

Research Article

Equivariant Hopf Bifurcation in a Ring of Identical Cells with Delay

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A kind of delay neural network with n elements is considered. By analyzing the distribution of the eigenvalues, a bifurcation set is given in an appropriate parameter space. Then by using the theory of equivariant Hopf bifurcations of ordinary differential equations due to Golubitsky et al. (1988) and delay differential equations due to Wu (1998), and combining the normal form theory of functional differential equations due to Faria and Magalhaes (1995), the equivariant Hopf bifurcation is completely analyzed.

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1. Introduction

The works of Golubitsky et al. [1, Chapter XVIII] have shown that rings of identical cells can lead to many interesting patterns of oscillation, which are predictable based on the theory of equivariant bifurcations. In a series of papers, Wu et al. [2–4] have extended the theory of equivariant Hopf bifurcation to delay differential equations. Recently, there has been interest in applying these results to neural networks, primarily to models related to the Hopfield-Cohen-Grossberg neural networks with time delays [5–9]. In a series of papers [10–23], the authors studied the Hopf bifurcation to a network with one or two delays. Particularly, Wu et al. [24] have investigated the synchronization and stable phase-locking in a delayed network with three identical neurons:

$$\dot{x}_i(t) = -x_i(t) + \alpha f(x_i(t - \tau)) + \beta [f(x_{i-1}(t - \tau)) + f(x_{i+1}(t - \tau))], \quad i(\text{mod}3) \quad (1.1)$$

where $f : R \rightarrow R$ is a sufficiently smooth sigmoid amplification function, normalized so that $f(0) = 0$ and $f'(0) = 1$, α , β , and τ are regarded as parameters. They have shown that in the region $A_{\text{as}} = \{(\alpha, \beta) : |\alpha - \beta| < 1\}$ of the normalized parameters, system (4.1) is absolutely synchronous in the sense that every solution is convergent to the set of all

synchronized phase states independent of the size of the time delay. The ω -limit set of a given orbit can be either a synchronized equilibrium or a synchronized periodic solution, depending on the connection topology of the network, the strength of the self-connection and the neighborhood-interaction, and the size of the delay. On the other hand, they have shown that, in the region $\alpha - \beta < -1$, there exists a continuous surface $\tau = \tau(\alpha, \beta)$ where Hopf bifurcation of either a stable synchronized periodic solution or two stable phase-locked periodic solutions and six unstable periodic waves take place (more precisely, three mirror-reflecting waves and three standing waves). A question of mathematical and biological interest is whether the dynamics of (4.1) are possible for the following neural network model with n elements:

$$\dot{x}_i(t) = -x_i(t) + \alpha f(x_i(t - \tau)) + \beta [f(x_{i-1}(t - \tau)) + f(x_{i+1}(t - \tau))], \quad i(\text{mod } n). \quad (1.2)$$

The purpose of the paper is to provide a detailed analysis of this question in the case that n is an odd number. More precisely, we shall extend the main results of Wu et al. [24] to the system (1.2) with n being an odd number. We are going to regard the delay τ as a parameter to investigate the dynamics of (1.2). According to the properties of the function $\tanh x$, we assume that the transfer function f is adequately smooth, for example, $f \in C^3$, and satisfies the following normalization, monotonicity, concavity, and boundedness conditions:

- (C1) $f(0) = 0$, $f'(0) = 1$, and $f'(x) > 0$ for all $x \in R$;
- (C2) $f''(0) = 0$, and $xf''(x) < 0$ for all $x \neq 0$;
- (C3) $-\infty < \lim_{x \rightarrow \pm\infty} f(x) < \infty$;
- (C4) $f'''(0) < 0$.

At first, by analyzing the distribution of the eigenvalues, we give a bifurcation set in an appropriate parameter space to describe the stability of the equilibrium of the system (1.2) of how to change as the parameters change. Meanwhile, the equivariant Hopf bifurcations are found. Then, by employing the center manifold and normal form theory, the direction of the bifurcation and the stability of the bifurcating periodic solutions are determined.

The rest of this paper is organized as follows. In Section 2, the characteristic equation of the linearization of system (1.2) at the zero equilibrium is derived. In Section 3, by analyzing the distribution of the eigenvalues, a bifurcation set is given in the (α, β) -plane. Then by using the theory of equivariant Hopf bifurcation to ordinary and delay differential equations due to Golubitsky et al. [1] and Wu [4], respectively, the local equivariant Hopf bifurcation of the trivial solution is completely analyzed. The appendix contain the detailed calculations of the normal forms on center manifolds of system (1.2) near Hopf bifurcation points.

We would like to mention that there are several articles on the bifurcation for n -dimensional neural network models with delays; we refer the reader to [16, 17, 22] and references therein.

2. The Characteristic Equations

From conditions (C1)–(C4), we know that the linearization of system (1.2) at an equilibrium $(0, 0, \dots, 0)^T$ is given by

$$\dot{x}_i(t) = -x_i(t) + \alpha x_i(t - \tau) + \beta [x_{i-1}(t - \tau) + x_{i+1}(t - \tau)], \quad i(\text{mod } n). \quad (2.1)$$

The characteristic matrix is

$$M_n(0, \lambda) = (\lambda + 1)\text{Id} - \alpha e^{-\lambda\tau}\text{Id} - \beta e^{-\lambda\tau}\delta, \quad (2.2)$$

where Id is $n \times n$ identity matrix and

$$\delta = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}. \quad (2.3)$$

Using a computation process similar to that of Wu et al. [24], Yuan and Campbell [25], we can obtain the following results.

Let

$$\chi = e^{2\pi i/n}, \quad v_j = \left(1, \chi^j, \chi^{2j}, \dots, \chi^{(n-1)j}\right)^T \quad (j = 0, 1, \dots, n-1). \quad (2.4)$$

From

$$\chi^{nj} = 1, \quad \chi^{(n-1)j} = \chi^{-j}, \quad \chi^{(n-2)j} = \chi^{-2j}, \dots, \dots, \quad (2.5)$$

it follows that

$$\delta v_j = \begin{pmatrix} \chi^j + \chi^{(n-1)j} \\ 1 + \chi^{2j} \\ \chi^j + \chi^{3j} \\ \vdots \\ 1 + \chi^{(n-2)j} \end{pmatrix} = (\chi^j + \chi^{-j})v_j = 2 \cos\left(\frac{2\pi j}{n}\right)v_j. \quad (2.6)$$

Hence,

$$\begin{aligned} M_n(0, \lambda)v_j &= \left[\lambda + 1 - \left(\alpha + \beta(\chi^j + \chi^{-j})\right)e^{-\lambda\tau}\right]v_j \\ &= \left[\lambda + 1 - \left(\alpha + 2\beta \cos\frac{2\pi j}{n}\right)e^{-\lambda\tau}\right]v_j, \end{aligned} \quad (2.7)$$

and the characteristic equation is

$$\begin{aligned}
 \det M_n(0, \lambda) &= \prod_{j=0}^{n-1} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} \right] \\
 &= \Delta_0(\lambda) \times \prod_{j=1}^{n-1} \Delta_j(\lambda) \\
 &= 0,
 \end{aligned} \tag{2.8}$$

where

$$\Delta_j(\lambda) = \lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau}, \quad (j = 0, 1, 2, \dots, n-1). \tag{2.9}$$

We observe that (2.7) can be simplified, further elucidating the structure of characteristic equation.

(I) When n is an odd number, we obtain

$$\begin{aligned}
 \det M_n(0, \lambda) &= \Delta_0(\lambda) \times \prod_{j=1}^{(n-1)/2} \Delta_j^2(\lambda) \\
 &= \left[\lambda + 1 - (\alpha + 2\beta) e^{-\lambda\tau} \right] \times \prod_{j=1}^{(n-1)/2} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} \right]^2.
 \end{aligned} \tag{2.10}$$

(II) When n is an even number, we obtain

$$\begin{aligned}
 \det M_n(0, \lambda) &= \Delta_0(\lambda) \Delta_{n/2}(\lambda) \times \prod_{j=1}^{(n-2)/2} \Delta_j^2(\lambda) \\
 &= \left[\lambda + 1 - (\alpha + 2\beta) e^{-\lambda\tau} \right] \left[\lambda + 1 - (\alpha - 2\beta) e^{-\lambda\tau} \right] \\
 &\quad \times \prod_{j=1}^{(n-2)/2} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} \right]^2.
 \end{aligned} \tag{2.11}$$

In the case when n is even, that is, $n = 2m$, $\det M_n(0, \lambda)$ may be further simplified into two subcases.

(i) When m is an odd number, we have

$$\begin{aligned}
 \det M_n(0, \lambda) &= \Delta_0(\lambda) \Delta_{n/2}(\lambda) \times \prod_{j=1}^{(m-1)/2} \Delta_j^2(\lambda) \Delta_{m-j}^2(\lambda) \\
 &= [\lambda + 1 - (\alpha + 2\beta)e^{-\lambda\tau}] [\lambda + 1 - (\alpha - 2\beta)e^{-\lambda\tau}] \\
 &\quad \times \prod_{j=1}^{(m-1)/2} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{\pi j}{m} \right) e^{-\lambda\tau} \right]^2 \left[\lambda + 1 - \left(\alpha - 2\beta \cos \frac{\pi j}{m} \right) e^{-\lambda\tau} \right]^2.
 \end{aligned} \tag{2.12}$$

(ii) When m is an even number, we have

$$\begin{aligned}
 \det M_n(0, \lambda) &= \Delta_0(\lambda) \Delta_{n/2}(\lambda) \Delta_{n/4}^2(\lambda) \times \prod_{j=1}^{(m-2)/2} \Delta_j^2(\lambda) \Delta_{m-j}^2(\lambda) \\
 &= [\lambda + 1 - (\alpha + 2\beta)e^{-\lambda\tau}] [\lambda + 1 - (\alpha - 2\beta)e^{-\lambda\tau}] (\lambda + 1 - \alpha e^{-\lambda\tau})^2 \\
 &\quad \times \prod_{j=1}^{(m-2)/2} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{\pi j}{m} \right) e^{-\lambda\tau} \right]^2 \left[\lambda + 1 - \left(\alpha - 2\beta \cos \frac{\pi j}{m} \right) e^{-\lambda\tau} \right]^2.
 \end{aligned} \tag{2.13}$$

From (2.8)–(2.13), we obtain the characteristic equation of (1.2) given by the following.

