(A)$: capital investment required to achieve ordering cost A, $0 < A \leq A_0$,

θ: fractional opportunity cost of capital per unit time,

ξ: percentage decrease in ordering cost A per dollar increase in investment $I(A)$.

The assumptions of the model are exactly the same as those in Ouyang and Chuang [13] who expect the following assumptions: the reorder point r = expected demand during the lead time + safety stock (SS), and SS = k × (standard deviation of lead time demand), that is, $r = \mu_* L + k\sigma_*\sqrt{L}$, where $\mu_* = p\mu_1 + (1-p)\mu_2$, $\sigma_* = (1 + p(1-p)\eta^2)^{1/2}\sigma$, $\mu_1 = \mu_* + (1-p)\eta\sigma/\sqrt{L}$, $\mu_2 = \mu_* - p\eta\sigma/\sqrt{L}$ (it means that $\mu_1 - \mu_2 = \eta\sigma/\sqrt{L}$, $\eta \in R$), and k is the safety factor. Moreover, the mixtures of distribution functions are unimodal for all p if $(\mu_1 - \mu_2)^2 < 27\sigma^2/(8L)$ (or $\eta < \sqrt{27/8}$) (see [32]). Besides, the reorder point must satisfy the following equation which implies a service level constraint $P(X > r) = q$, where q represents the allowable stock-out probability during L.

In addition, we assume that the capital investment, $I(A)$, in reducing ordering cost is a logarithmic function of the ordering cost A. That is,

$$I(A) = \nu \ln\left(\frac{A_0}{A}\right) \quad \text{for } 0 < A \leq A_0, \text{ where } \nu = \xi^{-1}. \tag{2.2}$$

This function is consistent with the Japanese experience as reported in Hall [33], and has been utilized in many articles (see [16, 17, 20–24, 34], etc.).

In this study, we relax the restriction about the form of the mixtures of d.f. of lead time demand, that is, we assume here that the lead time demand X has the mixtures of d.f. $F_* = pF_1 + (1-p)F_2$, where F_1 has finite mean $\mu_1 L$ and standard deviation $\sigma\sqrt{L}$ and F_2 has finite mean $\mu_2 L$ and standard deviation $\sigma\sqrt{L}$, $\mu_1 - \mu_2 = \eta\sigma/\sqrt{L}$, $\eta \in R$. Then the expected shortage at the end of the cycle is defined by $B(r) = E(X-r)^+$. Thus, the expected number of backorders per cycle is $\beta B(r)$ and the expected lost sales per cycle is $(1-\beta)B(r)$. Hence, the expected annual stock-out cost is $(D/Q)[\pi_x\beta + \pi_0(1-\beta)]B(r)$.

The expected net inventory level just before the order arrives is

$$\begin{aligned}
&E\left[(X-r)^- I_{(0<X<r)}\right] - \beta B(r) \\
&\quad = \int_0^r (r-x)\,dF_*(x) - \beta B(r) \\
&\quad = \int_0^r (r-x)\,dF_*(x) - \int_r^\infty (x-r)\,dF_*(x) + \int_r^\infty (x-r)\,dF_*(x) - \beta B(r) \\
&\quad = \int_0^\infty (r-x)\,dF_*(x) + E[X-r]^+ - \beta B(r) \\
&\quad = E\left[(r-x) I_{(0<x<\infty)}\right] + (1-\beta)B(r) \\
&\quad = r - \mu_* L + (1-\beta)B(r),
\end{aligned} \tag{2.3}$$

and the expected net inventory level at the beginning of the cycle is

$$Q + r - \mu_* L + (1-\beta)B(r). \tag{2.4}$$

Therefore, the expected annual holding cost is

$$h\left\{\frac{Q}{2} + r - \mu_* L + (1-\beta)B(r)\right\}. \tag{2.5}$$

Finally, the total expected annual cost (EAC) can be expressed as follows:

$$\begin{aligned}\mathrm{EAC}(Q,\beta,L) &= \text{ordering cost} + \text{holding cost} + \text{stock-out cost}\\ &\quad + \text{lead time crashing cost}\\ &= A\frac{D}{Q} + h\left\{\frac{Q}{2} + r - \mu_* L + (1-\beta)B(r)\right\}\\ &\quad + \frac{D}{Q}\left[\pi_x\beta + \pi_0(1-\beta)\right]B(r) + \frac{D}{Q}R(L).\end{aligned} \tag{2.6}$$

In practical situations, as shortage occurs, the longer the length of lead time is, the larger the amount of shortage is, the smaller the proportion of customers can wait, and hence the smaller the backorder rate would be; in addition, the larger backorder price discount is, hence the larger the backorder rate would be. Therefore, we also consider the backorder rate that is proposed by combining Ouyang and Chuang [13] (or Lee [14]) with Pan and Hsiao [15] at the same time. Thus, we define $\beta = \beta_0\pi_x/\pi_0$, where

$$\beta_0 = \frac{\delta}{1+\varepsilon E(X-r)^+}, \quad 0 \le \delta \le 1,\ 0 \le \varepsilon < \infty. \tag{2.7}$$

Hence, the total expected annual cost (2.6) reduces to

$$\begin{aligned}\mathrm{EAC}(Q,\pi_x,L) &= A\frac{D}{Q} + h\left\{\frac{Q}{2} + r - \mu_* L + \left(1 - \frac{\pi_x}{\pi_0}\frac{\delta}{1+\varepsilon E(X-r)^+}\right)B(r)\right\}\\ &\quad + \frac{D}{Q}\pi_0\left[1 - \frac{\pi_x}{\pi_0}\left(1-\frac{\pi_x}{\pi_0}\right)\frac{\delta}{1+\varepsilon E(X-r)^+}\right]B(r) + \frac{D}{Q}R(L).\end{aligned} \tag{2.8}$$

Besides, we also consider that the ordering cost can be reduced through capital investment and the ordering cost A as a decision variable. Hence, we seek to minimize the sum of capital investment cost of reducing ordering cost A and the inventory costs (as expressed in (2.8)) by optimizing over Q, A, π_x, and L constrained on $0 < A \le A_0$. That is, the objective of our problem is to minimize the following total expected annual cost:

$$\begin{aligned}\min \mathrm{EAC}(Q,A,\pi_x,L) &= \theta I(A) + \mathrm{EAC}(Q,\pi_x,L) = \theta v \ln\left(\frac{A_0}{A}\right) + A\frac{D}{Q}\\ &\quad + h\left\{\frac{Q}{2} + r - \mu_* L + \left(1 - \frac{\pi_x}{\pi_0}\frac{\delta}{1+\varepsilon E(X-r)^+}\right)B(r)\right\}\\ &\quad + \frac{D}{Q}\pi_0\left[1 - \frac{\pi_x}{\pi_0}\left(1-\frac{\pi_x}{\pi_0}\right)\frac{\delta}{1+\varepsilon E(X-r)^+}\right]B(r) + \frac{D}{Q}R(L),\end{aligned} \tag{2.9}$$

subject to $0 < A \le A_0$.

