Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 147195, 11 pages
doi:10.1155/2010/147195

Research Article

Controllability of Second-Order Equations in $L^2(\Omega)$

Hugo Leiva[1] and Nelson Merentes[2]

[1] *Departamento de Matemática, Facultad de Ciencias, Universidad de los Andes, Mérida 5101, Venezuela*
[2] *Departamento de Matemática, Facultad de Ciencias, Universidad Central de Venezuela, Caracas 1020, Venezuela*

Correspondence should be addressed to Hugo Leiva, hleiva@ula.ve

Received 24 August 2010; Accepted 13 November 2010

Academic Editor: Christos H. Skiadas

We present a simple proof of the interior approximate controllability for the following broad class of second-order equations in the Hilbert space $L^2(\Omega)$: $\ddot{y} + Ay = 1_\omega u(t)$, $t \in (0,\tau]$, $y(0) = y_0$, $\dot{y}(0) = y_1$, where Ω is a domain in $\mathbb{R}^N (N \geq 1)$, $y_0, y_1 \in L^2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control u belongs to $L^2(0,\tau; L^2(\Omega))$, and $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ is an unbounded linear operator with the following spectral decomposition: $Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k}\rangle \phi_{j,k}$, with the eigenvalues λ_j given by the following formula: $\lambda_j = j^{2m}\pi^{2m}$, $j = 1,2,3,\ldots$ and $m \geq 1$ is a fixed integer number, multiplicity γ_j is equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors (eigenfunctions) of A. Specifically, we prove the following statement: if for an open nonempty set $\omega \subset \Omega$ the restrictions $\phi^{\omega}_{j,k} = \phi_{j,k}|_\omega$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, then for all $\tau \geq 2/\pi^{m-1}$ the system is approximately controllable on $[0,\tau]$. As an application, we prove the controllability of the 1D wave equation.

1. Introduction

This paper has been motivated by the work in [1] and the articles [2, 3], where a new technique is used to prove the interior approximate controllability of some diffusion process. Particularly in [3], where the authors prove the interior approximate controllability of the following broad class of reaction diffusion equations in the Hilbert space $Z = L^2(\Omega)$ given by

$$z' = -Az + 1_\omega u(t), \quad t \in [0,\tau], \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^n$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in L^2(0,\tau; L^2(\Omega))$ and $A : D(A) \subset Z \to Z$ is an unbounded linear operator with the spectral decomposition

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \tag{1.2}$$

The eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors of A. The operator $-A$ generates a strongly continuous semigroup $\{T_A(t)\}_{t\geq 0}$ given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \tag{1.3}$$

As a consequence of this result, the controllability of the following heat equation follows trivially by putting $A = -\Delta$

$$\begin{aligned} z_t &= \Delta z + 1_\omega u(t,x), \quad \text{in } (0,\tau) \times \Omega, \\ z &= 0, \quad \text{on } (0,\tau) \times \partial\Omega, \\ z(0,x) &= z_0(x), \quad \text{in } \Omega. \end{aligned} \tag{1.4}$$

Following [1–3], in this paper, we study the interior approximate controllability of the following broad class of second-order equations in the Hilbert space $L^2(\Omega)$:

$$\begin{aligned} \ddot{y} + Ay &= 1_\omega u(t), \qquad t \in (0,\tau], \\ y(0) &= y_0, \qquad \dot{y}(0) = y_1, \end{aligned} \tag{1.5}$$

where the eigenvalues λ_j of the operator of A are given by the following formula:

$$\lambda_j = j^{2m}\pi^{2m}, \quad j = 1,2,3,\ldots, \ m \geq 1 \text{ is a fixed integer number.} \tag{1.6}$$

Specifically, we prove the following statement: if for an open nonempty set $\omega \subset \Omega$ the restrictions $\phi_{j,k}^{\omega} = \phi_{j,k}|_{\omega}$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, then for all $\tau \geq 2/\pi^{m-1}$ the system (1.5) is approximately controllable on $[0,\tau]$. Moreover, we can exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time (see Theorem 2.8).

This result implies the interior controllability of the following well-known examples of partial differential equations.