(1) When n is an odd number,

$$[\lambda + 1 - (\alpha + 2\beta)e^{-\lambda\tau}] \prod_{j=1}^{(n-1)/2} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} \right]^2 = 0. \tag{2.14}$$

(2) When n is an even number and n is not the multiple of 4,

$$\begin{aligned}
 &[\lambda + 1 - (\alpha + 2\beta)e^{-\lambda\tau}] \prod_{j=1}^{(n-2)/4} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} \right]^2 \\
 &\quad \times [\lambda + 1 - (\alpha - 2\beta)e^{-\lambda\tau}] \prod_{j=1}^{(n-2)/4} \left[\lambda + 1 - \left(\alpha - 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} \right]^2 \\
 &= 0.
 \end{aligned} \tag{2.15}$$

(3) When n is the multiple of 4,

$$\begin{aligned} & \left[\lambda + 1 - (\alpha + 2\beta)e^{-\lambda\tau} \right] \prod_{j=1}^{(n-4)/4} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} \right]^2 \\ & \quad \times \left[\lambda + 1 - (\alpha - 2\beta)e^{-\lambda\tau} \right] \prod_{j=1}^{(n-4)/4} \left[\lambda + 1 - \left(\alpha - 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} \right]^2 \times (\lambda + 1 - \alpha e^{-\lambda\tau})^2 \\ & = 0. \end{aligned} \tag{2.16}$$

3. Bifurcation Analysis with n being an Odd Number

In this section, we are going to analyze the distribution of the root of the characteristic equations given in previous section, respectively.

In this case, the characteristic equation of (1.2) is

$$\left[\lambda + 1 - (\alpha + 2\beta)e^{-\lambda\tau} \right] \prod_{j=1}^{(n-1)/2} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} \right]^2 = 0. \tag{3.1}$$

Clearly, $i\omega$ ($\omega > 0$) is a root of (3.1) if and only if there exists a

$$j \in \left\{ 0, 1, 2, \dots, \frac{n-1}{2} \right\} \tag{3.2}$$

such that ω satisfies

$$i\omega + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-i\omega\tau} = 0, \tag{3.3}$$

that is,

$$\cos \omega\tau = \frac{1}{\alpha + 2\beta \cos(2\pi j/n)}, \quad \sin \omega\tau = -\frac{\omega}{\alpha + 2\beta \cos(2\pi j/n)}. \tag{3.4}$$

This leads to

$$\omega_j = \sqrt{\left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right)^2 - 1}. \tag{3.5}$$

This shows that the necessary condition for (3.1) possessing purely imaginary roots is that there exists $j \in \{0, 1, 2, \dots, (n-1)/2\}$ such that $|\alpha + 2\beta \cos(2\pi j/n)| > 1$, that is,

$$\alpha + 2\beta \cos \frac{2\pi j}{n} > 1 \quad \text{or} \quad \alpha + 2\beta \cos \frac{2\pi j}{n} < -1. \tag{3.6}$$

From (3.1), conclusion then follows immediately.

Proposition 3.1. Equation (3.1) has at most $(n + 1)/2$ pairs of purely imaginary roots when n is an odd number.

Without loss of generality, one assumes the (3.1) has $(n + 1)/2$ pairs of purely imaginary roots $\pm\omega_j$ ($j = 0, 1, 2, \dots, (n - 1)/2$) exactly. Define

$$\tau_k^{(j)} = \frac{1}{\omega_j} \left[\arccos \frac{1}{\alpha + 2\beta \cos(2\pi j/n)} + 2k\pi \right], \quad j = 0, 1, \dots, \frac{n-1}{2}; k = 0, 1, 2, \dots \quad (3.7)$$

Then $\pm i\omega_j$ are purely imaginary roots of (3.1) with $\tau = \tau_k^{(j)}$ ($k = 0, 1, 2, \dots$), respectively. one notes that $\omega_j \tau_k^{(j)} \in (0, \pi]$ (resp., $\in (\pi, 2\pi]$) when

$$\sin(\omega_j \tau_k^{(j)}) = -\frac{\omega_j}{\alpha + 2\beta \cos(2\pi j/n)} > 0 \quad (\text{resp. } < 0). \quad (3.8)$$

We can obtain the following result with no difficulty.

Proposition 3.2. Let $\lambda(\tau) = \sigma(\tau) \pm i\omega(\tau)$ be the root of (3.1) near $\tau = \tau_k^{(j)}$ satisfying $\sigma(\tau_k^{(j)}) = 0$, $\omega(\tau_k^{(j)}) = \omega_j$ ($j = 0, 1, \dots, (n - 1)/2; k = 0, 1, 2, \dots$). Then

$$\left. \frac{d\sigma(\tau)}{d\tau} \right|_{\tau=\tau_k^{(j)}} > 0. \quad (3.9)$$

Proof. From (3.1) we have

$$\lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda\tau} = 0. \quad (3.10)$$

Differentiating both sides with respect to τ , we have

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = -\frac{e^{\lambda\tau}}{\lambda(\alpha + 2\beta \cos(2\pi j/n))} - \frac{\tau}{\lambda}. \quad (3.11)$$

Notice that $\lambda(\tau_k^{(j)}) = i\omega_j$, it follows that

$$\operatorname{Re} \left. \frac{d\lambda}{d\tau} \right|_{\tau=\tau_k^{(j)}} = \left. \frac{d\sigma(\tau)}{d\tau} \right|_{\tau=\tau_k^{(j)}} = \frac{\omega_j^2}{\left(1 + \tau_k^{(j)}\right)^2 + \left(\tau_k^{(j)} \omega_j\right)^2} > 0. \quad (3.12)$$

This completes the proof. \square

When $\tau = 0$, (3.1) becomes

$$[\lambda + 1 - (\alpha + 2\beta)] \prod_{j=1}^{(n-1)/2} \left[\lambda + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) \right]^2 = 0. \quad (3.13)$$

Its roots are given by

$$\begin{aligned} \lambda_0 &= \alpha + 2\beta - 1, \\ \lambda_1 &= \alpha + 2\beta \cos \frac{2\pi}{n} - 1, \\ \lambda_2 &= \alpha + 2\beta \cos \frac{4\pi}{n} - 1, \\ &\vdots \\ \lambda_{(n-1)/2} &= \alpha + 2\beta \cos \frac{(n-1)\pi}{n} - 1 \\ &= \alpha - 2\beta \cos \frac{\pi}{n} - 1. \end{aligned} \quad (3.14)$$

For convenience, denote

$$S := \left\{ (\alpha, \beta) \mid \alpha < 1, \frac{1-\alpha}{2 \cos((n-1)\pi/n)} < \beta < \frac{1-\alpha}{2} \right\}, \quad (3.15)$$

$$P := \left\{ (\alpha, \beta) \mid \frac{-1-\alpha}{2} < \beta < \frac{1-\alpha}{2}, \frac{1-\alpha}{2 \cos((n-1)\pi/n)} < \beta < \frac{-1-\alpha}{2 \cos((n-1)\pi/n)} \right\}, \quad (3.16)$$

and let $D := S/P$. One can find out that the region S is a sector, P is a parallelogram, and D is the outside of the parallelogram in the sector S . These are shown in Figure 1.

Proposition 3.3. *If $\alpha + 2\beta < 1$ and $\alpha + 2\beta \cos((n-1)\pi/n) < 1$ ($\alpha - 2\beta \cos(\pi/n) < 1$), that is $(\alpha, \beta) \in S$, then all roots of (3.13) have negative real parts.*

From (3.6), Proposition 3, and Corollary 2.4 of Ruan and Wei [19], we have the following result immediately.

Proposition 3.4. *If $|\alpha + 2\beta| < 1$ and $|\alpha + 2\beta \cos((n-1)\pi/n)| < 1$ ($|\alpha - 2\beta \cos(\pi/n)| < 1$), that is, $(\alpha, \beta) \in P$, then all roots of (3.1) have negative real parts for all $\tau \geq 0$.*

From Proposition 2 and Corollary 2.4 of Ruan and Wei [19], conclusion then follows immediately.

Proposition 3.5. *If there exist $j \in \{0, 1, 2, \dots, (n-1)/2\}$ such that $\alpha + 2\beta \cos(2\pi j/n) > 1$, that is, (α, β) is located outside of the sector S , then (3.1) has at least one root with positive part for all $\tau \geq 0$.*

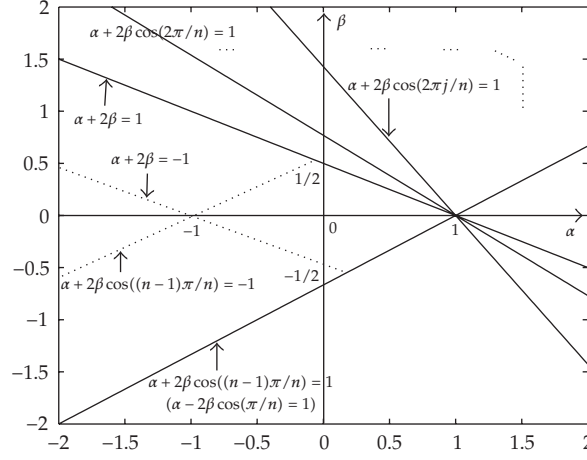


Figure 1: Bifurcation set in (α, β) -plane. The outside of the sector S is unstable region; the parallelogram P is the absolutely stable region; the outside of the parallelogram in the sector is conditionally stable region.

Proposition 3.6. Suppose that one of the following conditions is satisfied (i.e., $(\alpha, \beta) \in D$):

- (1) $|\alpha + 2\beta| < 1$ and $\alpha - 2\beta \cos(\pi/n) < -1$;
- (2) $\alpha + 2\beta < -1$ and $|\alpha - 2\beta \cos(\pi/n)| < 1$;
- (3) $\alpha + 2\beta < -1$ and $\alpha - 2\beta \cos(\pi/n) < -1$.

Then there exist at most $(n + 1)/2$ sequences of critical value

$$\tau_k^{(j)}, \quad j = 0, 1, \dots, \frac{n-1}{2}; k = 0, 1, 2, \dots \quad (3.17)$$

defined by (3.7) such that

- (1) all roots of (3.1) have negative real parts for all $\tau \in [0, \tau_0)$, where

$$\tau_0 = \min \left\{ \tau_0^{(j)} : 0 \leq j \leq \frac{n-1}{2} \right\}; \quad (3.18)$$

- (2) equation (3.1) has at least one root with positive part when $\tau > \tau_0$;
- (3) at $\tau = \tau_k^{(j)}$ ($j = 0, 1, \dots, (n-1)/2; k = 0, 1, \dots$), (3.1) has a pair of purely imaginary roots $\pm i\omega_j$.

From Propositions 3.1–3.6, we have the following results immediately.

Theorem 3.7. (i) If $|\alpha + 2\beta| < 1$ and $|\alpha - 2\beta \cos(\pi/n)| < 1$, then the zero equilibrium of (1.2) is asymptotically stable for all $\tau \geq 0$.

- (ii) If j ($j = 0, 1, 2, \dots, (n-1)/2$) exist such that $\alpha + 2\beta \cos(2\pi j/n) > 1$, then the zero equilibrium of (1.2) is unstable for all $\tau \geq 0$.

(iii) Suppose that one of the following conditions holds:

- (1) $|\alpha + 2\beta| < 1$ and $\alpha - 2\beta \cos(\pi/n) < -1$;
- (2) $\alpha + 2\beta < -1$ and $|\alpha - 2\beta \cos(\pi/n)| < 1$;
- (3) $\alpha + 2\beta < -1$ and $\alpha - 2\beta \cos(\pi/n) < -1$.