Now, we attempt to use a minimax criterion to solve this problem. If we let Ω be the class of all single c.d.f. (included F_1 and F_2) with finite mean and standard deviation, then the minimax criterion for our problem is to find the most unfavorable c.d.f.s F_1 and F_2 in Ω for each decision variable and then to minimize over the decision variables; that is, our problem is to solve

$$\min_{Q>0,\,0<A\le A_0,\,0\le\pi_x\le\pi_0,\,L>0}\ \max_{F_1,F_2\in\Omega} \mathrm{EAC}(Q,A,\pi_x,L). \tag{2.10}$$

In addition, we also need the following Proposition which was asserted by Gallego and Moon [35] to solve the above problem.

Proposition 2.1. *For any $F \in \Omega$,*

$$E[X-r]^{+} \le \frac{1}{2}\left\{\sqrt{\sigma^2 L + (r-\mu L)^2} - (r-\mu L)\right\}. \tag{2.11}$$

Moreover, the upper bound (2.11) *is tight. In other words, we can always find a distribution in which the above bound is satisfied with equality for every r.*

Using the inequality (2.11) for F_1 and F_2, we obtain

$$\begin{aligned}
B(r) &= E(X-r)^{+} \\
&= \int_r^{\infty}(x-r)\,dF_* \\
&= \int_r^{\infty}(x-r)\,d(pF_1+(1-p)F_2) \\
&= p\int_r^{\infty}(x-r)\,dF_1 + (1-p)\int_r^{\infty}(x-r)\,dF_2 \\
&\le p\cdot\frac{1}{2}\cdot\left\{\sqrt{\sigma^2L+(r-\mu_1L)^2}-(r-\mu_1L)\right\} \\
&\quad + (1-p)\cdot\frac{1}{2}\cdot\left\{\sqrt{\sigma^2L+(r-\mu_2L)^2}-(r-\mu_2L)\right\} \\
&= \frac{1}{2}(\mu_*L-r) + \frac{p}{2}\left[\sqrt{\sigma^2L+(r-\mu_1L)^2}\right] + \frac{(1-p)}{2}\left[\sqrt{\sigma^2L+(r-\mu_2L)^2}\right],
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
r-\mu_1L &= r-\mu_*L-(1-p)\eta\sigma\sqrt{L}, \\
r-\mu_2L &= r-\mu_*L+p\eta\sigma\sqrt{L} \quad \left(\because \mu_1-\mu_2=\frac{\eta\sigma}{\sqrt{L}}\right).
\end{aligned} \tag{2.13}$$

Then, the problem (2.10) is equivalent to minimize

$$\begin{aligned}
\mathrm{EAC}(Q, A, \pi_x, L) = {} & \theta v \ln\left(\frac{A_0}{A}\right) + A\frac{D}{Q} \\
& + h\left\{\frac{Q}{2} + k\sigma_* \sqrt{L} + \left(1 - \frac{\pi_x}{\pi_0}\frac{\delta}{1 + \varepsilon E(X - r)^+}\right)B(r)\right\} \\
& + \frac{D}{Q}\pi_0\left[1 - \frac{\pi_x}{\pi_0}\left(1 - \frac{\pi_x}{\pi_0}\right)\frac{\delta}{1 + \varepsilon E(X - r)^+}\right]B(r) + \frac{D}{Q}R(L),
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
B(r) = E(X - r)^+ = {} & \frac{1}{2}\left[-k\sqrt{1 + p(1 - p)\eta^2}\sigma\sqrt{L}\right] \\
& + \frac{p}{2}\sigma\sqrt{L}\left[\sqrt{1 + \left[k\sqrt{1 + p(1 - p)\eta^2} - (1 - p)\eta\right]^2}\right] \\
& + \frac{(1 - p)}{2}\sigma\sqrt{L}\left[\sqrt{1 + \left[k\sqrt{1 + p(1 - p)\eta^2} + p\eta\right]^2}\right],
\end{aligned} \tag{2.15}$$

subject to $0 < A \le A_0$.

In order to solve this nonlinear programming problem, we first ignore the restriction $0 < A \le A_0$ and take the first partial derivatives of $\mathrm{EAC}(Q, A, \pi_x, L)$ with respect to Q, A, π_x and $L \in (L_i, L_{i-1})$, respectively. We can obtain

$$\begin{aligned}
\frac{\partial \mathrm{EAC}(Q, A, \pi_x, L)}{\partial Q} &= -\frac{AD}{Q^2} + \frac{h}{2} - \frac{D}{Q^2}\pi_0\left(1 - \frac{\theta_2}{1 + \Delta(L)}\right)B(r) - \frac{D}{Q^2}R(L), \\
\frac{\partial \mathrm{EAC}(Q, A, \pi_x, L)}{\partial A} &= -\frac{\theta v}{A} + \frac{D}{Q}, \\
\frac{\partial \mathrm{EAC}(Q, A, \pi_x, L)}{\partial \pi_x} &= -\left(\frac{h}{\pi_0} + \frac{D}{Q}\left[1 - 2\frac{\pi_x}{\pi_0}\right]\right)\frac{\delta}{1 + \Delta(L)} \times B(r),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\frac{\partial \mathrm{EAC}(Q, A, \pi_x, L)}{\partial L} = {} & \frac{hk}{2\sqrt{L}}\left[\sqrt{1 + p(1 - p)\eta^2}\sigma\right] \\
& + \left\{h\left(1 - \frac{\theta_1}{[1 + \Delta(L)]^2}\right) + \frac{D}{Q}\pi_0\left(1 - \frac{\theta_2}{[1 + \Delta(L)]^2}\right)\right\}\frac{B(r)}{2L} - \frac{D}{Q}c_i,
\end{aligned} \tag{2.17}$$

where $\theta_1 = (\pi_x/\pi_0)\delta$, $\theta_2 = (\pi_x/\pi_0)(1 - \pi_x/\pi_0)\delta$, $\Delta(L) = \varepsilon E(X - r)^+ = \varepsilon B(r)$, and $B(r)$ is expressed as (2.15).

