Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 957061, 10 pages
doi:10.1155/2009/957061

Research Article

On the Solvability of a Problem of Stationary Fluid Flow in a Helical Pipe

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Received 28 March 2009; Accepted 24 May 2009

Recommended by Mehrdad Massoudi

We consider a flow of incompressible Newtonian fluid through a pipe with helical shape. We suppose that the flow is governed by the prescribed pressure drop between pipe's ends. Such model has relevance to some important engineering applications. Under small data assumption, we prove the existence and uniqueness of the weak solution to the corresponding Navier-Stokes system with pressure boundary condition. The proof is based on the contraction method.

1. Introduction

Engineering practice requires extensive knowledge of flow through curved pipes. Helically coiled pipes are well-known types of curved pipes which have been used in wide variety of applications. They cover a large number of devices such as pipelines, air conditioners, refrigeration systems, central heating radiators, and chemical reactors. Therefore, numerous researchers have studied the fluid flow in helical pipes with circular cross-section both theoretically and experimentally. Let us just mention some of them. Wang [1] proposed a nonorthogonal coordinate system to investigate the effects of curvature and torsion on the low-Reynolds number flow in a helical pipe. The introduction of an orthogonal system of coordinates along a spatial curve allowed Germano [2, 3] to explore in a simpler way the effects of pipe's geometry on a helical pipe flow. Yamamoto et al. [4] first studied experimentally the effects of torsion and curvature on the flow characteristics in a helical tube. The results obtained by the experiments were then compared with those obtained from the model of Yamamoto et al. in [5]. Hüttl and Friedrich [6, 7] applied the second-order finite volume method for solving the incompressible Navier-Stokes equations to study the turbulent flow in helically coiled tubes. More numerical simulations on helical pipe flow can be found in the works of Yamamoto et al. [8] and Wang and Andrews [9].

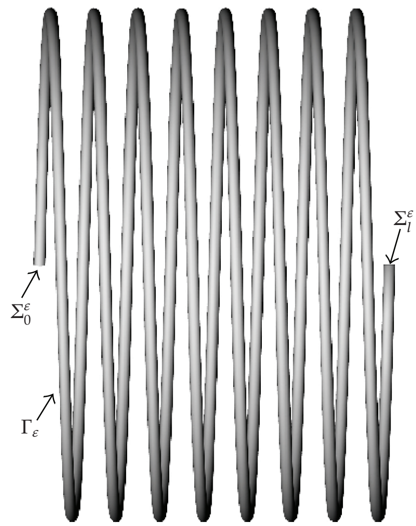

Figure 1: Helical pipe.

In this paper, we study the stationary flow of incompressible Newtonian fluid through a helical pipe with prescribed pressures at its ends. We suppose that the pipe's thickness and the helix step have the same small order $\mathcal{O}(\varepsilon)$, $0 < \varepsilon \ll 1$, while the diameter of the helix is larger, of order $\mathcal{O}(1)$ (see Figure 1). Such assumptions cover a large variety of realistic coiled pipes and they appear naturally in many devices as, for instance, the Liebig cooler, cooling channel in the nozzle of a rocket engine, particle separators used in the mineral processing industry, and so forth.

It is well-known that the flow of incompressible viscous Newtonian fluid is described by Navier-Stokes system of nonlinear PDEs. Their nonlinearity makes most boundary-value problems difficult or impossible to solve and thus the concept of weak solutions is introduced. When the velocity is prescribed on the whole boundary (Dirichlet condition), the existence of the weak solution can be proved by constructing approximate solutions via Galerkin method (see, e.g., Temam [10], although the main ideas goes back to Leray [11]). However, applications (as this one in the present paper) often give rise to problems where it is natural to prescribe the value of pressure on some part of the boundary. In case of (linear) Stokes system with pressure boundary condition, the existence and uniqueness of weak solution is obtained in Conca et al. [12]. For the Navier-Stokes system with boundary conditions involving pressure we only have some partial results and the full proof of existence is still unknown. The main difficulty lies in the fact that the integral

$$\int_\Omega (\mathbf{u}\nabla)\mathbf{u}\mathbf{u} = \frac{1}{2}\int_{\Gamma_p} |\mathbf{u}|^2(\mathbf{u}\cdot\mathbf{n}) \tag{1.1}$$

no longer equals to zero due to pressure boundary condition prescribed on $\Gamma_p \subset \partial\Omega$. (In our case $\Gamma_p = \Sigma_0^\varepsilon \cup \Sigma_\ell^\varepsilon$. see Figure 1) Here $\mathbf{u}$ denotes the velocity of the fluid which occupies a bounded domain $\Omega \subset \mathbf{R}^3$, while $\mathbf{n}$ stands for the unit outward normal on Γ_p.