Example 1.1. The 1D Wave Equation

$$\begin{gathered} y_{tt} - \Delta y = 1_{\omega} u(t,x), \quad \text{in } (0,\tau] \times \Omega, \\ y = 0, \quad \text{on } (0,\tau) \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in \Omega, \end{gathered} \tag{1.7}$$

where ω is an open nonempty subset of $\Omega = [0,1]$, 1_{ω} denotes the characteristic function of the set ω, and the distributed control $u \in L^2(0,\tau; L^2(\Omega))$.

Example 1.2. The Model of Vibrating String Equation

$$\begin{gathered} w_{tt} + \Delta^2 w = 1_{\omega} u(t,x), \quad \text{in } (0,\tau] \times \Omega, \\ w = \Delta w = 0, \quad \text{on } (0,\tau) \times \partial\Omega, \\ w(0,x) = \phi_0(x), \quad w_t(0,x) = \psi_0(x), \quad x \in \Omega, \end{gathered} \tag{1.8}$$

where $\Omega = [0,1]$, ω is an open nonempty subset of Ω, $u \in L^2([0,\tau]; L^2(\Omega))$, $\phi_0, \psi_0 \in L^2(\Omega)$.

2. Main Results

In this section, we will prove the main result of this work; to this end, we consider by $X = U = L^2(\Omega)$ and the linear unbounded operator $A : D(A) \subset X \to X$ can be written as follows.

(a) For all $x \in D(A)$, we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j E_j \xi, \tag{2.1}$$

where

$$E_j x = \sum_{k=1}^{\gamma_j} \langle \xi, \phi_{j,k} \rangle \phi_{j,k}. \tag{2.2}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in X and $x = \sum_{j=1}^{\infty} E_j x,\ x \in X$.

(b) The semigroup $\{T_A(t)\}$ generated by $-A$ can be written as follows:

$$T_A(t)x = \sum_{j=1}^{\infty} e^{-\lambda_j t} \mathrm{E}_j x. \tag{2.3}$$

(c) The fractional powered spaces X^r are given by

$$X^r = D(A^r) = \left\{ x \in X : \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 < \infty \right\}, \quad r \geq 0, \tag{2.4}$$

with the norm

$$\|x\|_{\mathrm{r}} = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r,$$
$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \tag{2.5}$$

Also, for $r \geq 0$, we define $Z_r = X^r \times X$, which is a Hilbert space endowed with the norm given by

$$\left\| \begin{bmatrix} y \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|y\|_r^2 + \|v\|^2. \tag{2.6}$$

Proposition 2.1. *The operator $P_j : Z_r \longrightarrow Z_r$, $j \geq 0$, defined by*

$$p_j = \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix}, \quad j \geq 1, \tag{2.7}$$

is a continuous (bounded) orthogonal projections in the Hilbert space Z_r.

Proof. First we will show that $P_j(Z_r) \subset Z_r$, which is equivalent to show that $E_j(X^r) \subset X^r$. In fact, let x be in X^r and consider $E_j x$. Then,

$$\sum_{n=1}^{\infty} \lambda_n^{2r} \left\| E_n E_j x \right\|^2 = \lambda_j^{2r} \left\| E_j x \right\|^2 < \infty. \tag{2.8}$$

Therefore, $E_j x \in X^r$, for all $x \in X^r$.

Now, we will prove that this projection is bounded. In fact, from the continuous inclusion $X^r \subset X$, there exists a constant $k > 0$ such that

$$\|x\| \leq k\|x\|_r, \quad \forall x \in X^r. \tag{2.9}$$

Then, for all $x \in X^r$, we have the following estimate

$$\left\|E_j x\right\|_r^2 = \sum_{n=1}^{\infty} \lambda_n^{2r} \left\| E_n E_j x \right\|^2 = \lambda_j^{2r} \left\| E_j x \right\|^2 \leq \lambda_j^{2r} \|x\|^2 \leq \lambda_j^{2r} k^2 \|x\|_r^2. \tag{2.10}$$

Hence, $\|E_j x\| \leq \lambda_j^r k \|x\|_r$, which implies the continuity of $E_j : X^r \to X^r$. So, P_j is a continuous projection on Z_r. □

Hence, with the change of variable $y' = v$, the system (1.5) can be written as a first-order system of ordinary differential equations in the Hilbert space $Z_{1/2} = X^{1/2} \times X$ as follows:

$$z' = \mathcal{A}z + B_\omega u, \quad z(0) = z_0, \quad z \in Z_{1/2}, \quad t \in (0, \tau], \tag{2.11}$$

where

$$z = \begin{bmatrix} y \\ v \end{bmatrix}, \qquad B_\omega = \begin{bmatrix} 0 \\ 1_\omega \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -A & 0 \end{bmatrix} \tag{2.12}$$

is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times D(A^{1/2})$.