Then there exist at most $(n+1)/2$ sequences of critical value $\tau_k^{(j)}$ ($j = 0, 1, \dots, (n-1)/2$; $k = 0, 1, 2, \dots$) defined by (3.7) such that the zero equilibrium of (1.2) is asymptotically stable for $\tau \in [0, \tau_0)$, where $\tau_0 = \min\{\tau_0^{(j)} : 0 \leq j \leq (n-1)/2\}$, unstable for all $\tau > \tau_0$, and the system undergoes a Hopf bifurcation with $\tau = \tau_k^{(j)}$ ($j = 0, 1, \dots, (n-1)/2$; $k = 0, 1, \dots$).

Proposition 3.6 shows that the region D , which is the outside of the parallelogram in the sector shown as in Figure 1, is a conditionally stable region. This means that the stability of the zero equilibrium of (1.2) is dependent on the delay. Namely, the distribution of the roots of (3.1) is dependent on the delay. Particularly, for $(\alpha, \beta) \in D$, (3.1) with some critical value τ has simple pure imaginary roots or multiply pure imaginary roots. It is well know that the Hopf bifurcations are different between the simple and multiple pure imaginary roots. So it is necessary to observe the region D . In fact, we have the following conclusions on occurrence of purely imaginary roots of (3.1) for $(\alpha, \beta) \in D$.

(1) In the region

$$E_0 = \left\{ (\alpha, \beta) \mid \alpha + 2\beta < -1; \alpha - 2\beta \cos \frac{\pi}{n} < 1; \alpha + 2\beta \cos \frac{2\pi}{n} > -1 \right\}, \quad (3.19)$$

Equation (3.1) with $\tau_k^{(0)}$ has one pair of purely imaginary roots exactly, denoted by $\pm i\omega_0$, which are simple, where $\omega_0 = \sqrt{(\alpha + 2\beta)^2 - 1}$ and

$$\tau_k^{(0)} = \frac{1}{\omega_0} \left(\arccos \frac{1}{\alpha + 2\beta} + 2k\pi \right) \quad (k = 0, 1, 2, \dots). \quad (3.20)$$

(2) In the region

$$E_1 = \left\{ (\alpha, \beta) \mid \alpha + 2\beta \cos \frac{2\pi}{n} < -1; \alpha - 2\beta \cos \frac{\pi}{n} < 1; \alpha + 2\beta \cos \frac{4\pi}{n} > -1 \right\}, \quad (3.21)$$

Equation (3.1) has two pairs of purely imaginary roots exactly, denoted by $\pm i\omega_0$ and $\pm i\omega_1$, when $\tau = \tau_k^{(1)}$ and $\tau = \tau_k^{(0)}$, respectively. Here $\pm i\omega_0$ are simple, $\pm i\omega_1$ are double, and $\omega_1 = \sqrt{(\alpha + 2\beta \cos(2\pi/n))^2 - 1}$, $\tau_k^{(0)}$ is defined by (3.20) and

$$\tau_k^{(1)} = \frac{1}{\omega_1} \left(\arccos \frac{1}{\alpha + 2\beta \cos(2\pi/n)} + 2k\pi \right) \quad (k = 0, 1, 2, \dots). \quad (3.22)$$

Furthermore,

$$\tau_0^{(0)} = \frac{1}{\omega_0} \left(\arccos \frac{1}{\alpha + 2\beta} \right) > \tau_0^{(1)} = \frac{1}{\omega_1} \left(\arccos \frac{1}{\alpha + 2\beta \cos(2\pi/n)} \right). \quad (3.23)$$

(3) In the region

$$E_2 = \left\{ (\alpha, \beta) \mid \alpha + 2\beta \cos \frac{4\pi}{n} < -1; \alpha - 2\beta \cos \frac{\pi}{n} < 1; \alpha + 2\beta \cos \frac{6\pi}{n} > -1 \right\}, \quad (3.24)$$

Equation (3.1) has three pairs of purely imaginary roots exactly, denoted by $\pm i\omega_0$, $\pm i\omega_1$, and $\pm i\omega_2$, when $\tau = \tau_k^{(0)}$, $\tau = \tau_k^{(1)}$, and $\tau = \tau_k^{(2)}$, respectively. Here $\pm i\omega_0$ are simple, $\pm i\omega_1$ and $\pm i\omega_2$ are double, $\omega_2 = \sqrt{(\alpha + 2\beta \cos(4\pi/n))^2 - 1}$, $\tau_k^{(0)}$ and $\tau_k^{(1)}$ are defined by (3.20) and (3.22), respectively, and

$$\tau_k^{(2)} = \frac{1}{\omega_2} \left(\arccos \frac{1}{\alpha + 2\beta \cos(4\pi/n)} + 2k\pi \right) \quad (k = 0, 1, 2, \dots). \quad (3.25)$$

Furthermore,

$$\tau_0^{(0)} > \tau_0^{(1)} > \tau_0^{(2)} = \frac{1}{\omega_2} \left(\arccos \frac{1}{\alpha + 2\beta \cos(4\pi/n)} \right). \quad (3.26)$$

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Naturally, the region $E_{(n-1)/4}$ (when $n-1$ is multiple of 4) or $E_{(n-3)/4}$ (when $n-1$ is not multiple of 4) can be defined by

$$E_{(n-1)/4} = \left\{ (\alpha, \beta) \mid \alpha + 2\beta \cos \frac{(n-1)\pi}{2n} < -1; \alpha - 2\beta \cos \frac{\pi}{n} < 1; \alpha + 2\beta \cos \frac{2\pi}{n} < -1 \right\}, \quad (3.27)$$

$$E_{(n-3)/4} = \left\{ (\alpha, \beta) \mid \alpha + 2\beta \cos \frac{(n-3)\pi}{2n} < -1; \alpha - 2\beta \cos \frac{\pi}{n} < 1; \alpha + 2\beta \cos \frac{2\pi}{n} < -1 \right\}.$$

★A: Obviously, in the region $E_{(n-1)/4}$ (or $E_{(n-3)/4}$), we have that

$$\tau_0^{(0)} > \tau_0^{(1)} > \tau_0^{(2)} > \tau_0^{(3)} > \dots > \tau_0^{((n-1)/4)} \quad \left(\text{or } \tau_0^{((n-3)/4)} \right). \quad (3.28)$$

(4) In the region

$$E_{((n-1)/2)} = \left\{ (\alpha, \beta) \mid \alpha + 2\beta \cos \frac{(n-1)\pi}{n} < -1; \alpha + 2\beta < 1; \alpha + 2\beta \cos \frac{(n-3)\pi}{n} > -1 \right\} \quad (3.29)$$

$$= \left\{ (\alpha, \beta) \mid \alpha - 2\beta \cos \frac{\pi}{n} < -1; \alpha + 2\beta < 1; \alpha - 2\beta \cos \frac{3\pi}{n} > -1 \right\},$$

Equation (3.1) has a pair of purely imaginary roots exactly, denoted by $\pm i\omega_{(n-1)/2}$, which are double, where $\omega_{(n-1)/2} = \sqrt{(\alpha + 2\beta \cos((n-1)\pi/n))^2 - 1} = \sqrt{(\alpha - 2\beta \cos(\pi/n))^2 - 1}$ and

$$\tau_k^{((n-1)/2)} = \frac{1}{\omega_{(n-1)/2}} \left(\arccos \frac{1}{\alpha - 2\beta \cos(\pi/n)} + 2k\pi \right) \quad (k = 0, 1, 2, \dots). \quad (3.30)$$

(5) In the region

$$E_{((n-3)/2)} = \left\{ (\alpha, \beta) \mid \alpha - 2\beta \cos \frac{3\pi}{n} < -1; \alpha + 2\beta < 1; \alpha - 2\beta \cos \frac{5\pi}{n} > -1 \right\}, \quad (3.31)$$

Equation (3.1) has two pairs of purely imaginary roots exactly, denoted by $\pm i\omega_{(n-1)/2}$ and $\pm i\omega_{(n-3)/2}$, when $\tau = \tau_k^{((n-1)/2)}$ and $\tau = \tau_k^{((n-3)/2)}$, respectively, and the two pairs of purely imaginary roots are all double. Here

$$\omega_{(n-3)/2} = \sqrt{\left(\alpha + 2\beta \cos \frac{(n-3)\pi}{n}\right)^2 - 1} = \sqrt{\left(\alpha - 2\beta \cos \frac{3\pi}{n}\right)^2 - 1}, \quad (3.32)$$

$\tau_k^{((n-1)/2)}$ is defined by (3.30), and

$$\tau_k^{((n-3)/2)} = \frac{1}{\omega_{(n-3)/2}} \left(\arccos \frac{1}{\alpha - 2\beta \cos(3\pi/n)} + 2k\pi \right) \quad (k = 0, 1, 2, \dots). \quad (3.33)$$

Moreover,

$$\tau_0^{((n-1)/2)} = \frac{1}{\omega_{(n-1)/2}} \left(\arccos \frac{1}{\alpha - 2\beta \cos(\pi/n)} \right) > \tau_0^{((n-3)/2)} = \frac{1}{\omega_{(n-3)/2}} \left(\arccos \frac{1}{\alpha - 2\beta \cos(3\pi/n)} \right). \quad (3.34)$$

.....

Naturally, the region $E_{(n+3)/4}$ (when $n-1$ is multiple of 4) or $E_{(n+1)/4}$ (when $n-1$ is not multiple of 4) can be defined by

$$E_{(n+3)/4} = \left\{ (\alpha, \beta) \mid \alpha + 2\beta \cos \frac{2\pi}{n} < -1; \alpha + 2\beta < 1; \alpha + 2\beta > -1 \right\}, \quad (3.35)$$

$$E_{(n+1)/4} = \left\{ (\alpha, \beta) \mid \alpha + 2\beta \cos \frac{2\pi}{n} < -1; \alpha + 2\beta < 1; \alpha + 2\beta > -1 \right\}.$$

★B: Obviously, in the region $E_{(n+3)/4}$ (or $E_{(n+1)/4}$), we have

$$\tau_0^{((n-1)/2)} > \tau_0^{((n-3)/2)} > \tau_0^{((n-5)/2)} > \dots > \tau_0^{((n+3)/4)} \quad \left(\text{or } \tau_0^{((n+1)/4)} \right). \quad (3.36)$$

To carry out our work we need some background from the theory of functional differential equation. Let $C = C([- \tau, 0], R^n)$ denote the Banach space of continuous mappings from $[- \tau, 0]$ into R^n equipped with the supernorm. Let $x(t)$ be a solution of (1.2) and define $x_t(\theta) = x(t + \theta)$, $- \tau \leq \theta \leq 0$. If $x(t)$ is continuous, then $x_t(\theta) \in C$. With this structure, we may write the model as the following functional differential equation:

$$\dot{x}(t) = F(x_t), \quad (3.37)$$

where $F : C \rightarrow R^n$ is defined via

$$F_i(\phi) = -\phi_i(0) + \alpha f(\phi_i(-\tau)) + \beta[f(\phi_{i-1}(-\tau)) + f(\phi_{i+1}(-\tau))], \quad i(\bmod n). \quad (3.38)$$

Similarly, the linearization of (3.37) at equilibrium $(0, 0, \dots, 0)^T$ may be written as

$$\dot{u}(t) = L(\tau)u_t, \quad (3.39)$$

where the linear operator $L(\tau) : C \rightarrow R^n$ is defined via

$$L(\tau)\phi = -\text{Id}\phi(0) + \alpha\text{Id}\phi(-\tau) + \beta\delta\phi(-\tau). \quad (3.40)$$

It is well known that a linear functional differential equation such as (3.39) generates a strongly continuous semigroup of linear operators with infinitesimal generator A given by

$$A\phi = \dot{\phi}, \quad \phi \in \text{Dom}(A), \quad (3.41)$$

where $\text{Dom}(A) = \phi \in C, \phi(0) = L(\tau)\phi$.