As a consequence, the nonlinear term in corresponding variational formulation does not vanish (and we do not know how to control it) causing the absence of the energy equality. Described technical difficulty can be elegantly overcome by prescribing the so-called dynamic (Bernoulli) pressure $p + (1/2)|\mathbf{u}|^2$ as proposed by Conca et al. [12] (see also Łukaszewicz [13] for nonstationary flow). Nevertheless, it should be mentioned that there is no physical justification for prescribing such boundary condition in the case of viscous fluid. In the case of ideal fluid the interpretation of the dynamic pressure is given by the Bernoulli law.

Another possibility is to restrict to the case of small boundary data as in Heywood et al. [14] or Marušić-Paloka [15] (see also Marušić-Paloka [16]) and we follow such approach. By doing that, we manage to control the inertial term in Navier-Stokes equations and to obtain existence and uniqueness of the solution for the physically realistic situation (without prescribing the dynamic pressure).

The paper is organized as follows. First we describe pipe's geometry using parametrization $\mathbf{r}_\varepsilon(x_1) = (x_1, a\cos(x_1/\varepsilon), a\sin(x_1/\varepsilon))$ of its central curve. Here the small parameter ε stands for the distance between two coils of the helix. We assume that the pipe has (constant) circular cross-section of size ε and use Frenet basis attached to a helix to formally define our domain. Then we state the boundary-value problem describing the flow of a Newtonian fluid inside of the pipe and introduce the corresponding variational formulation. The main result is formulated in Theorem 3.3 where we precisely establish the assumption on the prescribed pressure drop under which the weak solution exists and then prove its uniqueness. The proof is based on the auxiliary result on sharp Sobolev constants (Lemma 3.1) and the contraction method.

We end this introduction by giving few more bibliographic remarks. The asymptotic behavior of the fluid flow through a helical pipe was investigated in Marušić-Paloka and Pažanin [17, 18]. Using the techniques from Marušić-Paloka [19], enabling the treatment of the curved geometry, the asymptotic approximation of the solution is built and rigorously justified by proving the error estimate in terms of the small parameter ε. Last but not least, let us mention that in Marušić-Paloka and Pažanin [20] the nonisothermal flow of Newtonian fluid through a general curved pipe has been considered. In such flow temperature changes cannot be neglected so the Navier-Stokes equations are coupled with heat conducting equation. A simplified model showing explicitly the effects of pipe's geometry is derived via rigorous asymptotic analysis with respect to the pipe's thickness.

2. Setting of the Problem

2.1. Pipe's Geometry

We start by defining the helix whose parametrization has the form

$$\mathbf{r}_\varepsilon(x_1) = \left(x_1, a\cos\frac{x_1}{\varepsilon}, a\sin\frac{x_1}{\varepsilon}\right), \quad x_1 \in [0, \ell], \tag{2.1}$$

serving to define the center curve of the pipe. It should be noted that the curve is not parameterized by its arc length (i.e., we do not use the natural parametrization) since in that case interval length ℓ would depend on the small parameter ε which is inconvenient for further analysis. At each point of the helix we compute the Frenet basis as follows

$$\begin{aligned}
\mathbf{t}_\varepsilon(x_1) &= \frac{\mathbf{r}'_\varepsilon(x_1)}{|\mathbf{r}'_\varepsilon(x_1)|} = \frac{1}{\sqrt{a^2+\varepsilon^2}}\left(\varepsilon, -a\sin\frac{x_1}{\varepsilon}, a\cos\frac{x_1}{\varepsilon}\right),\\
\mathbf{n}_\varepsilon(x_1) &= \frac{\mathbf{t}'_\varepsilon(x_1)}{|\mathbf{t}'_\varepsilon(x_1)|} = \left(0, -\cos\frac{x_1}{\varepsilon}, -\sin\frac{x_1}{\varepsilon}\right),\\
\mathbf{b}_\varepsilon(x_1) &= \mathbf{t}_\varepsilon(x_1)\times\mathbf{n}_\varepsilon(x_1) = \frac{1}{\sqrt{a^2+\varepsilon^2}}\left(a, \varepsilon\sin\frac{x_1}{\varepsilon}, -\varepsilon\cos\frac{x_1}{\varepsilon}\right).
\end{aligned} \tag{2.2}$$