Since $B(r)$ is the expected shortage quantity at the end of cycle, we know that $B(r) > 0$ if shortages occur; $B(r) = 0$, otherwise. It is clear that $B(r)$ is positive. By examining the second-order sufficient conditions (SOSCs), it can be easily verified that $\mathrm{EAC}(Q, A, \pi_x, L)$ is not a convex function of (Q, A, π_x, L). However, for fixed Q, A, and π_x, $\mathrm{EAC}(Q, \pi_x, L)$ is concave in $L \in (L_i, L_{i-1})$ because

$$\begin{aligned}
&\frac{\partial^2 \mathrm{EAC}(Q, A, \pi_x, L)}{\partial L^2} \\
&\quad = -\frac{1}{4}\frac{hk}{L^{3/2}}\left[\sqrt{1 + p(1-p)\eta^2}\sigma\right] - \frac{1}{4}\frac{\sigma}{L^{3/2}} \\
&\qquad \times \left\{ h\frac{(1-\theta_1)[1 + 3\Delta(L)] + [3 + \Delta(L)][\Delta(L)]^2}{[1 + \Delta(L)]^3} \right. \\
&\qquad\qquad \left. + \frac{D}{Q}\pi_0 \frac{(1-\theta_2)[1 + 3\Delta(L)] + [3 + \Delta(L)][\Delta(L)]^2}{[1 + \Delta(L)]^3} \right\} \times \frac{B(r)}{\sigma\sqrt{L}}.
\end{aligned} \tag{2.18}$$

Therefore, for fixed Q, A, and π_x, the minimum total EAC will occur at the end points of the interval (L_i, L_{i-1}). On the other hand, for a given value of $L \in (L_i, L_{i-1})$, by setting (2.16) equal to zero, we obtain

$$Q = \left\{ \frac{2D}{h}\left(A + \left[\frac{\pi_x^2}{\pi_0}\beta_0 + \pi_0\left(1 - \frac{\pi_x}{\pi_0}\beta_0\right)\right] B(r) + R(L) \right) \right\}^{1/2}, \tag{2.19}$$

$$A = \frac{\theta v Q}{D}, \tag{2.20}$$

$$\pi_x = \frac{1}{2}\left(\frac{hQ}{D} + \pi_0\right), \tag{2.21}$$

where $\beta_0 = \delta/(1 + \varepsilon E(X-r)^+)$ and $B(r)$ is expressed as (2.15).

Theoretically, for fixed $L \in (L_i, L_{i-1})$, from (2.19)–(2.21), we can get the values of Q, A, and π_x (we denote these values by Q^*, A^*, and π_x^*).

Moreover, it can be shown that the SOSCs are satisfied since the Hessian matrix is positive definite at point (Q^*, A^*, π_x^*) (see the appendix for the proof). Hence, for a fixed $L \in (L_i, L_{i-1})$, the point (Q^*, A^*, π_x^*) is the local optimal solution such that the total expected annual cost has minimum value.

We now consider the constraint $0 < A \le A_0$. From (2.20), we note that A^* is positive. Also, if $A^* < A_0$, then (Q^*, A^*, π_x^*) is an interior optimal solution for given $L \in (L_i, L_{i-1})$. However, if $A^* \ge A_0$, then it is unrealistic to invest in changing the current ordering cost level. For this special case, the optimal ordering cost is the original ordering cost, that is, $A^* = A_0$, and our model reduces to (2.8) (i.e., the model of Lee et al. [36] with any mixture of distribution functions, not just mixture of normal distributions).

Substituting (2.20) and (2.21) into (2.19), we get

$$Q = \frac{\theta v + \sqrt{(\theta v)^2 + 2hD\left[1 - (h\beta_0/2D\pi_0)B(r)\right]\left\{\pi_0\left[1 - (1/4)\beta_0\right]B(r) + R(L)\right\}}}{h\left[1 - (h\beta_0/2D\pi_0)B(r)\right]}, \tag{2.22}$$

where $\beta_0 = \delta/(1 + \varepsilon E(X-r)^+)$ and $B(r)$ is expressed as (2.15).

Theoretically, for fixed A_0, D, h, π_0, σ, η, δ, p, q, ε, θ, v and each L_i ($i = 1, 2, \ldots, n$), the optimal $(Q_i, A_i, \pi_{x_i}, L_i)$ pair given L_i can be obtained by solving (2.22) iteratively until convergence. The convergence of the procedure can be shown. Furthermore, using (2.9), we can obtain the corresponding total expected annual cost $\mathrm{EAC}(Q_i, A_i, \pi_{x_i}, L_i)$. Hence, the minimum total expected annual cost is $\min_{i=0,1,2,\ldots,n} \mathrm{EAC}(Q_i, A_i, \pi_{x_i}, L_i)$. However, in practice, since the p.d.f. f_X of the lead time demand X is unknown, even if the value of q is given, we cannot get the exact value of k. Thus, in order to find the value of k, we need the following proposition.

Proposition 2.2. *Let Y be a random variable which has a p.d.f. $f_Y(y)$ with finite mean μL and standard deviation $\sigma\sqrt{L}(> 0)$, then for any real number $c > \mu L$,*

$$P(X > c) \le \frac{\sigma^2 L}{\sigma^2 L + (c - \mu L)^2}. \tag{2.23}$$

So, by using $F_* = pF_1 + (1-p)F_2$, the recorder point $r = \mu_* L + k\sigma_*\sqrt{L}$, and Proposition 2.2, we get

$$\begin{aligned} P(X > r) &\le p\frac{\sigma^2 L}{\sigma^2 L + (r - \mu_1 L)^2} + (1-p)\frac{\sigma^2 L}{\sigma^2 L + (r - \mu_2 L)^2} \\ &= p\frac{1}{1 + (r - \mu_1 L/\sigma\sqrt{L})^2} + (1-p)\frac{1}{1 + (r - \mu_2 L/\sigma\sqrt{L})^2} \\ &= \frac{p}{1 + \left[k\sqrt{1 + p(1-p)\eta^2} - (1-p)\eta\right]^2} + \frac{1-p}{1 + \left[k\sqrt{1 + p(1-p)\eta^2} + p\eta\right]^2}. \end{aligned} \tag{2.24}$$

Further, it is assumed that the allowable stock-out probability q during lead time is given, that is, $q = P(X > r)$, then from (2.24), we get $0 \le k \le \sqrt{(1/q) - 1} + |\eta|$. It is easy to verify that $\mathrm{EAC}(Q, A, \pi_x, L)$ has a smooth curve for $k \in [0, \sqrt{(1/q) - 1} + |\eta|]$. Hence, we can establish the following algorithm to obtain the suitable k and hence the optimal Q, A, π_x, and L.

Algorithm 2.3.

Step 1. Input the values of A_0, D, h, η, σ, π_0, θ, v, p, q, δ, ε, a_i, b_i, and c_i, $i = 1, 2, \ldots, n$.

Step 2. For a given q, we divide the interval $[0, \sqrt{(1/q) - 1} + |\eta|]$ into m equal subintervals, where m is large enough. And we let $k_0 = 0$, $k_N = \sqrt{(1/q) - 1} + |\eta|$ and $k_j = k_{j-1} + (k_N - k_0)/m$, $j = 1, 2, \ldots, m - 1$.

Step 3. Use the a_i, b_i, c_i, to compute L_i, $i = 1, 2, \ldots, n$.