To explore the possible (spatial) symmetry of (1.2), we need to introduce three compact Lie groups. One is the cycle group S^1 ; another is Z_n , the cyclic group of order n , which corresponds to rotations of $2\pi/n$, denoting the generator of this group by ρ , then action on R^n is given by $(\rho x)_i = x_{i+1}$; the third is the dihedral group D_n of order $2n$, which corresponds to the group of symmetries of an n -gon. It can be shown that D_n is generated by ρ and κ , where κ is the flip of order 2 or reflection, and it acts on R^n by $(\kappa x)_i = x_{n+2-i}$.

Definition 3.8. Let $F : C \rightarrow R^n$ and Γ be a compact group. The system (3.37) is said to be Γ -equivariant if $F(\gamma x_t) = \gamma F(x_t)$ for all $\gamma \in \Gamma$.

Proposition 3.9. *The nonlinear system (3.37) and linear system (3.39) are D_n equivariant.*

Proof. We begin with (3.37); that is, we let F be as in (3.38) and $\phi \in C$. We need only check the equivariant condition on the generators, ρ, κ , of D_n :

$$\begin{aligned} F_i(\rho\phi) &= -(\rho\phi)_i(0) + \alpha f((\rho\phi)_i(-\tau)) + \beta[f((\rho\phi)_{i-1}(-\tau)) + f((\rho\phi)_{i+1}(-\tau))] \\ &= -\phi_{i+1}(0) + \alpha f(\phi_{i+1}(-\tau)) + \beta[f(\phi_i(-\tau)) + f(\phi_{i+2}(-\tau))] \\ &= F_{i+1}(\phi) \\ &= \rho F_i(\phi), \\ F_i(\kappa\phi) &= -(\kappa\phi)_i(0) + \alpha f((\kappa\phi)_i(-\tau)) + \beta[f((\kappa\phi)_{i-1}(-\tau)) + f((\kappa\phi)_{i+1}(-\tau))] \\ &= -\phi_{n+2-i}(0) + \alpha f(\phi_{n+2-i}(-\tau)) + \beta[f(\phi_{n+3-i}(-\tau)) + f(\phi_{n+1-i}(-\tau))] \\ &= F_{n+2-i}(\phi) \\ &= \kappa F_i(\phi). \end{aligned} \quad (3.42)$$

Thus (3.37) is D_n equivariant.

From (3.39), we begin by noting that L may be written componentwise as follows:

$$L_i(\tau)\phi = -\phi_i(0) + a\phi_i(-\tau) + \beta[\phi_{i-1}(-\tau) + \phi_{i+1}(-\tau)], \quad i(\bmod n). \quad (3.43)$$

The rest of the proof is similar to that for (3.37). This completes the proof. \square

In the case where $(\alpha, \beta) \in E_0$, we can apply the standard Hopf bifurcation theorem of delay differential equations to obtain a Hopf bifurcation of synchronous periodic solutions. In the case where $(\alpha, \beta) \in E_j$ ($1 \leq j \leq (n-1)/2$), the aforementioned standard Hopf bifurcation theorem does not apply since $\pm i\omega_j$ are double eigenvalues. On the other hand, the considered system is equivariant with respect to the D_n -action where the Z_n subgroup acts by permutation (sending x_i to x_{i+1}) and the flip acts by interchanging (sending x_i to x_{n+2-i}). This allows us to apply the symmetric Hopf bifurcation theorem for delay differential equations established in [4] (as an extension of the well-known Golubitsky-Stewart Theorem [1] for symmetric ordinary differential equations) to obtain $2(n+1)$ branches of asynchronous periodic solutions. More precisely, we have the following theorem.

Theorem 3.10. *Assume that $(\alpha, \beta) \in D$ ($D = S/P$, where the regions S and P are defined by (3.15) and (3.16), resp.). Then*

- (1) *in case $(\alpha, \beta) \in E_0$, near $\tau = \tau_0^{(0)}$ there exists a supercritical bifurcation of stable synchronous periodic solutions of period p_τ near $(2\pi/\omega_0)$, bifurcated from the zero solution of system (1.2);*
- (2) *in case $(\alpha, \beta) \in E_j$ ($1 \leq j \leq (n-1)/2$), near $\tau = \tau_0^{(j)}$ there exist $2(n+1)$ branches of asynchronous periodic solutions of period p_τ near $(2\pi/\omega_j)$, bifurcated simultaneously from the zero solution of system (1.2), and there are*
 - (a) *two stable phase-locked oscillations: $x_i(t) = x_{i+1}(t \pm (p_\tau/n))$, for $i(\bmod n)$ and $t \in \mathbb{R}$,*
 - (b) *n unstable mirror-reflecting waves: $x_i(t) = x_{n+2k-i}(t)$ for $i(\bmod n)$ and $t \in \mathbb{R}$, where $k = 1, 2, \dots, n$,*
 - (c) *n unstable standing waves: $x_i(t) = x_{n+2k-i}(t - (p_\tau/2))$ for $i(\bmod n)$ and $t \in \mathbb{R}$, where $k = 1, 2, \dots, n$.*

Proof. (1) The existence is an immediate application of the standard Hopf bifurcation theorem for functional differential equations. Let $\tau^* = \tau_0^{(0)}$, $\mu = \tau - \tau_0^{(0)}$. According to the calculations in Part 2 of the appendix, the normal form of (1.2) on the center manifold can be written in polar coordinates as

$$\dot{\rho} = (a_1\mu + b_1\rho^2)\rho + O(\mu^2\rho) + O(\rho^4), \quad (3.44)$$

where

$$a_1 = \frac{\omega_0\tau_0^{(0)}}{(1 + \tau_0^{(0)})^2 + \omega_0^2} > 0,$$

$$b_1 = \frac{f'''(0) \tau_0^{(0)} (1 + \tau_0^{(0)} + \omega_0 \tau_0^{(0)})}{2 (1 + \tau_0^{(0)})^2 + \omega_0^2} < 0. \quad (3.45)$$

Conclusion (1) then follows immediately.

(2) Let $\tau^* = \tau_0^{(j)}$, $u^* = \omega_j / \tau_0^{(j)}$, $\mu = \tau - \tau_0^{(j)}$. We obtain from the calculations in Part 1 of the appendix the normal form of (1.2) on the center manifolds that is given by

$$\begin{aligned} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{pmatrix} &= u^* \tau^* \begin{pmatrix} \omega_2 \\ -\omega_1 \end{pmatrix} + \mu u^* \begin{pmatrix} \operatorname{Im}(a^{-1})\omega_1 + \operatorname{Re}(a^{-1})\omega_2 \\ -\operatorname{Re}(a^{-1})\omega_1 + \operatorname{Im}(a^{-1})\omega_2 \end{pmatrix} \\ &+ 3b\tau^* (\rho_1^2 + 2\rho_2^2) \begin{pmatrix} \operatorname{Re}(a^{-1}(1 - iu^*))\omega_1 - \operatorname{Im}(a^{-1}(1 - iu^*))\omega_2 \\ \operatorname{Im}(a^{-1}(1 - iu^*))\omega_1 + \operatorname{Re}(a^{-1}(1 - iu^*))\omega_2 \end{pmatrix} \\ &+ O(\mu^2|\omega|) + O(|\omega|^4), \\ \begin{pmatrix} \dot{\omega}_3 \\ \dot{\omega}_4 \end{pmatrix} &= u^* \tau^* \begin{pmatrix} \omega_4 \\ -\omega_3 \end{pmatrix} + \mu u^* \begin{pmatrix} \operatorname{Im}(a^{-1})\omega_3 + \operatorname{Re}(a^{-1})\omega_4 \\ -\operatorname{Re}(a^{-1})\omega_3 + \operatorname{Im}(a^{-1})\omega_4 \end{pmatrix} \\ &+ 3b\tau^* (\rho_2^2 + 2\rho_1^2) \begin{pmatrix} \operatorname{Re}(a^{-1}(1 - iu^*))\omega_3 - \operatorname{Im}(a^{-1}(1 - iu^*))\omega_4 \\ \operatorname{Im}(a^{-1}(1 - iu^*))\omega_3 + \operatorname{Re}(a^{-1}(1 - iu^*))\omega_4 \end{pmatrix} \\ &+ O(\mu^2|\omega|) + O(|\omega|^4), \end{aligned} \quad (3.46)$$

where

$$\begin{aligned} \rho_1 &= \sqrt{\omega_1^2 + \omega_2^2}, \\ \rho_2 &= \sqrt{\omega_3^2 + \omega_4^2}, \\ a &= 1 + \tau^* - iu^* \tau^*, \\ b &= \frac{1}{3!} f'''(0). \end{aligned} \quad (3.47)$$

Introducing the periodic-scaling parameter ω and letting

$$\begin{aligned} z_1(t) &= \omega_1(s) + i\omega_2(s), \\ z_2(t) &= \omega_3(s) + i\omega_4(s) \end{aligned} \quad (3.48)$$

with

$$s = [(1 + \omega)u^* \tau^*]^{-1} t, \quad (3.49)$$

and using a computation process similar to that of Wu et al. [24], we obtain the normal form

$$\begin{aligned}(1 + \omega)\dot{z}_1 &= -iz_1 - i\mu(\tau^*)^{-1}a^{-1}z_1 + 3b(u^*)^{-1}a^{-1}(1 - iu^*)\left(|z_1|^2 + 2|z_2|^2\right)z_1, \\ (1 + \omega)\dot{z}_2 &= -iz_2 - i\mu(\tau^*)^{-1}a^{-1}z_2 + 3b(u^*)^{-1}a^{-1}(1 - iu^*)\left(|z_2|^2 + 2|z_1|^2\right)z_2.\end{aligned}\tag{3.50}$$