It can be easily verified that the curvature (flexion) κ and the torsion τ are constant and given by

$$\kappa = \frac{a}{a^2+\varepsilon^2}, \qquad \tau = \frac{\varepsilon}{a^2+\varepsilon^2}. \tag{2.3}$$

For small parameter $\varepsilon > 0$ and a unit circle $B = B(0,1) \subset \mathbf{R}^2$, we introduce a thin straight pipe with circular cross-section:

$$S_\varepsilon = \left\{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 \in (0,\ell),\ \ x' = (x_2, x_3) \in \varepsilon B\right\} = (0,\ell)\times\varepsilon B. \tag{2.4}$$

We define the mapping $\mathbf{\Phi}_\varepsilon : S_\varepsilon \to \mathbf{R}^3$ by

$$\mathbf{\Phi}_\varepsilon(x) = \mathbf{r}_\varepsilon(x_1) + x_2\mathbf{n}_\varepsilon(x_1) + x_3\mathbf{b}_\varepsilon(x_1), \tag{2.5}$$

and we put

$$\Omega_\varepsilon = \mathbf{\Phi}_\varepsilon(S_\varepsilon). \tag{2.6}$$

Such domain is our thin pipe with helical shape filled with a viscous incompressible fluid. Finally, the lateral boundary of the pipe and its ends are denoted by

$$\Gamma_\varepsilon = \mathbf{\Phi}_\varepsilon((0,\ell)\times\varepsilon\partial B), \qquad \Sigma_i^\varepsilon = \mathbf{\Phi}_\varepsilon(\{i\}\times\varepsilon B), \quad i = 0,\ell. \tag{2.7}$$

2.2. The Equations

As mentioned before, the fluid inside of the pipe is assumed to be Newtonian so the velocity $\mathbf{u}_\varepsilon$ and pressure p_ε satisfy the following Navier-Stokes equations:

$$\begin{aligned}
-\mu\Delta\mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon\nabla)\mathbf{u}_\varepsilon + \nabla p_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon,\\
\operatorname{div}\mathbf{u}_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon.
\end{aligned} \tag{2.8}$$

Here $\mu > 0$ and we have added the subscript ε in our notation in order to stress the dependence of the solution on the small parameter. The above system must be completed by the boundary conditions:

$$\mathbf{u}_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \tag{2.9}$$

$$\mathbf{u}_\varepsilon \times \mathbf{t}_\varepsilon = 0, \quad \text{on } \Sigma_i^\varepsilon, \tag{2.10}$$

$$p_\varepsilon = q_i \quad \text{on } \Sigma_i^\varepsilon, \; i = 0, \ell. \tag{2.11}$$

The fluid flow is governed by the pressure drop between pipe's ends so in (2.11) we prescribe constant pressures q_0 and q_ℓ, $q_\ell < q_0$. In addition to the value of pressure, we need to prescribe something more on the boundary in order to assure that the problem is well possed. Thus, we take the tangential velocity to be zero on Σ_i^ε, while we keep the classical no-slip condition for the velocity on Γ_ε. Imposing that the tangential component of the velocity equals to zero is not a serious restriction since the only part that counts is the normal part, due to the Saint-Venant principle for thin domains (see, e.g., Marušić-Paloka [19]). Let us mention that instead of these conditions, one could prescribe the whole normal stress, including the viscous part, on the boundary (see, e.g., Heywood et al. [14]). Such situation can be treated using the same method, with a slight change of functional space in the corresponding variational formulation.