The proof of the following theorem follows in the same way as [4, Theorem 3.1], by putting $c = 0$ and $d = 1$ or directly from [5, lemma 2.1] or [6, Lemma 3.1].

Theorem 2.2. *The operator $\mathcal{A}$ given by* (2.12) *is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\in\mathbb{R}}$ given by*

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, \; t \geq 0, \tag{2.13}$$

where $\{P_j\}_{j\geq 1}$ is a complete family of orthogonal projections in the Hilbert space $Z_{1/2}$ given by

$$\begin{gathered} P_j = \operatorname{diag}[E_j, E_j], \quad j \geq 1, \\ A_j = R_j P_j, \qquad R_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j & 0 \end{bmatrix}. \end{gathered} \tag{2.14}$$

Also,

$$A_j^* = R_j^* P_j, \qquad R_j^* = \begin{bmatrix} 0 & -1 \\ \lambda_j & 0 \end{bmatrix}. \tag{2.15}$$

Moreover, $e^{A_j s} = e^{R_j s} P_j$ and the eigenvalues of R_j are $\sqrt{\lambda_j} i$ and $-\sqrt{\lambda_j} i$.

Now, before proving the main theorem, we will give the definition of approximate controllability for this system. To this end, for all $z_0 \in Z_{1/2}$ and $u \in L^2(0, \tau; U)$, the initial value problem

$$\begin{gathered} z' = \mathcal{A}z + B_\omega u(t), \quad z \in Z, \; t \in (0, \tau], \\ z(0) = z_0, \end{gathered} \tag{2.16}$$

where the control function u belongs to $L^2(0,\tau;U)$, admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)B_\omega u(s)ds, \quad t \in [0,\tau]. \tag{2.17}$$

Definition 2.3 (Approximate Controllability). *The system* (2.16) *is said to be approximately controllable on* $[0,\tau]$ *if for every* z_0, $z_1 \in Z_{1/2}$, $\varepsilon > 0$ *there exists* $u \in L^2(0,\tau;U)$ *such that the solution* $z(t)$ *of* (2.17) *corresponding to* u *verifies*

$$z(0) = z_0, \qquad \|z(\tau) - z_1\| < \varepsilon. \tag{2.18}$$

Consider the following bounded linear operator:

$$G : L^2(0,\tau;Z) \longrightarrow Z, \qquad Gu = \int_0^\tau T(\tau-s)B_\omega u(s)ds, \tag{2.19}$$

whose adjoint operator $G^* : Z \to L^2(0,\tau;Z)$ is given by

$$(G^*z)(s) = B_\omega^* T^*(\tau-s)z, \quad \forall s \in [0,\tau],\ \forall z \in Z. \tag{2.20}$$

The following lemma is trivial

Lemma 2.4. *The equation* (2.16) *is approximately controllable on* $[0,\tau]$ *if, and only if,* $\overline{\mathrm{Rang}(G)} = Z$.

The following result is well known from linear operator theory.

Lemma 2.5. *Let* W *and* Z *be Hilbert spaces and* $G^* \in L(Z,W)$ *the adjoint operator of the linear operator* $G \in L(W,Z)$. *Then,*

$$\overline{\mathrm{Rang}(G)} = Z \Longleftrightarrow \mathrm{Ker}(G^*) = \{0\}. \tag{2.21}$$

As a consequence of the foregoing Lemma, one can prove the following result.