Let $g : C \oplus C \oplus R \rightarrow C \oplus C$ be given so that $-g(z_1, z_2, \mu)$ is the right-hand side of (3.50). Then (3.50) can be written as

$$(1 + \omega)\dot{z} + g(z, \mu) = 0.\tag{3.51}$$

Note that

$$D_z g(0, 0)(z_1, z_2) = i(z_1, z_2), \quad z = (z_1, z_2) \in C \oplus C.\tag{3.52}$$

Also note that $g(\cdot, \mu) : C \oplus C \rightarrow C \oplus C$ is $D_n \times S^1$ -equivariant with respect to the following $D_n \times S^1$ -action on $C \oplus C$:

$$\begin{aligned}\rho(z_1, z_2) &= \left(e^{i(2k\pi/n)}z_1, e^{-i(2k\pi/n)}z_2\right), \quad Z_n = \langle \rho \rangle \leq D_n, \\ \kappa(z_1, z_2) &= (z_2, z_1), \quad Z_2 = \langle \kappa \rangle \leq D_n, \\ e^{i\theta}(z_1, z_2) &= \left(e^{i\theta}z_1, e^{i\theta}z_2\right), \quad e^{i\theta} \in S^1.\end{aligned}\tag{3.53}$$

Using a computation process similar to that of Wu et al. [24], we obtain

$$\begin{aligned}A_0 &= iu^*(\tau^*)^{-1}a^{-1} - i\omega, \\ A_N &= -6b(u^*)^{-1}a^{-1}(1 - iu^*), \\ B_0 &= 3b(u^*)^{-1}a^{-1}(1 - iu^*).\end{aligned}\tag{3.54}$$

By the results of [1, page 376], we know that the bifurcation of phase-locked oscillation is supercritical (resp., subcritical) and depends on whether $\text{Re}(A_N + B_0) > 0$ (resp., $\text{Re}(A_N + B_0) < 0$), and these are orbitally asymptotically stable if $\text{Re}(A_N + B_0) > 0$ and $\text{Re} B_0 < 0$.

Note that

$$\begin{aligned}
\operatorname{Re}(A_N + B_0) &= \operatorname{Re}\left(-3b(u^*)^{-1}a^{-1}(1 - iu^*)\right) \\
&= -3b(u^*)^{-1} \operatorname{Re}\left[\frac{1 + \tau^* + iu^*\tau^*}{(1 + \tau^*)^2 + (u^*\tau^*)^2}(1 - iu^*)\right] \\
&= -3b(u^*)^{-1} \frac{1 + \tau^* + (u^*)^2\tau^*}{(1 + \tau^*)^2 + (u^*\tau^*)^2} \\
&> 0, \\
\operatorname{Re}(B_0) &= \operatorname{Re}\left(3b(u^*)^{-1}a^{-1}(1 - iu^*)\right) \\
&= 3b(u^*)^{-1} \frac{1 + \tau^* + (u^*)^2\tau^*}{(1 + \tau^*)^2 + (u^*\tau^*)^2} \\
&< 0.
\end{aligned} \tag{3.55}$$

Consequently, the bifurcation of phase-locked oscillations is supercritical and orbitally asymptotically stable.

Note also that

$$\begin{aligned}
\operatorname{Re}(2A_N + B_0) &= \operatorname{Re}\left(-9b(u^*)^{-1}a^{-1}(1 - iu^*)\right) \\
&= -9b(u^*)^{-1} \frac{1 + \tau^* + (u^*)^2\tau^*}{(1 + \tau^*)^2 + (u^*\tau^*)^2} \\
&> 0, \\
\operatorname{Re}(B_0) &< 0.
\end{aligned} \tag{3.56}$$

We infer from the results of [1, page 376] again that the bifurcations of mirror-reflecting waves and standing waves are supercritical and unstable. This completes the proof. \square

Using a proof process similar to that of Yuan and Campbell [25], by using Liapunov's second method we can obtain the followings.

Theorem 3.11. *If $|\alpha| + 2|\beta| < 1$, then the trivial solution of (1.2) is global asymptotically stable.*

4. Computer Simulations

To demonstrate the properties of the Hopf bifurcation in Section 3, we carry out some numerical simulations for a particular case of (1.2) as in following form:

$$\dot{x}_i(t) = -x_i(t) + \alpha[\tanh(x_i(t - \tau))] + \beta[\tanh(x_{i-1}(t - \tau)) + \tanh(x_{i+1}(t - \tau))], \quad i(\bmod 3). \tag{4.1}$$

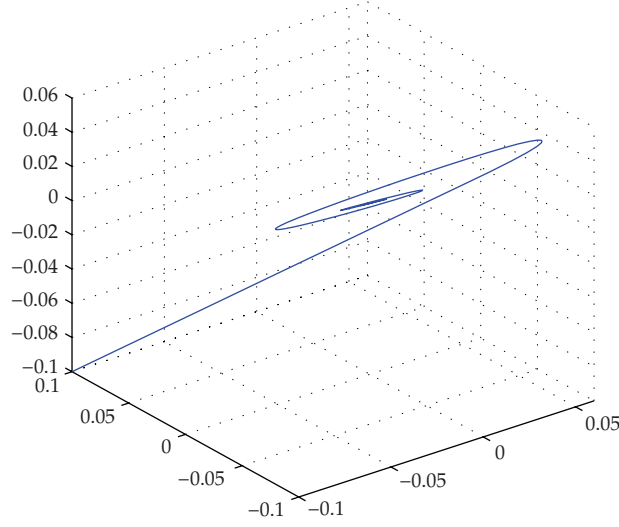


Figure 2: Simulations for the asymptotically stable equilibrium to system (4.1) with $\tau = 0.5 < \tau_0 = 0.955185$.

Clearly, the origin $(0, 0, 0)$ is a fixed point to (4.1). Choosing $\alpha = -2$ and $\beta = 1/3$ we obtain that $\tau_0^{(0)} = 2.73273$ and $\tau_0^{(1)} = 0.955185$, thus $\tau_0 = 0.955185$.

From Theorem 3.7(iii), it follows that the zero equilibrium is asymptotically stable if $\tau \in [0, \tau_0)$, it is unstable if $\tau > \tau_0$, and the system undergoes a Hopf bifurcation with $\tau = \tau_k^{(j)}$ ($j = 0, 1, \dots, (n-1)/2$; $k = 0, 1, \dots$). Simulate the solutions of system (4.1) for $\tau = 0.5$ and 1.0 . In Figure 2, it is shown that the zero equilibrium is asymptotically stable for $\tau = 0.5 < \tau_0 = 0.955185$. In Figure 3, for the data $\tau = 1.0 > \tau_0 = 0.955185$, it is shown that there exists a periodic orbit which is orbitally asymptotically stable.

Appendix

A. The Calculation of Normal Forms on Center Manifolds with n being an Odd Number

In this appendix, we employ the algorithm and notations of Faria and Magalhães [26] to derive the normal forms of system (1.2) on the center manifolds.

We first rescale the time by $t \mapsto (t/\tau)$ to normalize the delay so that (1.2) can be written as

$$\dot{x}(t) = F(x_t, \tau) \quad (\text{A.1})$$

in the phase space $C = C([-1, 0]; R^n)$, where for $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T \in C$, we have

$$(F(\phi, \tau))_i = -\tau\phi_i(0) + \alpha\tau f(\phi_i(-1)) + \beta\tau [f(\phi_{i-1}(-1)) + f(\phi_{i+1}(-1))] \quad (\text{A.2})$$

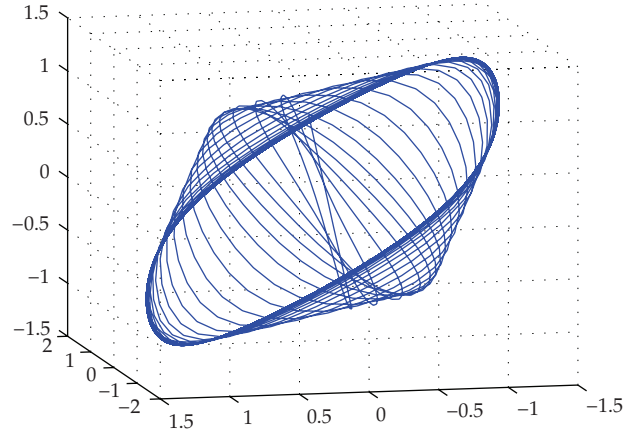


Figure 3: Simulations for an orbitally asymptotically stable periodic orbit to system (4.1) with $\tau = 1.0 > \tau_0 = 0.99185$.

with $i(\text{mod } n)$. We also assume that

$$f(x) = x + bx^3 + \text{h.o.t.} \tag{A.3}$$

with

$$b = \frac{1}{3!} f'''(0). \tag{A.4}$$

The linearized equation at zero for system (A.1) is

$$\dot{x}(t) = L(\tau)x_t, \tag{A.5}$$

where

$$L(\tau)(\phi) = -\tau\phi(0) + \tau\alpha\phi(-1) + \tau\beta\delta(\phi(-1)),$$

$$\delta = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}. \tag{A.6}$$

The characteristic equation of (A.5) at $(0, 0, \dots, 0)^T$ is $\det M_n(0, \lambda/\tau) = 0$, that is,

$$\left[\frac{\lambda}{\tau} + 1 - (\alpha + 2\beta)e^{-\lambda} \right] \prod_{j=1}^{(n-1)/2} \left[\frac{\lambda}{\tau} + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-\lambda} \right]^2 = 0, \quad (\text{A.7})$$

where $\det M_n(0, \lambda) = 0$ is the characteristic equation of the linearization of (1.2) at $(0, 0, \dots, 0)^T$.

Part 1.