3. Existence and Uniqueness of the Solution

3.1. Variational Formulation

Let us introduce the following natural functional space:

$$V_\varepsilon = \left\{ \mathbf{v} \in H^1(\Omega_\varepsilon)^3 : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_\varepsilon, \; \mathbf{v} = 0 \text{ on } \Gamma_\varepsilon, \; \mathbf{v} \times \mathbf{t}_\varepsilon = 0 \text{ on } \Sigma_i^\varepsilon, \; i = 0, \ell \right\}. \tag{3.1}$$

The space V_ε is equipped with the norm

$$\|\mathbf{v}\|_{V_\varepsilon} = \|\nabla \mathbf{v}\|_{L^2(\Omega_\varepsilon)}, \tag{3.2}$$

which is equivalent to the $H^1(\Omega_\varepsilon)^3$-norm due to Poincaré's inequality. Now we can write the variational formulation of our problem (2.8)–(2.11):

$$\begin{gathered}\text{find } \mathbf{u}_\varepsilon \in V_\varepsilon \text{ such that} \\ \mu \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon \nabla \mathbf{v} + \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \nabla) \mathbf{u}_\varepsilon \mathbf{v} + q_0 \int_{\Sigma_0^\varepsilon} (\mathbf{v} \cdot \mathbf{t}_\varepsilon) + q_\ell \int_{\Sigma_\ell^\varepsilon} (\mathbf{v} \cdot \mathbf{t}_\varepsilon) = 0 \quad \forall \mathbf{v} \in V_\varepsilon.\end{gathered} \tag{3.3}$$

Notice that we have eliminated the pressure p_ε. Furthermore, it should be observed that replacing q_i by $q_i + C$ (C = const.) $i = 0, \ell$ changes nothing so the relevant quantity is the pressure drop $q_0 - q_\ell$. As a consequence, the pressure p_ε is determined only up to an additive constant. The equivalence between the variational (3.3) and differential (2.8)–(2.11) formulation, in case of smooth solutions, is discussed in the usual way (see, e.g., Conca et al. [12]).

3.2. *The Main Result*

The goal of this paper is to prove the existence and uniqueness of the solution for the variational problem (3.3), under small data assumption. In order to accomplish that, we need the following auxiliary result.

Lemma 3.1. *There exist constants $C_1, C_2 > 0$, independent of ε, such that*

$$\|\varphi\|_{L^2(\Omega_\varepsilon)} \leq C_1 \varepsilon \|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}, \tag{3.4}$$

$$\|\varphi\|_{L^4(\Omega_\varepsilon)} \leq C_2 \varepsilon^{1/4} \|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}, \tag{3.5}$$

for all $\varphi \in H^1(\Omega_\varepsilon)$ satisfying $\varphi = 0$ on Γ_ε.

Remark 3.2. It is well known that Poincaré's constant depends on the geometry of the domain Ω_ε. Inequality (3.4) gives the precise dependence of that constant on the small parameter ε. Inequality (3.5) enable us to estimate the inertial term in the variational formulation (3.3).

Proof of Lemma 3.1. Let $\varphi \in H^1(\Omega_\varepsilon)$ be such that $\varphi = 0$ on Γ_ε. We introduce new functions

$$\Theta(x) = \varphi(z), \qquad \Psi(x_1, y') = \Theta(x_1, \varepsilon y'), \tag{3.6}$$

with $z = \mathbf{\Phi}_\varepsilon(x)$. It is clear that $\Psi \in H^1(S)$, where $S = (0,\ell) \times B$ and $\Psi = 0$ on $\Gamma = (0,\ell) \times \partial B$. Now we extend Ψ by zero on a strip-like domain $S_{a,b} = (0,\ell) \times (a,b)^2$, where (a,b) is chosen such that $B \subset (a,b)^2$. A simple integration formula yields

$$\Psi(x_1, y') = \int_a^{y_2} \frac{\partial\Psi}{\partial y_2}(x_1, \xi, y_3)\, d\xi. \tag{3.7}$$

Using the Cauchy-Schwarz inequality, one can easily obtain

$$\int_{S_{a,b}} |\Psi(x_1, y')|^2 dx_1 dy' \leq \frac{(b-a)^2}{2} \int_{S_{a,b}} \left| \frac{\partial\Psi}{\partial y_2}(x_1, \xi, y_3) \right|^2 dx_1 d\xi\, dy_3. \tag{3.8}$$