Lemma 2.6. *Let* W *and* Z *be Hilbert spaces and* $G^* \in L(Z,W)$ *the adjoint operator of the linear operator* $G \in L(W,Z)$. *Then,* $\overline{\mathrm{Rang}(G)} = Z$ *if, and only if, one of the following statements holds:*

(a) $\mathrm{Ker}(G^*) = \{0\}$,

(b) $\langle GG^*z, z\rangle > 0$, $z \neq 0$ *in* Z,

(c) $\lim_{\alpha\to 0^+} \alpha(\alpha I + GG^*)^{-1}z = 0$,

(d) $\sup_{\alpha>0} \|\alpha(\alpha I + GG^*)^{-1}\| \leq 1$.

The following theorem follows directly from (2.20) and Lemmas 2.4 and 2.6.

Theorem 2.7. *The equation* (2.16) *is approximately controllable on* $[0,\tau]$ *iff*

$$B_\omega^* T^*(t)z = 0, \quad \forall t \in [0,\tau], \Longrightarrow z = 0. \tag{2.22}$$

Now, we are ready to formulate and prove the main theorem of this work.

Theorem 2.8 (Main Theorem). *If for an open nonempty set $\omega \subset \Omega$ the restrictions $\phi^{\omega}_{j,k} = \phi_{j,k}|_{\omega}$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, then for all $\tau \geq 2/\pi^{m-1}$ the system* (2.16) *is approximately controllable on* $[0,\tau]$*. Moreover, a sequence of controls steering the system* (2.16) *from initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$ is given by*

$$u_{\alpha}(t) = B^{*}_{\omega}T^{*}(\tau - t)(\alpha I + GG^{*})^{-1}(z_1 - T(\tau)z_0), \tag{2.23}$$

and the error of this approximation E_{α} is given by

$$E_{\alpha} = \alpha(\alpha I + GG^{*})^{-1}(z_1 - T(\tau)z_0). \tag{2.24}$$

Proof. We will apply Theorem 2.7 to prove the controllability of system (2.16). To this end, we observe that the adjoint of operator B_{ω} is by

$$\begin{gathered} B^{*}_{\omega} = \begin{bmatrix} 0 & 1_{\omega} \end{bmatrix}, \\ T^{*}(t)z = \sum_{j=1}^{\infty} e^{A^{*}_{j}t}P_j z, \quad z \in Z,\ t \geq 0. \end{gathered} \tag{2.25}$$

Therefore,

$$B^{*}_{\omega}T^{*}(t)z = \sum_{j=1}^{\infty} B^{*}_{\omega}e^{R^{*}_{j}t}P_j z. \tag{2.26}$$

On the other hand, we have that

$$e^{R^{*}_{j}t} = \begin{bmatrix} \cos\sqrt{\lambda_j}t & -\dfrac{1}{\sqrt{\lambda_j}}\sin\sqrt{\lambda_j}t \\ \sqrt{\lambda_j}\sin\sqrt{\lambda_j}t & \cos\sqrt{\lambda_j}t \end{bmatrix} = \begin{bmatrix} \cos j^m\pi^m t & -\dfrac{1}{j^m\pi^m}\sin j^m\pi^m t \\ j^m\pi^m \sin j^m\pi^m t & \cos j^m\pi^m t \end{bmatrix}. \tag{2.27}$$

Suppose for all $z \in Z_{1/2}$ that

$$B^{*}_{\omega}T^{*}(t)z = \sum_{j=1}^{\infty}\left\{ j^m\pi^m \sin(j^m\pi^m t)(1_{\omega}E_j z_1) + \cos(j^m\pi^m t)(1_{\omega}E_j z_2)\right\} = 0, \quad \forall t \in [0,\tau]. \tag{2.28}$$

Then, if we make the change of variable $s = \pi^m t$, we obtain that

$$\sum_{j=1}^{\infty}\left\{ j^m\pi^m \sin(j^m s)(1_{\omega}E_j z_1) + \cos(j^m s)(1_{\omega}E_j z_2)\right\} = 0, \quad \forall s \in \left[0, \pi^k\tau\right]. \tag{2.29}$$

Since $\tau \geq 2/\pi^{m-1}$ we get that

$$\sum_{j=1}^{\infty}\left\{j^m\pi^m \sin\left(j^m s\right)\left(1_\omega E_j z_1\right) + \cos\left(j^m s\right)\left(1_\omega E_j z_2\right)\right\} = 0, \quad \forall s \in [0, 2\pi]. \tag{2.30}$$