Case 1 $((\alpha, \beta) \in E_j (1 \leq j \leq (n-1)/2)$. In this case, at $\tau = \tau^*$ the characteristic equation of (A.5) has imaginary zeros $\pm iu^*\tau^*$ which are double, where $\tau^* = \tau_0^{(j)}$ and

$$u^*\tau^* = \omega_j = \sqrt{\left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right)^2 - 1}. \quad (\text{A.8})$$

Since

$$M_n(0, iu^*)v_j = \left[iu^* + 1 - \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) e^{-iu^*\tau^*} \right] v_j = 0, \quad (\text{A.9})$$

the center space at $\tau = \tau^*$ and in complex coordinates is $X = \text{span}(\phi_1, \phi_2, \phi_3, \phi_4)$, where

$$\begin{aligned} \phi_1(\theta) &= e^{iu^*\tau^*\theta} v_j, & \theta \in [-1, 0], \\ \phi_2(\theta) &= e^{-iu^*\tau^*\theta} \bar{v}_j, & \theta \in [-1, 0], \\ \phi_3(\theta) &= e^{iu^*\tau^*\theta} \bar{v}_j, & \theta \in [-1, 0], \\ \phi_4(\theta) &= e^{-iu^*\tau^*\theta} v_j, & \theta \in [-1, 0], \end{aligned} \quad (\text{A.10})$$

$$v_j = \left(1, e^{(2\pi i/n)j}, e^{(2\pi i/n)2j}, \dots, e^{(2\pi i/n)(n-1)j} \right)^T, \quad j = 1, 2, \dots, n-1.$$

Let

$$\Phi = (\phi_1, \phi_2, \phi_3, \phi_4). \quad (\text{A.11})$$

Note that

$$\bar{v}_i^T v_j = \begin{cases} n, & i = j, \\ 0, & i \neq j, \quad i, j = 1, 2, \dots, n-1. \end{cases} \quad (\text{A.12})$$

$$\bar{v}_j = v_{n-j}.$$

It is easy to check that a basis for the adjoint space X^* is

$$\Psi(s) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \\ \psi_3(s) \\ \psi_4(s) \end{pmatrix} = \frac{1}{n} \begin{pmatrix} (\bar{a})^{-1} e^{-iu^* \tau^* s} \bar{v}_j^T \\ a^{-1} e^{iu^* \tau^* s} v_j^T \\ (\bar{a})^{-1} e^{-iu^* \tau^* s} v_j^T \\ a^{-1} e^{iu^* \tau^* s} \bar{v}_j^T \end{pmatrix}, \quad s \in [0, 1] \quad (\text{A.13})$$

with $(\Psi, \Phi) = \text{Id}$ (the 4×4 identity matrix) for the adjoint bilinear form on $C^* \times C$ defined in [27], where

$$a = 1 + \tau^* - iu^* \tau^*. \quad (\text{A.14})$$

It is useful to note the following:

$$\Psi(0) = \frac{1}{n} \begin{pmatrix} (\bar{a})^{-1} \bar{v}_j^T \\ a^{-1} v_j^T \\ (\bar{a})^{-1} v_j^T \\ a^{-1} \bar{v}_j^T \end{pmatrix}, \quad (\text{A.15})$$

and for $x \in C^4$, we have

$$\begin{aligned} \Phi(0)x &= [v_j, \bar{v}_j, \bar{v}_j, v_j]x = (x_1 + x_4)v_j + (x_2 + x_3)\bar{v}_j, \\ \Phi(-1)x &= \left(e^{-iu^* \tau^*} x_1 + e^{iu^* \tau^*} x_4 \right) v_j + \left(e^{iu^* \tau^*} x_2 + e^{-iu^* \tau^*} x_3 \right) \bar{v}_j, \\ \delta(\Phi(-1)x) &= \left(e^{-iu^* \tau^*} x_1 + e^{iu^* \tau^*} x_4 \right) \delta v_j + \left(e^{iu^* \tau^*} x_2 + e^{-iu^* \tau^*} x_3 \right) \delta \bar{v}_j = 2 \cos \frac{2\pi j}{n} \Phi(-1)x. \end{aligned} \quad (\text{A.16})$$

Introducing the new parameter

$$\mu = \tau - \tau^*, \quad (\text{A.17})$$

we can rewrite (A.1) as

$$\dot{z}(t) = L(\tau^*)z_t + G(z_t, \mu), \quad (\text{A.18})$$

where

$$G(z_t, \mu) = L(\mu)z_t + b\alpha(\tau^* + \mu) \begin{pmatrix} z_1^3(t-1) \\ z_2^3(t-1) \\ \vdots \\ z_n^3(t-1) \end{pmatrix} + b\beta(\tau^* + \mu) \begin{pmatrix} z_2^3(t-1) + z_n^3(t-1) \\ z_1^3(t-1) + z_3^3(t-1) \\ \vdots \\ z_1^3(t-1) + z_{n-1}^3(t-1) \end{pmatrix} + \text{h.o.t.} \quad (\text{A.19})$$

Define the 4×4 matrix

$$B = iu^* \tau^* \text{diag}(1, -1, 1, -1). \quad (\text{A.20})$$

Using the decomposition $z_t = \Phi x(t) + y_t$, we can decompose (A.18) as

$$\begin{aligned} \dot{x} &= Bx + \Psi(0)G(\Phi x + y, \mu), \\ \dot{y} &= A_Q y + (I - \pi)X_0 G(\Phi x + y, \mu) \end{aligned} \quad (\text{A.21})$$

with $x \in C^4$, $y \in Q^1$. Here and throughout this appendix, we refer the readers to [26] for explanations of several notations involved. We will write the Taylor expansion

$$\Psi(0)G(\Phi x + y, \mu) = \sum_{j \geq 2} \frac{1}{j!} f_j^1(x, y, \mu), \quad (\text{A.22})$$

where $f_j^1(x, y, \mu)$ are homogeneous polynomials of degree j in (x, y, μ) with coefficients in C^4 . Then the normal form of (1.2) on the center manifold of the origin at $\mu = 0$ is given by

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x, 0, \mu) + \frac{1}{3!}g_3^1(x, 0, \mu) + \text{h.o.t.}, \quad (\text{A.23})$$

where g_2^1 and g_3^1 will be calculated in the following part of this section.

First of all, using $\delta(\Phi(-1)x) = 2 \cos(2\pi j/n)\Phi(-1)x$, we find that

$$\begin{aligned}
\frac{1}{2}f_2^1(x, 0, \mu) &= \Psi(0)L(\mu)(\Phi x) \\
&= \Psi(0)\mu[-\Phi(0)x + \alpha\Phi(-1)x + \beta\delta(\Phi(-1)x)] \\
&= \Psi(0)\mu\left\{[-(x_1 + x_4)v_j - (x_2 + x_3)\bar{v}_j] + \left(\alpha + 2\beta \cos \frac{2\pi j}{n}\right)\right. \\
&\quad \left.\times \left[\left(e^{-iu^*\tau^*}x_1 + e^{iu^*\tau^*}x_4\right)v_j + \left(e^{iu^*\tau^*}x_2 + e^{-iu^*\tau^*}x_3\right)\bar{v}_j\right]\right\} \quad (\text{A.24}) \\
&= i\mu u^*\Psi(0)\left[(x_1 - x_4)v_j + (-x_2 + x_3)\bar{v}_j\right] \\
&= i\mu u^*\begin{pmatrix} (\bar{a})^{-1}(x_1 - x_4) \\ a^{-1}(-x_2 + x_3) \\ (\bar{a})^{-1}(-x_2 + x_3) \\ a^{-1}(x_1 - x_4) \end{pmatrix}.
\end{aligned}$$

These are the second-order terms in (μ, x) of (A.22), and following Faria and Magalhaes [26], we have the second-order terms in (μ, x) of the normal form on center manifold as follows:

$$\frac{1}{2}g_2^1(x, 0, \mu) = \text{Proj}_{\ker(M_2^1)} \frac{1}{2}f_2^1(x, 0, \mu). \quad (\text{A.25})$$

Here we recall that

$$M_j^1(p)(x, \mu) = D_x p(x, \mu)Bx - Bp(x, \mu), \quad j \geq 2. \quad (\text{A.26})$$

In particular,

$$M_j^1(\mu x^q e_k) = iu^*\tau^*\mu\left(q_1 - q_2 + q_3 - q_4 + (-1)^k\right)x^q e_k, \quad |q| = j - 1, \quad (\text{A.27})$$

where $j \geq 2$, $1 \leq k \leq 4$, and $\{e_1, e_2, e_3, e_4\}$ are the canonical basis for C^4 . Therefore, if $|q| = 1$, then

$$\begin{aligned}
&\ker(M_2^1) \cap \text{span}\{\mu x^q e_k : |q| = 1, k = 1, 2, 3, 4\} \\
&= \text{span}\{\mu x_1 e_1, \mu x_3 e_1, \mu x_2 e_2, \mu x_4 e_2, \mu x_1 e_3, \mu x_3 e_3, \mu x_2 e_4, \mu x_4 e_4\}, \\
&\frac{1}{2}g_2^1(x, 0, \mu) = i\mu u^*\begin{pmatrix} (\bar{a})^{-1}x_1 \\ -a^{-1}x_2 \\ (\bar{a})^{-1}x_3 \\ -a^{-1}x_4 \end{pmatrix}. \quad (\text{A.28})
\end{aligned}$$

To compute $g_3^1(x, 0, \mu)$, we first note that from (A.26) it follows that

$$\begin{aligned} g_3^1(x, 0, \mu) &= \text{Proj}_{\ker(M_3^1)} \overline{f_3^1}(x, 0, \mu) \\ &= \text{Proj}_{\ker(M_3^1)} \overline{f_3^1}(x, 0, 0) + O(\mu^2|x|), \end{aligned} \quad (\text{A.29})$$

since $\mu x^q e_j$ is not in $\text{Ker}(M_3^1)$, for $|q| = 2, j = 1, 2, 3, 4$. Next, we define

$$\overline{f_3^1}(x, 0, \mu) = f_3^1(x, 0, \mu) + \frac{3}{2} \left[(D_x f_2^1) U_2^1 - (D_x U_2^1) g_2^1 \right] (x, 0, \mu) + \frac{3}{2} \left[(D_y f_2^1) h \right] (x, 0, \mu), \quad (\text{A.30})$$

where U_2^1 is the change of variables associated with the transformation from f_2^1 to g_2^1 , and h is such that $M_2^2(h) = g_2^2$, that is, $h = U_2^2$ is the change of variables associated with the transformation of the second-order terms in the second equation of system (A.21). For $\mu = 0$, $f_2^1(x, 0, 0) = g_2^1(x, 0, 0) = 0$, and we have simply

$$\frac{1}{3!} \overline{f_3^1}(x, 0, 0) = \frac{1}{3!} f_3^1(x, 0, 0) = b\tau^* \Psi(0) \left[\beta \delta \left((\Phi(-1)x)^3 \right) + \alpha (\Phi(-1)x)^3 \right], \quad (\text{A.31})$$

where we utilized the following notations:

$$(\Phi(-1)x)^3 = \begin{pmatrix} ((\Phi(-1)x)_1)^3 \\ ((\Phi(-1)x)_2)^3 \\ \vdots \\ ((\Phi(-1)x)_n)^3 \end{pmatrix}. \quad (\text{A.32})$$

Let

$$A_1 = e^{-iu^* \tau^*} x_1 + e^{iu^* \tau^*} x_4, \quad A_2 = e^{iu^* \tau^*} x_2 + e^{-iu^* \tau^*} x_3. \quad (\text{A.33})$$