Since $\Psi = 0$ on $S_{a,b} \setminus S$, it follows that

$$\|\Psi\|_{L^2(S)} \leq C \left\| \frac{\partial\Psi}{\partial y_2} \right\|_{L^2(S)}, \tag{3.9}$$

with C independent of ε. By a simple change of variables ($x' = \varepsilon y'$), we get

$$\int_{S_\varepsilon} |\Theta|^2 dx = \varepsilon^2 \int_S |\Psi|^2 dx_1 dy',$$
$$\int_{S_\varepsilon} \left|\frac{\partial \Theta}{\partial x_2}\right|^2 dx = \int_S \left|\frac{\partial \Psi}{\partial y_2}\right|^2 dx_1 dy'. \tag{3.10}$$

Direct calculation gives $\det \nabla \mathbf{\Phi}_\varepsilon = \sqrt{g_\varepsilon}$, where

$$g_\varepsilon = 1 + \frac{(x_2 - a)^2}{\varepsilon^2} - \frac{x_2^2}{a^2 + \varepsilon^2}, \tag{3.11}$$

implying the asymptotic behavior $\det \nabla \mathbf{\Phi}_\varepsilon = \sqrt{g_\varepsilon} = (a/\varepsilon)\sqrt{1 + \mathcal{O}(\varepsilon)}$. In view of that, by simple change of variables, we obtain

$$\int_{\Omega_\varepsilon} |\varphi|^2 dz = \int_{S_\varepsilon} |\Theta|^2 \sqrt{g_\varepsilon}\, dx \le \frac{C}{\varepsilon} \int_{S_\varepsilon} |\Theta|^2 dx,$$
$$\int_{S_\varepsilon} \left|\frac{\partial \Theta_\varepsilon}{\partial x_2}\right|^2 dx \le \int_{S_\varepsilon} |\nabla \Theta|^2 \sqrt{g_\varepsilon} \frac{1}{\sqrt{g_\varepsilon}} dx \le C\varepsilon \int_{\Omega_\varepsilon} |\nabla \varphi|^2 dz. \tag{3.12}$$

Now from (3.9)–(3.12) we deduce

$$\int_{\Omega_\varepsilon} |\varphi|^2 dz \le C\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla \varphi|^2 dz, \tag{3.13}$$

implying (3.4). Finally, using the interpolation inequality, the embedding $H^1(\Omega_\varepsilon) \hookrightarrow L^6(\Omega_\varepsilon)$, and the estimate (3.4) we obtain at once

$$\|\varphi\|_{L^4(\Omega_\varepsilon)} \le \|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/4} \|\varphi\|_{L^6(\Omega_\varepsilon)}^{3/4} \le C\varepsilon^{1/4} \|\nabla \varphi\|_{L^2(\Omega_\varepsilon)}. \tag{3.14}$$

□

Our main result can be stated as follows.

Theorem 3.3. *Assume that the pressure drop and the helix step are such that*

$$\varepsilon^2 (q_0 - q_\ell) < \frac{2\mu^2 \ell}{9 C_1 C_2^2}. \tag{3.15}$$

Then the problem (3.3) *admits at least one solution* $\mathbf{u}_\varepsilon \in V_\varepsilon$. *Moreover, such solution is unique in the ball:*

$$B_\varepsilon = \left\{ \mathbf{v} \in V_\varepsilon : \|\nabla \mathbf{v}\|_{L^2(\Omega_\varepsilon)} \le \frac{\mu}{3 C_2^2 \sqrt{\varepsilon}} \right\}, \tag{3.16}$$

where $C_1, C_2 > 0$ *are the constants from Lemma 3.1.*

Proof. The idea of the proof is to introduce the mapping $T : B_\varepsilon \to H^1(\Omega_\varepsilon)^3$ defined by $T(\mathbf{w}) = \mathbf{u}$, where $\mathbf{u}$ is the solution of the variational problem:

$$\begin{aligned}&\text{find } \mathbf{u} \in V_\varepsilon \text{ such that}\\ &\mu\int_{\Omega_\varepsilon} \nabla\mathbf{u}\nabla\mathbf{v} + \int_{\Omega_\varepsilon} (\mathbf{w}\nabla)\mathbf{u}\mathbf{v} + q_0\int_{\Sigma_0^\varepsilon} (\mathbf{v}\cdot\mathbf{t}_\varepsilon) + q_\ell\int_{\Sigma_\ell^\varepsilon} (\mathbf{v}\cdot\mathbf{t}_\varepsilon) = 0, \quad \forall\mathbf{v} \in V_\varepsilon.\end{aligned} \tag{3.17}$$