Step 4. For each L_i, $i = 1, 2, \ldots, n$, compute Q_i by using (2.22) for given $k_j \in \{k_0, k_1, \ldots, k_m\}$, $j = 1, 2, \ldots, m$. Then, compute A_i and π_{x_i} by using (2.20) and (2.21).

Step 5. Compare π_{x_i} and π_0. If $\pi_{x_i} < \pi_0$, then take π_{x_i} into Step 6; if $\pi_{x_i} \ge \pi_0$, then take $\pi_{x_i} = \pi_0$ into Step 6. Compare A_i and A_0. If $A_i < A_0$, then take A_i into Step 6; if $A_i \ge A_0$, then take $A_i = A_0$ into Step 6.

Table 1: Lead time data.

Lead time component i	Normal duration b_i (days)	Minimum duration a_i (days)	Unit crashing cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Step 6. For each pair $(Q_i, A_i, \pi_{x_i}, L_i)$ and $k_j \in \{k_0, k_1, \ldots, k_m\}$, compute the corresponding total expected annual cost $\mathrm{EAC}_{k_j}(Q_i, \pi_{x_i}, L_i)$, $i = 1, 2, \ldots, n$.

Step 7. Find $\min_{k_j \in \{k_0, k_1, \ldots, k_m\}} \mathrm{EAC}_{k_j}(Q_i, A_i, \pi_{x_i}, L_i)$. If $\mathrm{EAC}_{k_S^*}(Q_i, A_i, \pi_{x_i}, L_i) = \min_{k_j \in \{k_0, k_1, \ldots, k_m\}} \mathrm{EAC}_{k_j}(Q_i, A_i, \pi_{x_i} L_i)$, then find $\min_{i=0,1,2,\ldots,n} \mathrm{EAC}_{k_S^*}(Q_i, A_i, \pi_{x_i} L_i)$. If $\mathrm{EAC}(Q^*, A^*, \pi_x^*, L^*) = \min_{i=0,1,\ldots,n} \mathrm{EAC}_{k_S^*}(Q_i, A_i, \pi_{x_i}, L_i)$, then (Q^*, A^*, π_x^*, L^*) is the optimal solution; the value of $k_{s(i)}$ such that $\mathrm{EAC}(Q^*, A^*, \pi_x^*, L^*)$ exists is the optimal safety factor, and we denote it by k^*.

Step 8. Stop.

3. A Numerical Example

In order to illustrate the above solution procedure, let us consider an inventory system with the following data: $D = 600$ units/year, $A_0 = \$200$ per order, $h = \$20$, $\pi_0 = \$150$, $\mu_* = 11$ units/week, $\sigma = 7$ units/week, and the lead time has three components with data shown in Table 1. Besides, for ordering cost reduction, we take $\theta = 0.1$ per dollar per year and $v = 5800$.

We assume here that the lead time demand follows a mixture of distribution functions and want to solve the case when $p = 0, 0.2, 0.4, 0.6, 0.8, 1$, $q = 0.2$, $\delta = 0, 0.5, 1$, $\eta = 0.7$, and $\varepsilon = 0, 0.5, 1, 10, 20, 40, 80, 100, \infty$.

If we knew the form of the c.d.f.s F_1 and F_2, we could solve the problem optimally for that particular distribution. For example, if F_* is c.d.f. of mixture of normal distributions, then the total expected annual cost is

$$\begin{aligned}
&\mathrm{EAC}_n(Q, A, \pi_x, L) \\
&= \theta v \ln\left(\frac{A_0}{A}\right) + A\frac{D}{Q} + h\left(\frac{Q}{2} + \sigma\sqrt{L}\left\{p\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-p)\eta\right)\right]\right.\right. \\
&\qquad\qquad \left. + (1-p)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - p\eta\right)\right]\right\} \\
&\qquad\qquad \left. + (1-\beta)\sigma\sqrt{L}\Psi(r_1, r_2, p)\right) \\
&\quad + \frac{D}{Q}\left[\pi_x\beta + \pi_0(1-\beta)\right]\sigma\sqrt{L}\Psi(r_1, r_2, p) + \frac{D}{Q}R(L).
\end{aligned} \tag{3.1}$$

We can obtain the optimal $(Q_n, A_n, \pi_{x_n}, L_n)$ by the standard procedure and incur an expected cost $\mathrm{EAC}_n(Q_n, A_n, \pi_{x_n}, L_n)$ (see Lee et al. [30]). For fixed δ, if we use (Q^*, A^*, π_x^*, L^*)

instead of the optimal $(Q_n, A_n, \pi_{x_n}, L_n)$ for mixture of normal distributions, then we can get an expected cost $\mathrm{EAC}_n(Q^*, A^*, \pi_x^*, L^*)$. Hence, as the c.d.f. F_* is mixture cumulative of normal distributions, the added cost by using the minimax mixture of distributions free procedure instead of the standard procedure is $\mathrm{EAC}_n(Q^*, A^*, \pi_x^*, L^*) - \mathrm{EAC}_n(Q_n, A_n, \pi_{x_n}, L_n)$. This is the largest amount that we would be willing to pay for the knowledge of d.f. F_*. This quantity can be regarded as the expected value of additional information (EVPI). The results of the solution procedure are solved by using the subroutine ZREAL of IMSL from the computer software Compaq Visual Fortran V6.0 (Inclusive of IMSL) [37] and summarized in Table 2. From Table 2, we note that (i) the order quantity Q^*, the ordering cost A^*, the backorder price discount π_x^*, and the minimum total expected annual costs $\mathrm{EAC}(Q^*, A^*, \pi_x^*, L^*)$ increase as ε increases (i.e., the back-order rate β decreases) for $\delta = 0.5, 1.0$ and the fixed p; (ii) the minimum total expected annual cost $\mathrm{EAC}(Q^*, A^*, \pi_x^*, L^*)$ increase and then decrease as p increases for the fixed ε and δ, thus for $\delta = 0.5, 1.0$ and the fixed ε, when $p = 0$ or 1, the model considers only one kind of customers' demand; when $0 < p < 1$, the model considers two kinds of customers' demand. It implies that $\mathrm{EAC}(Q^*, A^*, \pi_x^*, L^*)$ of two kinds of customers' demand are larger than $\mathrm{EAC}(Q^*, A^*, \pi_x^*, L^*)$ of one kind of customers' demand, thus if the true distribution of the lead time demand is a mixture of normal distributions, we use a single distribution (such as [13]) to substitute the true distribution of the lead time demand then the minimum expected total annual cost will be underestimated; (iii) the order quantity Q^*, the ordering cost A^*, the backorder price discount π_x^* and the minimum total expected annual cost $\mathrm{EAC}(Q^*, A^*, \pi_x^*, L^*)$ decrease as δ increases for the fixed ε and p; (iv) no matter what values of p, the optimal lead time L^* is approached to a certain value (3 weeks) for $\delta = 0(0.5)1$ and $\varepsilon = 0, 0.5, 1, 10, 20, 40, 80, 100, \infty$; (v) while for the fixed p, EVAI increases and then decreases as ε increases (i.e., the backorder rate β decreases) for $\delta = 0.5$ and fixed p; (vi) EVAI increases and then decreases as p increases except $\varepsilon = 0$, when $\delta = 1.0$; (vii) the cost penalty $\mathrm{EAC}_n^{'}/\mathrm{EAC}_n$ of using the distribution free operating policy instead of the optimal one is increasing and then decreasing as ε increases (i.e., the backorder rate β decreases) for the fixed p except $\delta = 0.0$ and $\varepsilon = 0$, $p = 0.6, 0.8, 1.0$, when $\delta = 1.0$. Conveniently, we organize the above (i)–(vii) in Table 3.