We have

$$\begin{aligned}
(\Phi(-1)x)^3 &= \begin{pmatrix} (A_1 + A_2)^3 \\ (e^{(2\pi i/n)j} A_1 + e^{-(2\pi i/n)j} A_2)^3 \\ (e^{(2\pi i/n)2j} A_1 + e^{-(2\pi i/n)2j} A_2)^3 \\ \vdots \\ (e^{(2\pi i/n)(n-1)j} A_1 + e^{-(2\pi i/n)(n-1)j} A_2)^3 \end{pmatrix} \\
&= A_1^3 v_j^3 + 3A_1^2 A_2 v_j + A_2^3 \bar{v}_j^3 + 3A_1 A_2^2 \bar{v}_j, \\
\delta(\Phi(-1)x)^3 &= \begin{pmatrix} (e^{(2\pi i/n)j} A_1 + e^{-(2\pi i/n)j} A_2)^3 + (e^{(2\pi i/n)(n-1)j} A_1 + e^{-(2\pi i/n)(n-1)j} A_2)^3 \\ (A_1 + A_2)^3 + (e^{(2\pi i/n)2j} A_1 + e^{-(2\pi i/n)2j} A_2)^3 \\ (e^{(2\pi i/n)j} A_1 + e^{-(2\pi i/n)j} A_2)^3 + (e^{(2\pi i/n)3j} A_1 + e^{-(2\pi i/n)3j} A_2)^3 \\ \vdots \\ (A_1 + A_2)^3 + (e^{(2\pi i/n)(n-2)j} A_1 + e^{-(2\pi i/n)(n-2)j} A_2)^3 \end{pmatrix}.
\end{aligned} \tag{A.34}$$

Then, we have

$$\begin{aligned}
\Psi(0)(\Phi(-1)x)^3 &= \frac{1}{n} \begin{pmatrix} (\bar{a})^{-1} \bar{v}_j^T \\ a^{-1} v_j^T \\ (\bar{a})^{-1} v_j^T \\ a^{-1} \bar{v}_j^T \end{pmatrix} (A_1^3 v_j^3 + 3A_1^2 A_2 v_j + A_2^3 \bar{v}_j^3 + 3A_1 A_2^2 \bar{v}_j) \\
&= 3 \begin{pmatrix} (\bar{a})^{-1} A_1^2 A_2 \\ a^{-1} A_1 A_2^2 \\ (\bar{a})^{-1} A_1 A_2^2 \\ a^{-1} A_1^2 A_2 \end{pmatrix}, \\
\Psi(0)\delta(\Phi(-1)x)^3 &= [(\Psi(0)\delta)^T]^T (\Phi(-1)x)^3 \\
&= [\delta(\Psi(0))^T]^T (\Phi(-1)x)^3
\end{aligned}$$

$$\begin{aligned}
&= \left[\frac{1}{n} \delta \left((\bar{a})^{-1} \bar{v}_j, a^{-1} v_j, (\bar{a})^{-1} v_j, a^{-1} \bar{v}_j \right) \right]^T (\Phi(-1)x)^3 \\
&= \left[\frac{1}{n} 2 \cos \frac{2\pi j}{n} \left((\bar{a})^{-1} \bar{v}_j, a^{-1} v_j, (\bar{a})^{-1} v_j, a^{-1} \bar{v}_j \right) \right]^T (\Phi(-1)x)^3 \\
&= \frac{1}{n} 2 \cos \frac{2\pi j}{n} \begin{pmatrix} (\bar{a})^{-1} \bar{v}_j^T \\ a^{-1} v_j^T \\ (\bar{a})^{-1} v_j^T \\ a^{-1} \bar{v}_j^T \end{pmatrix} \left(A_1^3 v_j^3 + 3A_1^2 A_2 v_j + A_2^3 \bar{v}_j^3 + 3A_1 A_2^2 \bar{v}_j \right) \\
&= 6 \cos \frac{2\pi j}{n} \begin{pmatrix} (\bar{a})^{-1} A_1^2 A_2 \\ a^{-1} A_1 A_2^2 \\ (\bar{a})^{-1} A_1 A_2^2 \\ a^{-1} A_1^2 A_2 \end{pmatrix}.
\end{aligned} \tag{A.35}$$

Then, we have

$$\begin{aligned}
\frac{1}{3!} f_3^1(x, 0, 0) &= b\tau^* \Psi(0) \left[\beta \delta \left((\Phi(-1)x)^3 \right) + \alpha (\Phi(-1)x)^3 \right] \\
&= b\tau^* 3 \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) \begin{pmatrix} (\bar{a})^{-1} A_1^2 A_2 \\ a^{-1} A_1 A_2^2 \\ (\bar{a})^{-1} A_1 A_2^2 \\ a^{-1} A_1^2 A_2 \end{pmatrix}.
\end{aligned} \tag{A.36}$$

Note that

$$\begin{aligned}
A_1^2 A_2 &= \left(e^{-2iu^* \tau^*} x_1^2 + 2x_1 x_4 + e^{2iu^* \tau^*} x_4^2 \right) \left(e^{iu^* \tau^*} x_2 + e^{-iu^* \tau^*} x_3 \right), \\
A_1 A_2^2 &= \left(e^{-iu^* \tau^*} x_1 + e^{iu^* \tau^*} x_4 \right) \left(e^{2iu^* \tau^*} x_2^2 + 2x_2 x_3 + e^{-2iu^* \tau^*} x_3^2 \right).
\end{aligned} \tag{A.37}$$

Also note that

$$M_3^1(x^q e_j) = 0, \quad \text{with } |q| = 3 \text{ iff } q_1 - q_2 + q_3 - q_4 + (-1)^j = 0, \quad j = 1, 2, 3, 4. \tag{A.38}$$

Then

$$\begin{aligned}
\frac{1}{3!}g_3^1(x, 0, 0) &= \text{Proj}_{\ker(M_2^1)} \frac{1}{3!}f_3^1(x, 0, 0) \\
&= 3b\tau^* \left(\alpha + 2\beta \cos \frac{2\pi j}{n} \right) \begin{pmatrix} (\bar{a})^{-1} e^{-iu^*\tau^*} (x_1^2 x_2 + 2x_1 x_3 x_4) \\ a^{-1} e^{iu^*\tau^*} (x_1 x_2^2 + 2x_2 x_3 x_4) \\ (\bar{a})^{-1} e^{-iu^*\tau^*} (x_3^2 x_4 + 2x_1 x_2 x_3) \\ a^{-1} e^{iu^*\tau^*} (x_3 x_4^2 + 2x_1 x_2 x_4) \end{pmatrix} \\
&= 3b\tau^* \begin{pmatrix} (\bar{a})^{-1} (1 + iu^*) x_1 (x_1 x_2 + 2x_3 x_4) \\ a^{-1} (1 - iu^*) x_2 (x_1 x_2 + 2x_3 x_4) \\ (\bar{a})^{-1} (1 + iu^*) x_3 (x_3 x_4 + 2x_1 x_2) \\ a^{-1} (1 - iu^*) x_4 (x_3 x_4 + 2x_1 x_2) \end{pmatrix}.
\end{aligned} \tag{A.39}$$

Consequently, the normal form on the center manifold becomes

$$\begin{aligned}
\dot{x} &= iu^*\tau^* \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} x + i\mu u^* \begin{pmatrix} (\bar{a})^{-1} x_1 \\ -a^{-1} x_2 \\ (\bar{a})^{-1} x_3 \\ -a^{-1} x_4 \end{pmatrix} \\
&+ \frac{1}{3!}g_3^1(x, 0, 0) + O(\mu^2|x|) + O(|x|^4)
\end{aligned} \tag{A.40}$$

for $x \in \mathbb{C}^4$. Changing to real coordinates by the change of variables

$$x = S\omega, \quad \text{with } S = \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \end{pmatrix} \tag{A.41}$$

and letting

$$\begin{aligned}
\rho_1^2 &= x_1 x_2 = \omega_1^2 + \omega_2^2, \\
\rho_2^2 &= x_3 x_4 = \omega_3^2 + \omega_4^2,
\end{aligned} \tag{A.42}$$

we obtain

$$\begin{aligned}
\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{pmatrix} &= u^* \tau^* \begin{pmatrix} \omega_2 \\ -\omega_1 \end{pmatrix} + \mu u^* \begin{pmatrix} \operatorname{Im}(a^{-1})\omega_1 + \operatorname{Re}(a^{-1})\omega_2 \\ -\operatorname{Re}(a^{-1})\omega_1 + \operatorname{Im}(a^{-1})\omega_2 \end{pmatrix} \\
&+ 3b\tau^* (\rho_1^2 + 2\rho_2^2) \begin{pmatrix} \operatorname{Re}(a^{-1}(1-iu^*))\omega_1 - \operatorname{Im}(a^{-1}(1-iu^*))\omega_2 \\ \operatorname{Im}(a^{-1}(1-iu^*))\omega_1 + \operatorname{Re}(a^{-1}(1-iu^*))\omega_2 \end{pmatrix} \\
&+ O(\mu^2|\omega|) + O(|\omega|^4), \\
\begin{pmatrix} \dot{\omega}_3 \\ \dot{\omega}_4 \end{pmatrix} &= u^* \tau^* \begin{pmatrix} \omega_4 \\ -\omega_3 \end{pmatrix} + \mu u^* \begin{pmatrix} \operatorname{Im}(a^{-1})\omega_3 + \operatorname{Re}(a^{-1})\omega_4 \\ -\operatorname{Re}(a^{-1})\omega_3 + \operatorname{Im}(a^{-1})\omega_4 \end{pmatrix} \\
&+ 3b\tau^* (\rho_2^2 + 2\rho_1^2) \begin{pmatrix} \operatorname{Re}(a^{-1}(1-iu^*))\omega_3 - \operatorname{Im}(a^{-1}(1-iu^*))\omega_4 \\ \operatorname{Im}(a^{-1}(1-iu^*))\omega_3 + \operatorname{Re}(a^{-1}(1-iu^*))\omega_4 \end{pmatrix} \\
&+ O(\mu^2|\omega|) + O(|\omega|^4).
\end{aligned} \tag{A.43}$$

If we use double polar coordinates

$$\begin{aligned}
\omega_1 &= \rho_1 \cos \chi_1, \\
\omega_2 &= \rho_1 \sin \chi_1, \\
\omega_3 &= \rho_2 \cos \chi_2, \\
\omega_4 &= \rho_2 \sin \chi_2,
\end{aligned} \tag{A.44}$$

then we find

$$\begin{aligned}
\dot{\rho}_1 &= (a_1\mu + b_1\rho_1^2 + 2b_1\rho_2^2)\rho_1 + O(\mu^2|(\rho_1, \rho_2)|) + O(|(\rho_1, \rho_2)|^4), \\
\dot{\rho}_2 &= (a_1\mu + b_1\rho_2^2 + 2b_1\rho_1^2)\rho_2 + O(\mu^2|(\rho_1, \rho_2)|) + O(|(\rho_1, \rho_2)|^4), \\
\dot{\chi}_1 &= -u^* \tau^* + c_1\mu + d_1\rho_1^2 + 2d_1\rho_2^2 + O(\mu^2|(\rho_1, \rho_2)|) + O(|(\rho_1, \rho_2)|^4), \\
\dot{\chi}_2 &= -u^* \tau^* + c_1\mu + d_1\rho_2^2 + 2d_1\rho_1^2 + O(\mu^2|(\rho_1, \rho_2)|) + O(|(\rho_1, \rho_2)|^4),
\end{aligned} \tag{A.45}$$

with

$$\begin{aligned}
a_1 &= \operatorname{Im}(a^{-1})u^*, & b_1 &= \operatorname{Re}(a^{-1}(1-iu^*))3b\tau^*, \\
c_1 &= -u^* \operatorname{Re}(a^{-1}), & d_1 &= 3b\tau^* \operatorname{Im}(a^{-1}(1-iu^*)).
\end{aligned} \tag{A.46}$$

As

$$a^{-1} = \frac{1 + \tau^* + iu^*\tau^*}{(1 + \tau^*)^2 + (u^*\tau^*)^2}, \quad (\text{A.47})$$

we have

$$\begin{aligned} a_1 &= \frac{(u^*)^2\tau^*}{(1 + \tau^*)^2 + (u^*\tau^*)^2}, \\ b_1 &= 3b\tau^* \frac{1 + \tau^* + (u^*)^2\tau^*}{(1 + \tau^*)^2 + (u^*\tau^*)^2}, \\ c_1 &= -\frac{u^*(1 + \tau^*)}{(1 + \tau^*)^2 + (u^*\tau^*)^2}, \\ d_1 &= -3b \frac{u^*\tau^*}{(1 + \tau^*)^2 + (u^*\tau^*)^2}. \end{aligned} \quad (\text{A.48})$$

Part 2.