(Although the operator$T$ depends onε we drop the indexε for the sake of notational simplicity.) The first step is to show that mapping T is well defined. For that purpose, let $a : V_\varepsilon \times V_\varepsilon \to \mathbf{R}$ be the following bilinear form

$$a(\mathbf{u},\mathbf{v}) = \mu\int_{\Omega_\varepsilon} \nabla\mathbf{u}\nabla\mathbf{v} + \int_{\Omega_\varepsilon} (\mathbf{w}\nabla)\mathbf{u}\mathbf{v}. \tag{3.18}$$

Using estimate (3.5), we get

$$a(\mathbf{u},\mathbf{u}) \geq \left(\mu - C_2^2\sqrt{\varepsilon}\|\nabla\mathbf{w}\|_{L^2(\Omega_\varepsilon)}\right)\|\nabla\mathbf{u}\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.19}$$

Since $\mathbf{w} \in B_\varepsilon$, we conclude

$$a(\mathbf{u},\mathbf{u}) \geq \frac{2\mu}{3}\|\nabla\mathbf{u}\|^2_{L^2(\Omega_\varepsilon)}, \quad \forall\mathbf{u} \in V_\varepsilon, \tag{3.20}$$

implying that the form $a(\cdot,\cdot)$ is elliptic on V_ε. Analogously, we can easily obtain that

$$|a(\mathbf{u},\mathbf{v})| \leq \frac{4\mu}{3}\|\nabla\mathbf{u}\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{v}\|_{L^2(\Omega_\varepsilon)}. \tag{3.21}$$

Thus, due to Lax-Milgram Lemma, problem (3.17) has a unique solution and T is well defined on B_ε.

The next step is to prove that $T(B_\varepsilon) \subset B_\varepsilon$. Applying (3.4), we obtain

$$\begin{aligned}&\left|q_0\int_{\Sigma_0^\varepsilon} (\mathbf{v}\cdot\mathbf{t}_\varepsilon) + q_\ell\int_{\Sigma_\ell^\varepsilon} (\mathbf{v}\cdot\mathbf{t}_\varepsilon)\right|\\ &\quad = \left|\int_{\Omega_\varepsilon} \operatorname{div}\left(\left(q_0 + \frac{q_\ell - q_0}{\ell}x_1\right)\mathbf{v}\right)\right| \leq \frac{q_0 - q_\ell}{\ell}C_1\varepsilon\sqrt{\varepsilon}\|\nabla\mathbf{v}\|_{L^2(\Omega_\varepsilon)}.\end{aligned} \tag{3.22}$$

From (3.17) and (3.20), it follows that

$$\|\nabla T(\mathbf{w})\|_{L^2(\Omega_\varepsilon)} = \|\nabla\mathbf{u}\|_{L^2(\Omega_\varepsilon)} \leq \frac{3}{2\mu}\frac{q_0 - q_\ell}{\ell}C_1\varepsilon\sqrt{\varepsilon}. \tag{3.23}$$

Taking into account the condition (3.15), we obtain

$$\|\nabla T(\mathbf{w})\|_{L^2(\Omega_\varepsilon)} < \frac{\mu}{3C_2^2\sqrt{\varepsilon}}, \tag{3.24}$$

proving that $T(B_\varepsilon) \subset B_\varepsilon$.