4. Concluding Remarks

In this article, we consider that the backorder rate is dependent on the amount of the shortages and the backorder price discount by using the idea of Ouyang and Chuang [13] and Pan and Hsiao [15]. Hence, we regard the backorder rate as controllable variable. In addition, the ordering cost can be controlled and reduced through various efforts such as worker training, procedural changes, and specialized equipment acquisition. So, we also treat the ordering cost as a decision variable. Moreover, we make no assumption about the form of the mixtures of distribution functions of the lead time demand and apply the minimax criterion to solve the problem. We also develop an algorithmic procedure to find the optimal inventory policy.

In this study, we consider the service level constraint to satisfy the equation as $P(X > r) = q$. But, the reorder point r is a controllable variable. Hence, it would be interesting to treat the reorder point as a decision variable in future research.

Table 2: Summary of the optimal solution procedure (L_i in weeks and $\eta = 0.7$, $\delta = 0$).

ε	(Q^*, A^*, π_x^*, L^*)	EAC'	EAC_n'	$(Q_n, A_n, \pi_{x_n}, L_n)$	EAC_n	EVAI	EAC_n'/EAC_n
$p = 0.0$							
0	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
0.5	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
1	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
10	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
20	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
40	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
80	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
100	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
∞	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
$p = 0.2$							
0	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
0.5	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
1	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
10	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
20	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
40	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
80	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
100	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
∞	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
$p = 0.4$							
0	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
0.5	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
1	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
10	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
20	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
40	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
80	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
100	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
∞	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
$p = 0.6$							
0	(148,143, 77.463, 3)	3833.241	3590.772	(160,154, 77.659, 3)	3583.159	7.613	1.00212
0.5	(148,143, 77.463, 3)	3833.241	3590.772	(160,154, 77.659, 3)	3583.159	7.613	1.00212
1	(148,143, 77.463, 3)	3833.241	3590.772	(160,154, 77.659, 3)	3583.159	7.613	1.00212
10	(148,143, 77.463, 3)	3833.241	3590.772	(160,154., 77.659, 3)	3583.159	7.613	1.00212
20	(148,143, 77.463, 3)	3833.241	3590.772	(160,154, 77.659, 3)	3583.159	7.613	1.00212
40	(148,143, 77.463, 3)	3833.241	3590.772	(160,154, 77.659, 3)	3583.159	7.613	1.00212
80	(148,143, 77.463, 3)	3833.241	3590.772	(160,154, 77.659, 3)	3583.159	7.613	1.00212
100	(148,143, 77.463, 3)	3833.241	3590.772	(160,154, 77.659, 3)	3583.159	7.613	1.00212
∞	(148,143, 77.463, 3)	3833.241	3590.772	(160,154, 77.659, 3)	3583.159	7.613	1.00212

Table 2: Continued.

ε	(Q^*, A^*, π_x^*, L^*)	EAC'	EAC_n'	$(Q_n, A_n, \pi_{x_n}, L_n)$	EAC_n	EVAI	$\text{EAC}_n'/\text{EAC}_n$
$p = 0.8$							
0	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
0.5	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
1	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
10	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
20	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
40	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
80	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
100	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
∞	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
$p = 1.0$							
0	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
0.5	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
1	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
10	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
20	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
40	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
80	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
100	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
∞	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
$p = 0.0$							
0	(145,140, 77.420, 3)	3731.388	3430.876	(151,146, 77.521, 3)	3428.830	2.046	1.00060
0.5	(145,141, 77.424, 3)	3765.760	3475.731	(154,149, 77.562, 3)	3471.910	3.821	1.00110
1	(146,141, 77.431, 3)	3781.284	3494.414	(155,150, 77.580, 3)	3490.019	4.395	1.00126
10	(148,143, 77.460, 3)	3816.667	3532.005	(157,152, 77.615, 3)	3527.263	4.741	1.00134
20	(148,143, 77.464, 3)	3820.227	3535.434	(157,152, 77.619, 3)	3530.709	4.725	1.00134
40	(148,143, 77.466, 3)	3822.125	3537.240	(157,152, 77.620, 3)	3532.524	4.716	1.00134
80	(148,143, 77.467, 3)	3823.105	3538.166	(157,152, 77.621, 3)	3533.456	4.710	1.00133
100	(148,143, 77.467, 3)	3823.303	3538.354	(157,152, 77.621, 3)	3533.644	4.710	1.00133
∞	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133
$p = 0.2$							
0	(145,140, 77.415, 3)	3739.222	3477.626	(154,148, 77.559, 3)	3473.490	4.136	1.00119
0.5	(145,140, 77.421, 3)	3773.282	3526.203	(156,151, 77.603, 3)	3519.557	6.646	1.00189
1	(146,141, 77.428, 3)	3788.724	3545.816	(157,152, 77.621, 3)	3538.347	7.469	1.00211
10	(147,142, 77.457, 3)	3824.042	3583.914	(159,154, 77.656, 3)	3575.973	7.940	1.00222
20	(148,143, 77.460, 3)	3827.606	3587.301	(160,154, 77.660, 3)	3579.386	7.914	1.00221
40	(148,143, 77.462, 3)	3829.505	3589.078	(160,154, 77.661, 3)	3581.180	7.898	1.00221
80	(148,143, 77.463, 3)	3830.487	3589.989	(160,154, 77.662, 3)	3582.100	7.889	1.00220
100	(148,143, 77.463, 3)	3830.686	3590.173	(160,154, 77.662, 3)	3582.286	7.887	1.00220
∞	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220

Table 2: Continued.