Case 2 $((\alpha, \beta) \in E_0)$. In this case, at $\tau = \tau^*$ the characteristic equation of (A.5) has imaginary zeros $\pm iu^*\tau^*$ which are simple, where $\tau^* = \tau_0^{(0)}$ and

$$u^*\tau^* = \omega_0 = \sqrt{(\alpha + 2\beta)^2 - 1} \quad (\text{A.49})$$

are given by (3.5) and (3.7), respectively. Since

$$M_n(0, iu^*)v_0 = \left[iu^* + 1 - (\alpha + 2\beta)e^{-iu^*\tau^*} \right] v_0 = 0, \quad (\text{A.50})$$

where $v_0 = (1, 1, \dots, 1)^T$, the center space at $\tau = \tau^*$ and in complex coordinates is $X = \text{span}(\phi_1, \phi_2)$, where

$$\begin{aligned} \phi_1(\theta) &= e^{iu^*\tau^*\theta} v_0, \\ \phi_2(\theta) &= e^{-iu^*\tau^*\theta} v_0, \end{aligned} \quad \theta \in [-1, 0]. \quad (\text{A.51})$$

Let

$$\Phi = (\phi_1, \phi_2). \quad (\text{A.52})$$

From $v_0^T v_0 = n$, it is easy to check that a basis for the adjoint space X^* is

$$\Psi(s) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} = \frac{1}{n} \begin{pmatrix} (\bar{a})^{-1} e^{-iu^*\tau^*s} v_0^T \\ a^{-1} e^{iu^*\tau^*s} v_0^T \end{pmatrix}, \quad s \in [0, 1] \quad (\text{A.53})$$

with $(\Psi, \Phi) = \text{Id}$ (the 2×2 identity matrix) for the adjoint bilinear form on $C^* \times C$ defined in [27], where

$$a = 1 + \tau^* - iu^* \tau^*. \quad (\text{A.54})$$

It is useful to note the following:

$$\Psi(0) = \frac{1}{n} \begin{pmatrix} (\bar{a})^{-1} v_0^T \\ a^{-1} v_0^T \end{pmatrix}, \quad (\text{A.55})$$

and for $x \in C^2$, we have

$$\begin{aligned} \Phi(0)x &= [v_0, v_0] \quad x = (x_1 + x_2)v_0, \\ \Phi(-1)x &= \left(e^{-iu^* \tau^*} x_1 + e^{iu^* \tau^*} x_2 \right) v_0, \\ \delta(\Phi(-1))x &= 2 \left(e^{-iu^* \tau^*} x_1 + e^{iu^* \tau^*} x_2 \right) v_0. \end{aligned} \quad (\text{A.56})$$

For the new parameter $\mu = \tau - \tau^*$ and decomposition $z_t = \Phi x(t) + y_t$, $x \in C^2$, $y \in Q^1$, and with

$$B = \text{diag}(iu^* \tau^*, -iu^* \tau^*), \quad (\text{A.57})$$

the normal form of (1.2) on the center manifold of the origin at $\mu = 0$ is

$$\dot{x} = Bx + \frac{1}{2} g_2^1(x, 0, \mu) + \frac{1}{3!} g_3^1(x, 0, \mu) + \text{h.o.t.}, \quad (\text{A.58})$$

and we will compute the second- and third-order terms, that is, $g_2^1(x, 0, \mu)$ and $g_3^1(x, 0, \mu)$, as we have done above for Case 2 of Theorem 3.10 We have

$$\begin{aligned} \frac{1}{2} f_2^1(x, 0, \mu) &= \Psi(0)L(\mu)(\Phi x) \\ &= \Psi(0)\mu [-\Phi(0)x + \alpha\Phi(-1)x + \beta\delta(\Phi(-1)x)] \\ &= \Psi(0)\mu \left[-(x_1 + x_2) + (\alpha + 2\beta) \left(e^{-iu^* \tau^*} x_1 + e^{iu^* \tau^*} x_2 \right) \right] v_0 \\ &= \Psi(0)\mu \left[-(x_1 + x_2) + (1 + iu^*)x_1 + (1 - iu^*)x_2 \right] v_0 \\ &= i\mu u^* \begin{pmatrix} (\bar{a})^{-1} (x_1 - x_2) \\ a^{-1} (x_1 - x_2) \end{pmatrix}. \end{aligned} \quad (\text{A.59})$$

Since

$$\begin{aligned} \frac{1}{2}g_2^1(x, 0, \mu) &= \text{Proj}_{\ker(M_2^1)} \frac{1}{2}f_2^1(x, 0, \mu), \\ M_j^1(\mu x^q e_k) &= iu^* \tau^* \mu (q_1 - q_2 + (-1)^k) x^q e_k, \quad |q| = j - 1, k = 1, 2, j \geq 2 \end{aligned} \quad (\text{A.60})$$

for the canonical basis $\{e_1, e_2\}$ for C^2 , then

$$\begin{aligned} \ker(M_2^1) \cap \text{span}\{\mu x^q e_k : |q| = 1, k = 1, 2\} &= \text{span}\{\mu x_1 e_1, \mu x_2 e_2\}, \\ \frac{1}{2}g_2^1(x, 0, \mu) &= i\mu u^* \begin{pmatrix} (\bar{a})^{-1} x_1 \\ -a^{-1} x_2 \end{pmatrix}. \end{aligned} \quad (\text{A.61})$$

As for the previous case and for similar reasons,

$$\begin{aligned} g_3^1(x, 0, \mu) &= \text{Proj}_{\ker(M_3^1)} \overline{f_3^1}(x, 0, \mu) \\ &= \text{Proj}_{\ker(M_3^1)} f_3^1(x, 0, 0) + O(\mu^2 |x|), \end{aligned} \quad (\text{A.62})$$

where

$$\begin{aligned} \frac{1}{3!} f_3^1(x, 0, 0) &= b\tau^* \Psi(0) \left[\beta \delta \left((\Phi(-1)x)^3 \right) + \alpha (\Phi(-1)x)^3 \right] \\ &= b\tau^* \Psi(0) (\alpha + 2\beta) \left(e^{-iu^* \tau^*} x_1 + e^{iu^* \tau^*} x_2 \right)^3 v_0 \\ &= b\tau^* (\alpha + 2\beta) \left(e^{-iu^* \tau^*} x_1 + e^{iu^* \tau^*} x_2 \right)^3 \begin{pmatrix} (\bar{a})^{-1} \\ a^{-1} \end{pmatrix}. \end{aligned} \quad (\text{A.63})$$

Also note that

$$M_3^1(x^q e_j) = iu^* \tau^* (q_1 - q_2 + (-1)^j) x^q e_j, \quad |q| = 3, j = 1, 2, \quad (\text{A.64})$$

which implies that

$$\ker(M_3^1) \cap \text{span}\{x^q e_j : |q| = 3, j = 1, 2\} = \text{span}\{x_1^2 x_2 e_1, x_1 x_2^2 e_2\}. \quad (\text{A.65})$$

We can then derive

$$\frac{1}{3!} g_3^1(x, 0, 0) = 3b\tau^* \begin{pmatrix} (\bar{a})^{-1} (1 + iu^*) x_1^2 x_2 \\ a^{-1} (1 - iu^*) x_1 x_2^2 \end{pmatrix}. \quad (\text{A.66})$$

Consequently, the normal form on the center manifold becomes

$$\begin{aligned} \dot{x} &= iu^*\tau^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + i\mu u^* \begin{pmatrix} (\bar{a})^{-1}x_1 \\ -a^{-1}x_2 \end{pmatrix} \\ &+ \frac{1}{3!}g_3^1(x, 0, 0) + O(\mu^2|x|) + O(|x|^4) \end{aligned} \quad (\text{A.67})$$

for $x \in \mathbb{C}^2$. Changing to real coordinates by the change of variables

$$x = S\omega, \quad \text{with } S = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \quad (\text{A.68})$$

and letting

$$\rho^2 = x_1x_2 = \omega_1^2 + \omega_2^2, \quad (\text{A.69})$$

we obtain

$$\begin{aligned} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{pmatrix} &= u^*\tau^* \begin{pmatrix} \omega_2 \\ -\omega_1 \end{pmatrix} + \mu u^* \begin{pmatrix} \text{Im}(a^{-1})\omega_1 + \text{Re}(a^{-1})\omega_2 \\ -\text{Re}(a^{-1})\omega_1 + \text{Im}(a^{-1})\omega_2 \end{pmatrix} \\ &+ 3b\tau^*\rho^2 \begin{pmatrix} \text{Re}(a^{-1}(1-iu^*))\omega_1 - \text{Im}(a^{-1}(1-iu^*))\omega_2 \\ \text{Im}(a^{-1}(1-iu^*))\omega_1 + \text{Re}(a^{-1}(1-iu^*))\omega_2 \end{pmatrix} \\ &+ O(\mu^2|\omega|) + O(|\omega|^4). \end{aligned} \quad (\text{A.70})$$

If we use polar coordinates

$$\begin{aligned} \omega_1 &= \rho \cos \chi, \\ \omega_2 &= \rho \sin \chi, \end{aligned} \quad (\text{A.71})$$

then we find that

$$\begin{aligned} \dot{\rho} &= (a_1\mu + b_1\rho^2)\rho + O(\mu^2\rho) + O(\rho^4), \\ \dot{\chi} &= -u^*\tau^* + c_1\mu + d_1\rho^2 + O(\mu^2\rho) + O(\rho^4), \end{aligned} \quad (\text{A.72})$$

where $a_1, b_1, c_1,$ and d_1 are as in (A.48).

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