Now it remains to prove that T is a contraction which will enable us to use Banach fixed point theorem. Let $\mathbf{w}_1, \mathbf{w}_2 \in B_\varepsilon$, $\mathbf{w}_1 \neq \mathbf{w}_2$ be such that $T(\mathbf{w}_1) = \mathbf{u}_1$, $T(\mathbf{w}_2) = \mathbf{u}_2$, with $\mathbf{u}_1$ and $\mathbf{u}_2$ being the solutions of problem (3.17) for $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. Subtracting the corresponding equations it follows that

$$\mu\int_{\Omega_\varepsilon} \nabla(T(\mathbf{w}_1) - T(\mathbf{w}_2))\nabla\mathbf{v} + \int_{\Omega_\varepsilon} ((\mathbf{w}_1\nabla)T(\mathbf{w}_1) - (\mathbf{w}_2\nabla)T(\mathbf{w}_2))\mathbf{v} = 0, \quad \forall\mathbf{v} \in V_\varepsilon. \tag{3.25}$$

Setting $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$, we obtain

$$\begin{aligned} &\mu\|\nabla(T(\mathbf{w}_1) - T(\mathbf{w}_2))\|_{L^2(\Omega_\varepsilon)}^2 \\ &\quad = -\int_{\Omega_\varepsilon} ((\mathbf{w}_1 - \mathbf{w}_2)\nabla)T(\mathbf{w}_1)(T(\mathbf{w}_1) - T(\mathbf{w}_2)) \\ &\qquad - \int_{\Omega_\varepsilon} (\mathbf{w}_2\nabla)(T(\mathbf{w}_1) - T(\mathbf{w}_2))(T(\mathbf{w}_1) - T(\mathbf{w}_2)). \end{aligned} \tag{3.26}$$

For the first integral, we have

$$\begin{aligned} &\left|\int_{\Omega_\varepsilon} ((\mathbf{w}_1 - \mathbf{w}_2)\nabla)T(\mathbf{w}_1)(T(\mathbf{w}_1) - T(\mathbf{w}_2))\right| \\ &\quad \leq C_2^2\sqrt{\varepsilon}\|\nabla(\mathbf{w}_1 - \mathbf{w}_2)\|_{L^2(\Omega_\varepsilon)}\|\nabla T(\mathbf{w}_1)\|_{L^2(\Omega_\varepsilon)}\|\nabla(T(\mathbf{w}_1) - T(\mathbf{w}_2))\|_{L^2(\Omega_\varepsilon)} \\ &\quad \leq \frac{\mu}{3}\|\nabla(\mathbf{w}_1 - \mathbf{w}_2)\|_{L^2(\Omega_\varepsilon)}\|\nabla(T(\mathbf{w}_1) - T(\mathbf{w}_2))\|_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{3.27}$$

Here we used the estimate (3.5) and the fact that $T(\mathbf{w}_1) \in B_\varepsilon$. For the second integral, we proceed in the similar way and obtain

$$\left|\int_{\Omega_\varepsilon} (\mathbf{w}_2\nabla)(T(\mathbf{w}_1) - T(\mathbf{w}_2))(T(\mathbf{w}_1) - T(\mathbf{w}_2))\right| \leq \frac{\mu}{3}\|\nabla(T(\mathbf{w}_1) - T(\mathbf{w}_2))\|_{L^2(\Omega_\varepsilon)}^2, \tag{3.28}$$

leading to

$$\|\nabla(T(\mathbf{w}_1) - T(\mathbf{w}_2))\|_{L^2(\Omega_\varepsilon)} \leq \frac{1}{2}\|\nabla(\mathbf{w}_1 - \mathbf{w}_2)\|_{L^2(\Omega_\varepsilon)}, \tag{3.29}$$

implying that T is a contraction. Now Banach fixed point theorem provides the existence and uniqueness of $\mathbf{u}_\varepsilon$ finishing the proof. □

Remark 3.4. Regarding the uniqueness of the solution, it should be observed that we managed to prove only that the weak solution $\mathbf{u}_\varepsilon$ is unique in some ball B_ε around zero, under small data assumption (3.15). It means that we cannot exclude the existence of some other solutions with large norm. The existence and uniqueness (up to an additive constant) of the pressure $p \in L^2(\Omega_\varepsilon)$ satisfying the governing equations in the sense of the distributions can be obtained in the standard way using De Rham theorem (see, e.g., Temam [10]).

Acknowledgment

This research was supported by the Ministry of Science, Education and Sports, Republic of Croatia, Grant 037-0372787-2797.

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