On the order hand, it is well known that $\{1, \cos(ns), \sin(ns) : n = 1, 2, 3, \ldots\}$ is an orthogonal base of $L^2[0, 2\pi]$, which implies that $\{\cos(j^m s), \sin(j^m s) : j = 1, 2, 3, \ldots\}$ is an orthogonal set in $L^2[0, 2\pi]$, and therefore

$$\left(1_\omega E_j z_1\right)(x) = 0, \quad \left(1_\omega E_j z_2\right)(x) = 0, \quad \forall x \in \Omega,\ j = 1, 2, 3, \ldots, \tag{2.31}$$

that is,

$$\left(E_j z_1\right)(x) = 0, \quad \left(E_j z_2\right)(x) = 0, \quad \forall x \in \omega,\ j = 1, 2, 3, \ldots, \tag{2.32}$$

that is,

$$\sum_{k=1}^{\gamma_j}\langle z_1, \phi_{j,k}\rangle\phi_{j,k}(x) = 0, \quad \sum_{k=1}^{\gamma_j}\langle z_2, \phi_{j,k}\rangle\phi_{j,k}(x) = 0, \quad \forall x \in \omega,\ j = 1, 2, 3, \ldots. \tag{2.33}$$

Since the restrictions $\phi_{j,k}^{\omega} = \phi_{j,k}|_{\omega}$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, we get that

$$\langle z_1, \phi_{j,k}\rangle = 0, \quad \langle z_2, \phi_{j,k}\rangle = 0, \quad j = 1, 2, 3, \ldots;\ k = 1, 2, \ldots, \gamma_j. \tag{2.34}$$

Therefore,

$$P_j(z) = \begin{bmatrix} E_j(z_1) \\ E_j(z_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.35}$$

Hence, $z = \sum_{j=1}^{\infty} P_j z = 0$, and the proof of the approximate controllability of the system (2.16) is completed.

Now, given the initial and the final states z_0 and z_1, we consider the sequence of controls

$$\begin{aligned} u_\alpha(\cdot) &= B_\omega^* T^*(\tau - \cdot)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) \\ &= G^*(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0), \quad \alpha > 0. \end{aligned} \tag{2.36}$$

Then,

$$\begin{aligned} Gu_\alpha &= GG^*(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) \\ &= (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) \\ &= z_1 - T(\tau)z_0 - \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0). \end{aligned} \tag{2.37}$$

From part (c) of Lemma 2.6, we know that

$$\lim_{\alpha \to 0^+} \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) = 0. \tag{2.38}$$

Therefore,

$$\lim_{\alpha \to 0^+} Gu_\alpha = z_1 - T(\tau)z_0, \tag{2.39}$$

that is,

$$\lim_{\alpha \to 0^+} \left\{ T(\tau)z_0 + \int_0^\tau T(\tau - s)B_\omega u_\alpha(s)ds \right\} = z_1. \tag{2.40}$$

This completes the proof of the theorem. □

The following basic theorem will be used to prove an important consequence of the foregoing theorem.

Theorem 2.9 (see [7, Theorem 1.23, page 20]). *Suppose $\Omega \subset \mathbb{R}^n$ is open, nonempty, and connected set, and f is real analytic function in Ω with $f = 0$ on a nonempty open subset ω of Ω. Then, $f = 0$ in Ω.*

Corollary 2.10. *If $\phi_{j,k}$ are analytic functions on Ω, then for all open nonempty set $\omega \subset \Omega$ and all $\tau \geq 2/\pi^{m-1}$ the system* (2.16) *is approximately controllable on* $[0, \tau]$.

Proof. It is enough to prove that, for all open nonempty set $\omega \subset \Omega$ the restrictions $\phi^{\omega}_{j,k} = \phi_{j,k}|_\omega$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, which follows directly from Theorem 2.9. □

3. Applications

For the applications, we will use Corollary 2.10 and the following fact.

Theorem 3.1 (see [3]). *The eigenfunctions of the operator $-\Delta$ with Dirichlet boundary conditions on Ω are real analytic functions in Ω.*

In this section, we will prove the approximate controllability of (1.7) and (1.8). Specifically, we will prove the following theorem.