ε	(Q^*, A^*, π_x^*, L^*)	EAC'	EAC_n'	$(Q_n, A_n, \pi_{x_n}, L_n)$	EAC_n	EVAI	$\mathrm{EAC}_n'/\mathrm{EAC}_n$
$p = 0.4$							
0	(145,140, 77.414, 3)	3741.923	3488.279	(154,149, 77.565, 3)	3483.682	4.597	1.00132
0.5	(145,140, 77.419, 3)	3775.915	3537.551	(157,151, 77.609, 3)	3530.245	7.306	1.00207
1	(146,141, 77.427, 3)	3791.338	3557.216	(158,152, 77.627, 3)	3549.144	8.072	1.00227
10	(147,142, 77.454, 3)	3826.641	3595.519	(160,154, 77.663, 3)	3586.828	8.691	1.00242
20	(148,143, 77.459, 3)	3830.205	3598.779	(160,155, 77.666, 3)	3590.237	8.542	1.00238
40	(148,143, 77.461, 3)	3832.105	3600.552	(160,155, 77.668, 3)	3592.027	8.525	1.00237
80	(148,143, 77.462, 3)	3833.087	3601.460	(160,155, 77.669, 3)	3592.945	8.515	1.00237
100	(148,143, 77.462, 3)	3833.286	3601.643	(160,155, 77.669, 3)	3593.130	8.513	1.00237
∞	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
$p = 0.6$							
0	(145,140, 77.415, 3)	3740.986	3477.898	(153,148, 77.555, 3)	3473.982	3.916	1.00113
0.5	(145,140, 77.421, 3)	3775.037	3526.154	(156,151, 77.599, 3)	3519.769	6.385	1.00181
1	(146,141, 77.427, 3)	3790.476	3545.698	(157,152, 77.617, 3)	3538.497	7.200	1.00203
10	(147,142, 77.456, 3)	3825.792	3583.761	(159,154, 77.653, 3)	3576.090	7.671	1.00215
20	(148,143, 77.460, 3)	3829.356	3587.152	(159,154, 77.656, 3)	3579.506	7.646	1.00214
40	(148,143, 77.461, 3)	3831.256	3588.932	(159,154, 77.658, 3)	3581.302	7.630	1.00213
80	(148,143, 77.462, 3)	3832.237	3589.844	(160,154, 77.658, 3)	3582.222	7.622	1.00213
100	(148,143, 77.463, 3)	3832.436	3590.028	(160,154, 77.659, 3)	3582.409	7.620	1.00213
∞	(148,143, 77.463, 3)	3833.241	3590.772	(160,154, 77.659, 3)	3583.159	7.612	1.00212
$p = 0.8$							
0	(145,140, 77.417, 3)	3737.285	3456.812	(152,147, 77.539, 3)	3453.849	2.963	1.00086
0.5	(145,140, 77.422, 3)	3771.472	3503.469	(155,150, 77.582, 3)	3498.346	5.123	1.00146
1	(146,141, 77.430, 3)	3786.948	3522.514	(156,151, 77.599, 3)	3516.783	5.731	1.00163
10	(147,143, 77.458, 3)	3822.291	3560.458	(158,153, 77.635, 3)	3554.216	6.243	1.00176
20	(148,143, 77.461, 3)	3825.854	3563.868	(158,153, 77.638, 3)	3557.646	6.222	1.00175
40	(148,143, 77.463, 3)	3827.753	3565.660	(158,153, 77.640, 3)	3559.451	6.209	1.00174
80	(148,143, 77.464, 3)	3828.734	3566.579	(158,153, 77.641, 3)	3560.377	6.202	1.00174
100	(148,143, 77.465, 3)	3828.933	3566.666	(158,153, 77.641, 3)	3560.564	6.102	1.00171
∞	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
$p = 1.0$							
0	(145,140, 77.420, 3)	3731.388	3430.876	(151,146, 77.521, 3)	3428.831	2.046	1.00060
0.5	(145,141, 77.424, 3)	3765.760	3475.731	(154,149, 77.562, 3)	3471.910	3.821	1.00110
1	(146,141, 77.431, 3)	3781.284	3494.414	(155,150, 77.580, 3)	3490.020	4.394	1.00126
10	(148,143, 77.460, 3)	3816.667	3532.005	(157,152, 77.615, 3)	3527.264	4.741	1.00134
20	(148,143, 77.464, 3)	3820.227	3535.435	(157,152, 77.619, 3)	3530.709	4.726	1.00134
40	(148,143, 77.466, 3)	3822.125	3537.240	(157,152, 77.620, 3)	3532.524	4.716	1.00134
80	(148,143, 77.467, 3)	3823.105	3538.167	(157,152, 77.621, 3)	3533.456	4.711	1.00133
100	(148,143, 77.467, 3)	3823.303	3538.354	(157,152, 77.621, 3)	3533.645	4.709	1.00133
∞	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133

Table 2: Continued.