Theorem 3.2. *For all open nonempty set $\omega \subset \Omega = [0,1]$, we have the following statements.*

(a) *For all $\tau \geq 2$ the system* (1.7) *is approximately controllable on* $[0,\tau]$.

(b) *For all $\tau \geq 2/\pi$ the system* (1.8) *is approximately controllable on* $[0,\tau]$.

Proof. Let $X = L^2(\Omega)$ and consider the linear unbounded operator $-\Delta : D(-\Delta) \subset X \to X$ with $D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$. In this case, the eigenvalues and the eigenfunctions of $A = -\Delta$ are given, respectively, by

$$\lambda_j = j^2\pi^2, \quad \phi_j(x) = \sqrt{2}\sin(j\pi x), \quad j = 1,2,3,\ldots,$$
$$Ax = -\Delta x = \sum_{j=1}^{\infty} \lambda_j \langle x, \phi_j \rangle \phi_j = \sum_{j=1}^{\infty} j^2\pi^2 \langle x, \phi_j \rangle \phi_j. \tag{3.1}$$

Then, $m = 1$ and $\tau \geq 2$. So, (a) follows from Corollary 2.10.

To prove (b), we consider the operator

$$Ax = (-\Delta)^2 x = \sum_{j=1}^{\infty} \lambda_j^2 \langle x, \phi_j \rangle \phi_j = \sum_{j=1}^{\infty} j^4\pi^4 \langle x, \phi_j \rangle \phi_j. \tag{3.2}$$

Then, $m = 2$ and $\tau \geq 2/\pi$. So, (b) follows from Corollary 2.10. □

4. Final Remark

The result presented in this paper can be formulated in a more general setting. Indeed, we can consider the following second-order evolution equation in a general Hilbert space X:

$$\ddot{y} + Ay = Cu(t), \quad t \in (0,\tau],$$
$$y(0) = y_0, \qquad \dot{y}(0) = y_1, \tag{4.1}$$

where, $A : D(A) \subset X \to X$ is an unbounded linear operator in X with the spectral decomposition given by (1.2), the control $u \in L^2(0,\tau;X)$ and $C : X \to X$ is a linear and bounded operator (linear and continuous). In this case, the characteristic function set is a particular operator C, and the following theorem is a generalization of Theorem 2.8.

Theorem 4.1. *If the vectors $C^*\phi_{j,k}$ are linearly independent in X, then for all $\tau \geq 2/\pi^{m-1}$ the system* (4.1) *is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system* (2.16) *from initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$ is given by*

$$u_\alpha(t) = B^*T^*(\tau - t)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0), \tag{4.2}$$

and the error of this approximation E_α is given by

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0), \tag{4.3}$$

where the operator B is given by $B = \begin{bmatrix} 0 \\ C \end{bmatrix}$.

The novelty of this result is based on the fact that, it is general, rigorous, applicable, and easily comprehensible by those young mathematician who are located in places away from majors research center.

Acknowledgment

This work was supported by the CDCHT-ULA-project: C-1667-09-05-AA and BCV.

References

[1] S. Badraoui, "Approximate controllability of a reaction-diffusion system with a cross-diffusion matrix and fractional derivatives on bounded domains," *Boundary Value Problems*, vol. 2010, Article ID 281238, 14 pages, 2010.

[2] H. Larez and H. Leiva, "Interior controllability of a 2×2 reaction-diffusion system with cross-diffusion matrix," *Boundary Value Problems*, vol. 2009, Article ID 560407, 9 pages, 2009.

[3] H. Leiva and Y. Quintana, "Interior controllability of a broad class of reaction diffusion equations," *Mathematical Problems in Engineering*, vol. 2009, Article ID 708516, 8 pages, 2009.

[4] H. Leiva, "Exact controllability of the suspension bridge model proposed by Lazer and McKenna," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 404–419, 2005.

[5] H. Leiva, "A lemma on C_0-semigroups and applications," *Quaestiones Mathematicae*, vol. 26, no. 3, pp. 247–265, 2003.

[6] A. Carrasco and H. Leiva, "Variation of constants formula for functional parabolic partial differential equations," *Electronic Journal of Differential Equations*, vol. 2007, no. 130, pp. 1–20, 2007.

[7] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, vol. 137 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1992.