ε	(Q^*, A^*, π_x^*, L^*)	EAC'	EAC_n'	$(Q_n, A_n, \pi_{x_n}, L_n)$	EAC_n	EVAI	$\mathrm{EAC}_n'/\mathrm{EAC}_n$
$p = 0.0$							
0	(142,137, 77.367, 3)	3630.318	3319.556	(142,137, 77.364, 4)	3306.329	13.227	1.00400
0.5	(143,138, 77.379, 3)	3705.659	3411.048	(150,145, 77.501, 3)	3408.019	3.029	1.00089
1	(144,139, 77.394, 3)	3737.691	3449.035	(152,147, 77.536, 3)	3444.940	4.095	1.00119
10	(147,142, 77.452, 3)	3809.204	3524.967	(157,151, 77.608, 3)	3520.104	4.862	1.00138
20	(148,143, 77.460, 3)	3816.342	3531.755	(157,152, 77.615, 3)	3527.009	4.746	1.00135
40	(148,143, 77.464, 3)	3820.141	3535.369	(157,152, 77.618, 3)	3530.642	4.727	1.00134
80	(148,143, 77.466, 3)	3822.103	3537.223	(157,152, 77.620, 3)	3532.507	4.716	1.00134
100	(148,143, 77.466, 3)	3822.500	3537.598	(157,152, 77.621, 3)	3532.884	4.714	1.00133
∞	(148,143, 77.468, 3)	3824.107	3539.109	(157,152, 77.622, 3)	3534.405	4.705	1.00133
$p = 0.2$							
0	(142,137, 77.362, 3)	3638.703	3361.079	(144,139, 77.405, 4)	3354.245	6.834	1.00204
0.5	(142,138, 77.374, 3)	3713.324	3460.282	(152,147, 77.541, 3)	3454.668	5.613	1.00162
1	(143,139, 77.391, 3)	3745.187	3500.043	(155,149, 77.577, 3)	3492.971	7.072	1.00202
10	(147,142, 77.448, 3)	3816.571	3577.010	(159,154, 77.650, 3)	3568.894	8.116	1.00227
20	(147,142, 77.455, 3)	3823.715	3583.795	(159,154, 77.656, 3)	3575.733	8.062	1.00225
40	(148,143, 77.460, 3)	3827.519	3587.240	(160,154, 77.660, 3)	3579.324	7.916	1.00221
80	(148,143, 77.462, 3)	3829.483	3589.063	(160,154, 77.661, 3)	3581.164	7.898	1.00221
100	(148,143, 77.462, 3)	3829.881	3589.431	(160,154, 77.662, 3)	3581.536	7.895	1.00220
∞	(148,143, 77.464, 3)	3831.490	3590.915	(160,154, 77.663, 3)	3583.035	7.880	1.00220
$p = 0.4$							
0	(142,137, 77.362, 3)	3641.520	3370.810	(145,140, 77.411, 4)	3365.724	5.085	1.00151
0.5	(142,138, 77.374, 3)	3715.988	3471.297	(153,148, 77.547, 3)	3465.200	6.097	1.00176
1	(143,139, 77.390, 3)	3747.813	3511.377	(155,150, 77.584, 3)	3503.724	7.653	1.00218
10	(147,142, 77.447, 3)	3819.167	3588.520	(159,154, 77.657, 3)	3579.762	8.758	1.00245
20	(147,142, 77.454, 3)	3826.313	3595.290	(160,154, 77.663, 3)	3586.591	8.699	1.00243
40	(148,143, 77.459, 3)	3830.118	3598.719	(160,155, 77.666, 3)	3590.174	8.545	1.00238
80	(148,143, 77.461, 3)	3832.083	3600.536	(160,155, 77.668, 3)	3592.011	8.526	1.00237
100	(148,143, 77.461, 3)	3832.481	3600.903	(160,155, 77.668, 3)	3592.382	8.522	1.00237
∞	(148,143, 77.463, 3)	3834.091	3602.383	(160,155, 77.670, 3)	3593.878	8.505	1.00237
$p = 0.6$							
0	(142,137, 77.362, 3)	3640.473	3361.860	(144,139, 77.401, 4)	3355.785	6.075	1.00181
0.5	(142,138, 77.375, 3)	3715.083	3460.331	(152,147, 77.537, 3)	3454.970	5.361	1.00155
1	(143,139, 77.390, 3)	3746.940	3499.948	(154,149, 77.574, 3)	3493.147	6.802	1.00195
10	(147,142, 77.447, 3)	3818.320	3576.848	(159,153, 77.646, 3)	3569.003	7.845	1.00220
20	(147,142, 77.454, 3)	3825.465	3583.641	(159,154, 77.652, 3)	3575.848	7.792	1.00218
40	(148,143, 77.459, 3)	3829.269	3587.091	(159,154, 77.656, 3)	3579.443	7.648	1.00214
80	(148,143, 77.461, 3)	3831.233	3588.916	(159,154, 77.658, 3)	3581.285	7.631	1.00213
100	(148,143, 77.462, 3)	3831.631	3589.285	(159,154, 77.658, 3)	3581.658	7.627	1.00213
∞	(148,143, 77.463, 3)	3833.240	3590.772	(160,154, 77.659, 3)	3583.159	7.613	1.00212

Table 2: Continued.

ε	(Q^*, A^*, π_x^*, L^*)	EAC$'$	EAC_n'	$(Q_n, A_n, \pi_{x_n}, L_n)$	EAC_n	EVAI	$\text{EAC}_n'/\text{EAC}_n$
$p = 0.8$							
0	(142,137, 77.364, 3)	3636.534	3343.016	(143,138, 77.384, 4)	3334.111	8.905	1.00267
0.5	(143,138, 77.377, 3)	3711.456	3438.123	(151,146, 77.520, 3)	3433.970	4.153	1.00121
1	(143,139, 77.391, 3)	3743.388	3477.029	(153,148, 77.556, 3)	3471.557	5.472	1.00158
10	(147,142, 77.449, 3)	3814.824	3553.487	(158,152, 77.628, 3)	3547.095	6.392	1.00180
20	(147,143, 77.457, 3)	3821.965	3560.217	(158,153, 77.635, 3)	3553.968	6.249	1.00176
40	(148,143, 77.461, 3)	3825.767	3563.804	(158,153, 77.638, 3)	3557.581	6.223	1.00175
80	(148,143, 77.463, 3)	3827.731	3565.644	(158,153, 77.640, 3)	3559.434	6.209	1.00174
100	(148,143, 77.463, 3)	3828.128	3566.015	(158,153, 77.640, 3)	3559.808	6.207	1.00174
∞	(148,143, 77.466, 3)	3829.737	3567.416	(159,153, 77.642, 3)	3561.319	6.096	1.00171
$p = 1.0$							
0	(142,137, 77.367, 3)	3630.318	3319.556	(142,137, 77.364, 4)	3306.329	13.227	1.00400
0.5	(143,138, 77.379, 3)	3705.659	3411.049	(150,145, 77.501, 3)	3408.019	3.029	1.00089
1	(144,139, 77.394, 3)	3737.691	3449.036	(152,147, 77.536, 3)	3444.940	4.095	1.00119
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40	(148,143, 77.464, 3)	3820.141	3535.369	(157,152, 77.618, 3)	3530.642	4.727	1.00134
80	(148,143, 77.466, 3)	3822.102	3537.223	(157,152, 77.620, 3)	3532.507	4.716	1.00134
100	(148,143, 77.466, 3)	3822.500	3537.598	(157,152, 77.621, 3)	3532.884	4.714	1.00133
∞	(148,143, 77.468, 3)	3824.107	3539.110	(157,152, 77.622, 3)	3534.405	4.705	1.00133

Note: we obtain the optimal $(Q_n, A_n, \pi_{x_n}, L_n)$ by the standard procedure, F_* is mixture of normal distribution, and incur an expected annual cost $\text{EAC}_n(Q_n, A_n, \pi_{x_n}, L_n)$. (Q^*, A^*, π_x^*, L^*) stands for the optimal order quantity, the ordering cost, the back-order price discount, and the optimal lead time, respectively, that the demand in the lead time is mixture of free distribution; $\text{EAC}(Q^*, A^*, \pi_x^*, L^*)$ is the minimum total expected annual cost. We use (Q^*, A^*, π_x^*, L^*) instead of the optimal $(Q_n, A_n, \pi_{x_n}, L_n)$ for $\text{EAC}_n(Q^*, A^*, \pi_x^*, L^*)$. In other word, $\text{EAC}' = \text{EAC}(Q^*, A^*, \pi_x^*, L^*)$, $\text{EAC}_n' = \text{EAC}_n(Q^*, A^*, \pi_x^*, L^*)$, and $\text{EAC}_n = \text{EAC}_n(Q_n, A_n, \pi_{x_n}, L_n)$.

Table 3: The relationships among variables, objectives and parameters.

Fixed parameter	Variable parameters	Outcomes	
		Decision variables	Objective
$p, \delta = 0.5, 1.0$	$\varepsilon \uparrow$	$(Q^*, A^*, \pi_x^*) \uparrow$	EAC $\uparrow$
δ, ε	$p \uparrow$		EAC $\nearrow\searrow$
p, ε	$\delta \uparrow$	$(Q^*, A^*, \pi_x^*) \downarrow$	EAC $\downarrow$
δ, p, ε		$L^* = 3$	
$p, \delta = 0.5$	$\varepsilon \uparrow$		EVAI $\nearrow\searrow$
$\delta = 1.0, \varepsilon \neq 0$	$p \uparrow$		EVAI $\nearrow\searrow$

Appendix

For a given value of L, we first obtain the Hessian matrix H as follows:

$$H = \begin{bmatrix} \dfrac{\partial^2 \text{EAC}(Q, A, \pi_x, L)}{\partial Q^2} & \dfrac{\partial^2 \text{EAC}(Q, A, \pi_x, L)}{\partial Q \partial A} & \dfrac{\partial^2 \text{EAC}(Q, A, \pi_x, L)}{\partial Q \partial \pi_x} \\ \dfrac{\partial^2 \text{EAC}(Q, A, \pi_x, L)}{\partial A \partial Q} & \dfrac{\partial^2 \text{EAC}(Q, A, \pi_x, L)}{\partial A^2} & \dfrac{\partial^2 \text{EAC}(Q, A, \pi_x, L)}{\partial A \partial \pi_x} \\ \dfrac{\partial^2 \text{EAC}(Q, A, \pi_x, L)}{\partial \pi_x \partial Q} & \dfrac{\partial^2 \text{EAC}(Q, A, \pi_x, L)}{\partial \pi_x \partial A} & \dfrac{\partial^2 \text{EAC}(Q, A, \pi_x, L)}{\partial \pi_x^2} \end{bmatrix}, \tag{A.1}$$

where

$$\begin{aligned}
\frac{\partial^2 \mathrm{EAC}(Q,A,\pi_x,L)}{\partial Q^2} &= 2\frac{AD}{Q^3}+2\frac{D}{Q^3}\pi_0\left(1-\frac{\theta_2}{1+\Delta(L)}\right)\frac{\Delta(L)}{\varepsilon}+\frac{2D}{Q^3}R(L)>0,\\
\frac{\partial^2 \mathrm{EAC}(Q,A,\pi_x,L)}{\partial A^2} &= \frac{\theta v}{A^2}>0,\\
\frac{\partial^2 \mathrm{EAC}(Q,A,\pi_x,L)}{\partial \pi_x^2} &= 2\frac{D}{Q}\frac{\delta/\pi_0}{1+\Delta(L)}\frac{\Delta(L)}{\varepsilon}>0,\\
\frac{\partial^2 \mathrm{EAC}(Q,A,\pi_x,L)}{\partial Q\partial A} &= \frac{\partial^2 \mathrm{EAC}(Q,A,\pi_x,L)}{\partial A\partial Q}=-\frac{D}{Q^2},\\
\frac{\partial^2 \mathrm{EAC}(Q,A,\pi_x,L)}{\partial Q\partial \pi_x} &= \frac{\partial^2 \mathrm{EAC}(Q,A,\pi_x,L)}{\partial \pi_x\partial Q}=\frac{D}{Q^2}\left[1-2\frac{\pi_x}{\pi_0}\right]\frac{\delta}{1+\Delta(L)}\frac{\Delta(L)}{\varepsilon},\\
\frac{\partial^2 \mathrm{EAC}(Q,A,\pi_x,L)}{\partial A\partial \pi_x} &= \frac{\partial^2 \mathrm{EAC}(Q,A,\pi_x,L)}{\partial \pi_x\partial A}=0.
\end{aligned} \tag{A.2}$$

Then we proceed by evaluating the principal minor of H at point (Q^*, A^*, π_x^*). The first principal minor of H is

$$|H_{11}| = \frac{\partial^2 \mathrm{EAC}(Q^*,A^*,\pi_x^*,L)}{\partial Q^{*2}}>0. \tag{A.3}$$

The second principal minor of H is (note that from (2.16) $D/Q^* = \theta v/A^*$)

$$\begin{aligned}
|H_{22}| &= \frac{\partial^2 \mathrm{EAC}(Q^*,A^*,\pi_x^*,L)}{\partial Q^{*2}}\cdot\frac{\partial^2 \mathrm{EAC}(Q^*,A^*,\pi_x^*,L)}{\partial A^{*2}}-\left[\frac{\partial^2 \mathrm{EAC}(Q^*,A^*,\pi_x^*,L)}{\partial Q^*\partial A^*}\right]^2\\
&= \left\{\frac{2A^*D}{Q^{*3}}+\frac{2D}{Q^{*3}}\pi_0\left[1-\frac{\theta_2^*}{1+\Delta(L)}\right]\frac{\Delta(L)}{\varepsilon}+\frac{2D}{Q^{*3}}R(L)\right\}\times\frac{\theta v}{A^{*2}}-\left(-\frac{D}{Q^{*2}}\right)^2\\
&= \frac{D}{Q^{*3}}\left\{\frac{\theta v}{A^*}+\frac{2\theta v}{A^{*2}}\left(\pi_0\left[1-\frac{\theta_2^*}{1+\Delta(L)}\right]\frac{\Delta(L)}{\varepsilon}+R(L)\right)\right\}>0.
\end{aligned} \tag{A.4}$$

The third principal minor of H is

$$\begin{aligned}
|H_{33}| &= \frac{\partial^2 \mathrm{EAC}(Q^*,A^*,\pi_x^*,L)}{\partial \pi_x^{*2}}\cdot|H_{22}|-\left[\frac{\partial^2 \mathrm{EAC}(Q^*,A^*,\pi_x^*,L)}{\partial Q^*\partial \pi_x^*}\right]^2\cdot\frac{\partial^2 \mathrm{EAC}(Q^*,A^*,\pi_x^*,L)}{\partial A^{*2}}\\
&= 2\frac{D^2}{Q^{*4}}\left[\frac{\theta v}{A^*}+\frac{2\theta v}{A^{*2}}R(L)\right]\frac{\delta/\pi_0}{1+\Delta(L)}\frac{\Delta(L)}{\varepsilon}+\frac{D^2}{Q^{*4}}\frac{\theta v}{A^{*2}}\frac{\delta}{1+\Delta(L)}\left(\frac{\Delta(L)}{\varepsilon}\right)^2\left[4-\frac{\delta}{1+\Delta(L)}\right]>0,
\end{aligned} \tag{A.5}$$

where $\theta_2^* = (\pi_x^*/\pi_0)(1-\pi_x^*/\pi_0)\delta$, $\Delta(L) = \varepsilon B(r)$.

Therefore, from (A.3)–(A.5), it is clearly seen that the Hessian matrix H is positive definite at point (Q^*, A^*, π_x^*).

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